

# Geodesic Deviation of Photons in Einstein and Higher Derivative Gravity

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## Abstract

We derive the wave equation obeyed by electromagnetic fields in curved space-time. We find that there are Riemann and Ricci curvature coupling terms to the photon polarisation which result in a polarisation dependent deviation of the photon trajectories from null geodesics. Photons are found to have an effective mass in an external gravitational field and their velocity in an inertial frame is in general less than  $c$ . The effective photon mass in the Schwarzschild metric is  $m_\gamma = (2GM/r^3)^{1/2}$  and near the horizon it is larger than the Hawking temperature of the blackhole. Our result implies that Hawking radiation of photons would not take place. We also conclude that there is no superluminal photon velocity in higher derivative gravity theories (arising from QED radiative corrections), as has been claimed in literature. We show that these erroneous claims are due to the neglect of the Riemann and Ricci coupling terms which exist in Einstein's gravity.

A standard result of Einstein's gravity is that the trajectories of all massless particles are null geodesics. A question worth examining is whether there is a deviation from the null geodesics if the particles have a spin i.e. due to the interaction of spin with the Riemann and Ricci curvatures of the gravitational fields. In this letter we have studied this question for the case of photon propagation in a curved background. Starting with the action  $\sqrt{-g}F_{\mu\nu} F^{\mu\nu}$  for electromagnetic fields in a gravitational field, we derive the wave equation for electromagnetic field tensor  $F_{\mu\nu}$ , which turns out to be of the form (Noonan [1])

$$\nabla^\mu \nabla_\mu F_{\nu\lambda} + R_{\rho\mu\nu\lambda} F^{\rho\mu} - R^\rho{}_\lambda F_{\nu\rho} + R^\rho{}_\nu F_{\lambda\rho} = 0 \quad (1)$$

We see that the photon propagation depends upon the coupling between the Riemann and Ricci curvatures and the photon polarisation. This leads to a deviation of the photon trajectories from the null geodesic by amounts proportional to the Riemann and Ricci curvatures. The photon trajectories in the geometrical optics limit are described by the following generalisation of the geodesic equation

$$\frac{d^2 X^a}{ds^2} + \Gamma_{\beta\gamma}^\alpha \frac{dX^\beta}{ds} \frac{dX^\gamma}{ds} = \frac{1}{2} \nabla^\alpha [R^{\rho\mu}{}_{\nu\lambda} f_{\rho\mu} + R^\rho{}_\lambda f_{\nu\rho} - R^\rho{}_\nu f_{\lambda\rho}] \times \frac{f^{\nu\lambda}}{|f^2|} \quad (2)$$

The nonzero right hand side of the modified geodesic equation (2) leads to a polarisation dependence in the gravitational red shift, bending, and Shapiro time delay of light. The redshift in a photon's wavelength propagating in the gravitational field of a spherical mass (or a blackhole) is given by

$$\frac{\lambda_2}{\lambda_1} = \frac{(1 - 2GM/r_2)}{(1 - 2GM/r_1)} \left\{ \frac{1 - \frac{2GM}{\omega^2 r_1^3} \left(1 - \frac{2GM}{r_1}\right)}{1 - \frac{2GM}{\omega^2 r_2^3} \left(1 - \frac{2GM}{r_2}\right)} \right\}^{1/2} \quad (3)$$

The extra factor in the curly brackets is the correction to the standard red shift formula due to Riemann curvature coupling. This deviation is measurable when

the photon wavelength is of the same order as square root of the magnitude of the Riemann curvature. Near the horizon of blackholes, where the curvature can be large, this deviation is significant. We derive the dispersion equation of a photon in the Schwarzschild metric from the wave equation (1) and find that in a local inertial frame the dispersion relation is given by

$$\omega^2 - \vec{k}^2 = \frac{2GM}{r^3} \quad (4)$$

This implies that in the gravitational field of a blackhole, photons acquire an effective mass

$$m_\gamma = \left( \frac{2GM}{r^3} \right)^{1/2} \quad (5)$$

At the horizon  $r = 2GM$ , the effective photon mass is  $m_\gamma = 1/(2GM)$ . Since the photon mass is larger than the Hawking temperature  $T_H = 1/8\pi GM$ , our result implies that Hawking radiation of photons would not take place.

A curious phenomena discovered by Drummond and Hathrell [2] is that in higher derivative gravity which arises by QED radiative corrections, the photon velocity in local inertial frame can exceed the velocity of light in the Minkowski space. Due to the coupling of the electromagnetic fields with the Riemann and Ricci tensors in the Lagrangian, it was claimed that the photon velocity in the Schwarzschild, Robertson-Walker, gravitational wave and deSitter backgrounds is larger than the flat space velocity  $c$ . This result was extended by Daniels and Shore to charged [3] and rotating blackholes [4] with the same conclusions. Latorre et al [5] have shown a universal relation between the velocity shift of photons to the energy density which generates the background metric, and Shore [6] has related the velocity shift to the coefficients of conformal anomaly. Lafrance and Myers [7] interpret these results as the breakdown of the Equivalence principle in higher derivative gravity and Dolgov

and Khriptovich [8,9] derive this result from field theoretic dispersion relations. Finally Mende [10] has proposed this effect as a test for string theories of quantum gravity.

In this letter we show that claims of superluminal photon velocity are due to the neglect of the Riemann coupling terms in the wave equation which arises from the minimal  $F_{\mu\nu}F^{\mu\nu}$  Lagrangian. We find that in Einstein's gravity the photon velocity in a Schwarzschild blackhole metric is given by

$$v = \frac{c}{\left(1 + \frac{1}{|K^2|} \left(\frac{2m}{r^3}\right)\right)^{1/2}} \quad (6)$$

and for the Friedman-Robertson-Walker metric it is given by

$$v = \frac{c}{\left(1 + \frac{8\pi G}{6|K^2|} (3p + 5\rho)\right)^{1/2}} \quad (7)$$

Therefore the velocity of photons in a gravitational field is less than  $c$  in Einstein's gravity. We derive the wave equation by including higher derivative terms and show that Riemann coupling terms from the higher derivative gravity are always smaller in magnitude than the Riemann terms that already exists in Einstein's gravity. This analysis shows that the photon velocity never exceeds  $c$ , whether in Einstein's theory or by the inclusion of higher derivative terms in the Lagrangian.

The interaction of electromagnetic fields with gravity is given by the action

$$S = \int d^4x \sqrt{-g} F_{\mu\nu} F^{\mu\nu} \quad (8)$$

(We use the convention  $c = 1$ , signature  $-2$  and Greek letters denote spacetime indices  $0 - 3$ ). From (8) we obtain the equations of motion

$$\nabla^\mu F_{\mu\nu} = 0 \quad (9)$$

Equation (9) with the Bianchi identity

$$\nabla_\mu F_{\nu\lambda} + \nabla_\nu F_{\lambda\mu} + \nabla_\lambda F_{\mu\nu} = 0 \quad (10)$$

gives the Maxwell's equation in curved background. Operating on equation (9) by the  $\nabla_\lambda$  and using the Bianchi identity (10) the wave equation may be obtained as

$$\begin{aligned} \nabla^\mu \nabla_\mu F_{\nu\lambda} + \nabla_\lambda (\nabla^\mu F_{\nu\mu}) + [\nabla^\mu, \nabla_\nu] F_{\lambda\mu} \\ - [\nabla_\lambda, \nabla^\mu] F_{\mu\nu} = 0 \end{aligned} \quad (11)$$

The second term vanishes owing to (9). Using the identity for the commutator of covariant derivatives:

$$[\nabla^\mu, \nabla_\nu] F_{\alpha\mu} = R_{\rho\alpha\nu\mu} F^{\rho\mu} + R_{\rho\lambda} F_\nu^\rho \quad (12)$$

and the circular identity

$$R_{\rho\lambda\nu\mu} + R_{\rho\nu\mu\lambda} + R_{\rho\mu\lambda\nu} = 0 \quad (13)$$

the wave equation (11) reduces to the form

$$\nabla^\mu \nabla_\mu F_{\nu\lambda} + R_{\rho\mu\nu\lambda} F^{\rho\mu} + R^\rho_\lambda F_{\nu\rho} - R^\rho_\nu F_{\lambda\rho} = 0 \quad (14)$$

The Riemann and Ricci curvature coupling terms to the photon polarisation (or spin) give rise to the polarisation dependent deviation of photon orbits from null geodesics.

Photon trajectories are described by the eikonal solutions of the wave equation (14) of the form

$$F_{\mu\nu} = e^{iS(x)} f_{\mu\nu} \quad (15)$$

where the phase  $S$  varies more rapidly in spacetime than the amplitude  $f_{\mu\nu}$ . The wavenumber of the photon trajectories is given by the gradient of the phase

$$K_\mu = \nabla_\mu S \quad (16)$$

In the geometrical optics approximation where

$$\nabla_\mu F_{\alpha\beta} = iK_\mu F_{\alpha\beta} \quad (17)$$

the wave equation (14) can be written in the form

$$-K^\mu K_\mu f_{\nu\lambda} + R^{\rho\mu}{}_{\nu\lambda} f_{\rho\mu} - R^\rho{}_\lambda f_{\nu\rho} + R^\rho{}_\nu f_{\lambda\rho} = 0 \quad (18)$$

The dispersion relation may be written as

$$K^2 = \left( R^{\rho\mu}{}_{\nu\lambda} f_{\rho\mu} - R^\rho{}_\lambda f_{\nu\rho} + R^\rho{}_\nu f_{\lambda\rho} \right) \frac{f^{\nu\lambda}}{(f_{\alpha\beta} f^{\alpha\beta})} \quad (19)$$

Operating on (19) by  $\nabla^\alpha$ , the L.H.S. is

$$\nabla^\alpha K^2 = 2K^\mu \nabla_\mu K^\alpha \quad (20)$$

where we have used the identity  $\nabla^\alpha K_\mu = \nabla_\mu K^\alpha$  which follows from the definition of  $K_\mu$  as a gradient. Light rays are defined as the integral curves of the wave vector  $K_\mu$  i.e. the curves  $x_\mu(s)$  for which  $\frac{dX_\mu}{ds} = K_\mu$ . Substituting  $K_\mu = dX_\mu/ds$  in (20) we have for the L.H.S.

$$\nabla^\alpha K^2 = 2 \frac{dX^\mu}{ds} \nabla_\mu \left( \frac{dX^\alpha}{ds} \right) = 2 \left( \frac{d^2 X^\alpha}{ds^2} + \Gamma_{\mu\nu}^\alpha \frac{dX^\nu}{ds} \frac{dX^\mu}{ds} \right) \quad (21)$$

Using (19), (20) and (21) we obtain the modified geodesic equation for photon trajectories

$$\frac{d^2 X^\alpha}{ds^2} + \Gamma_{\beta\gamma}^\alpha \frac{dX^\beta}{ds} \frac{dX^\gamma}{ds} = \frac{1}{2} \nabla^\alpha (R^{\rho\mu}{}_{\nu\lambda} \cdot f_{\rho\mu} - R^\rho{}_\lambda f_{\nu\rho} + R^\rho{}_\nu f_{\lambda\rho}) \times \frac{f^{\nu\lambda}}{(f_{\alpha\beta} f^{\alpha\beta})} \quad (22)$$

To obtain the photon velocity in the curved space, we can use the dispersion relation (18) directly. Of the six components of  $F_{\mu\nu}$  in equation (18), only three are independent owing to the Bianchi identity (10). Choosing the components of the electric field vector  $E_i = f_{oi}$  as the independent components we have from the Bianchi identity (10)

$$K_o f_{ij} + K_i f_{jo} + K_j f_{oi} = 0 \quad (23)$$

Using (23) to substitute for  $f_{ij}$  in terms of the electric field components  $f_{jo}$  and  $f_{oi}$  in the wave equation (18) we obtain

$$\begin{aligned} \left[ \left( +K^\mu K_\mu + R^o_o + R^\ell_o \frac{K_\ell}{K_o} \right) \delta_i^j + \left( -2R^{oj}_{oi} + 4R^{\ell j}_{oi} \frac{K_\ell}{K_o} + R^j_i \right. \right. \\ \left. \left. - \frac{1}{K_o} R^j_o k_i \right) f_{oj} \right] = 0 \end{aligned} \quad (24)$$

which is the wave equation obeyed by the three components of the electric field vector (in (24) Latin indices are over three spacelike components and repeated indices are summed over). To simplify the notation we write (20) as

$$\left( K^2 \delta_i^j + \epsilon_i^j \right) f_{oj} = 0 \quad (25)$$

where

$$\epsilon_i^j \equiv \left( R^o_o + R^\ell_o \frac{K_\ell}{K_o} \right) \delta_i^j + \left( -2R^{oj}_{oi} + 4R^{\ell j}_{oi} \frac{K_\ell}{K_o} + R^j_i - \frac{1}{K_o} R^j_o K_i \right) \quad (26)$$

Of the three components of the electric field vector in the electromagnetic waves, only the transverse waves are observable in an asymptotically flat space. To obtain the equation for the transverse fields we first diagonalise the matrix  $\epsilon_i^j$  to give the equation for the three normal modes:

$$\left( k^2 + \epsilon_i \right) f_{oi} = 0 \quad (i = 1, 2, 3) \quad (27)$$

where  $\epsilon_i$  are the eigenvalues of the  $\epsilon_i^j$  matrix. The three equations (27) can be separated into the equation for the transverse fields

$$f_{oj}^{(T)} \equiv \left( \delta_j^i - n^i n_j \right) f_{oi} \quad (28)$$

and the longitudinal fields

$$f_{oj}^{(L)} \equiv n^i n_j f_{oi} \quad (29)$$

where  $n^i \equiv \frac{K_i}{|K|}$  are the components of the unit vector along the direction of propagation. The wave equation for the transverse photon is

$$\left[ K^2 \left( \delta_i^j - n^j n_i \right) + \epsilon_i \left( \delta_i^j - n^j n_i \right) \right] f_{oj} = 0 \quad (30)$$

and for the longitudinal photon the wave equation is

$$(K^2 + \epsilon_i) (n^j n_i) f_{oj} = 0 \quad (31)$$

### Non-Rotating Black Holes (Schwarzschild Metric):

In the gravitational field of non-rotating blackhole, the non-zero components of the  $\epsilon_j^i$  matrix (26) are

$$\epsilon_1^1 = 4 \frac{GM}{r^3}, \quad \epsilon_2^2 = \epsilon_3^3 = -2 \frac{GM}{r^3} \quad (32)$$

Considering radial trajectories with  $\vec{n} = (\hat{r}, \hat{\theta}, \hat{\phi}) = (1, 0, 0)$ , equation (30) yields the wave equations for the transverse fields  $E_2, E_3$ :

$$\begin{pmatrix} K^2 - 2GM/r^3 & 0 \\ 0 & K^2 - 2GM/r^3 \end{pmatrix} \begin{pmatrix} E_2 \\ E_3 \end{pmatrix} = 0 \quad (33)$$

The dispersion relation yields

$$\omega^2 - k_1^2 = 2 \frac{GM}{r^3} \quad (34)$$

The velocity of propagation of the transverse photon is

$$v_1^{(t)} = \frac{\partial \omega}{\partial k_1} = \frac{1}{\left(1 + \frac{1}{k_1^2} \left(\frac{2GM}{r^3}\right)\right)^{1/2}} < 1 \quad (35)$$

The photon velocity is subluminal and the transverse photons have an effective mass

$$m_\gamma = \left(\frac{2GM}{r^3}\right)^{1/2} \quad (36)$$

For tangential trajectories,  $\vec{n} = (0, 1, 0)$  and the equations for the transverse fields  $E_1$  and  $E_3$  yield:

$$\begin{pmatrix} K^2 + 4GM/r^3 & 0 \\ 0 & K^2 - 2GM/r^3 \end{pmatrix} \begin{pmatrix} E_1 \\ E_3 \end{pmatrix} = 0 \quad (37)$$



The dispersion relation is  $\left(K^2 + \frac{4GM}{r^3}\right) \left(K^2 - \frac{2GM}{r^3}\right) = 0$  and the root corresponding to the propagating mode is

$$K^2 = \omega^2 - k_2^2 = \frac{2GM}{r^3} \quad (38)$$

and the velocity of wave propagation is given by

$$v^{(T)} = \frac{\partial \omega}{\partial k_2} = \frac{1}{\left(1 + \frac{1}{k_1^2} \left(\frac{2GM}{r^3}\right)\right)^{1/2}} < 1 \quad (39)$$

Thus for both radial and tangential trajectories the photon velocity is subluminal and the effective photon mass is given by (36). From the dispersion relations (34) and (38) for the radial and tangential photons we see that a photon with frequency  $\omega$  (in the inertial frame) reaches a terminal radius  $r_{min} = (2GM/\omega^2)^{1/3}$  at which point its wavenumber becomes zero. A finite energy photon falling into a blackhole therefore would never encounter the singularity at the origin of the blackhole.

At the horizon of the blackhole the effective photon mass is  $m_\gamma = \frac{1}{2GM}$ . Comparing this with the Hawking temperature of the blackhole  $T_H = \frac{1}{8\pi GM}$ , we find that the photon mass is larger than the Hawking temperature. This implies that Hawking radiation from blackholes cannot be in the form of photons.

In the coordinate frame the dispersion relation obeyed by  $K_\mu = (\omega, k_r k_\theta, k_\rho)$  obtained from the wave equation (30), for a radial trajectory is given by

$$\left(1 - \frac{2GM}{r}\right)^{-1} \omega^2 - \left(1 - \frac{2GM}{r}\right) k_r^2 = \frac{2GM}{r^3} \quad (40)$$

Since  $\omega$  is a constant of motion (the metric being stationary), the wavenumber  $k_r$  changes with  $r$ . The redshift of the wavelength  $\lambda = 1/k_r$  is given by the expression

$$\frac{\lambda(r_2)}{\lambda(r_1)} = \frac{(1 - 2GM/r_2)}{(1 - 2GM/r_1)} \left\{ \frac{1 - (2GM/\omega^2 r_1^3)(1 - 2GM/r_1)}{1 - (2GM/\omega^2 r_2^3)(1 - 2GM/r_2)} \right\}^{1/2} \quad (41)$$

The extra factor in the curly brackets is the correction to the standard redshift formula due to the coupling of the Riemann curvature to the wave vector. The correction is significant when the photon wavelength is comparable to the Riemann curvature.

#### Expanding Universe (Friedmann-Robertson-Walker Metric):

For wave propagation in a general homogeneous, isotropic universe described by the line element  $ds^2 = +dt^2 - R^2(t) (dr^2 + r^2 d\theta^2 + \sin^2 \theta d\phi^2)$ , the non-zero components of the  $\epsilon_j^i$  matrix (26) are

$$\epsilon_1^1 = \epsilon_2^2 = \epsilon_3^3 = \frac{\ddot{R}}{R} - \frac{2\dot{R}^2}{R^2} = -\frac{8\pi G}{6} (3p + 5\rho) \quad (42)$$

where  $R$  is the scale factor, and  $p$  and  $\rho$  are the pressure and energy density respectively.

The dispersion relation for the transverse photons is

$$\omega^2 - k_i^2 = \frac{8\pi G}{6} (3p + 5\rho) \quad (43)$$

and the photon velocity is given by

$$v_i^{(T)} = \frac{\partial \omega}{\partial k_i} = \frac{1}{\left(1 + \frac{8\pi G}{6k_i^2} (3p + 5\rho)\right)^{1/2}} < 1 \quad (44)$$

and once again the photon velocity is subluminal.

#### Higher Derivative Gravity:

Drummond and Mathrell [1] obtained the higher derivative couplings which arise from the QED loop corrections to the graviton-photon vertex. The effective

Lagrangian valid in the frequency range  $\omega^2 < \alpha/m_e^2$  is given by

$$\begin{aligned}\mathcal{L} = & -\frac{1}{4}F_{\mu\nu} + aR F_{\mu\nu} F^{\mu\nu} + bR_{\mu\nu} F^{\mu\sigma} F^\nu_\sigma \\ & + cR_{\mu\nu\sigma\tau} F^{\mu\nu} F^{\sigma\tau}\end{aligned}\quad (45)$$

with the coefficients  $a = -\frac{5}{720} \frac{\alpha}{\pi m_e^2}$ ,  $b = \frac{26}{620} \frac{\alpha}{\pi m_e^2}$ ,  $c = -\frac{2}{720} \frac{\alpha}{\pi m_e^2}$  (where  $\alpha$  is the fine structure constant and  $m_e$  the electron mass). The equations of motion from (45) are given by

$$\nabla_\mu F^{\mu\nu} + X^\nu = 0 \quad (46)$$

where

$$\begin{aligned}X^\nu = & a(F^{\mu\nu}\nabla_\mu R + R\nabla_\mu F^{\mu\nu}) \\ & + 2b(F^{\sigma\nu}\nabla_\mu R^\mu_\sigma \nabla_\mu F^{\sigma\nu} - F^{\sigma\nu}\nabla_\mu R^\nu_\sigma - R^\nu_\sigma \nabla_\mu F^{\sigma\mu}) \\ & + 4c(F^{\sigma\tau}\nabla_\mu R^{\mu\nu}_{\sigma\tau} + R^{\mu\nu}_{\sigma\tau}\nabla_\mu F^{\sigma\tau}) = 0\end{aligned}\quad (47)$$

Operating on equation (46) by  $\nabla^\lambda$  and using the Bianchi identity (10) the commutator identity (12) and the circular identity (13), we obtain the wave equation in higher derivative gravity:

$$\begin{aligned}\nabla^\mu \nabla_\mu F_{\nu\lambda} + & \{R_{\rho\mu\nu\lambda} F^{\rho\mu} + R^\rho_\lambda F_{\nu\rho} - R^\rho_\nu F_{\lambda\rho}\} \\ & + [\nabla_\nu X_\lambda - \nabla_\lambda X_\nu] = 0\end{aligned}\quad (48)$$

In the derivations of [1-5] the terms in the curly bracket do not appear. This is because in the derivations, the eikonal approximation is made in the Maxwell's equation and therefore the Riemann and Ricci terms which arise on commuting the covariant derivatives do not appear. In (48) however we see that in the range of validity of the effective action (45),  $\omega^2 < \alpha/m_e^2$ , the Riemann and Ricci terms in the curly brackets of (48) which are present in Einstein's gravity are larger than the terms in the square brackets of (48) which arise from the loop corrections. The photon velocities are therefore given by the expression (31) for blackholes, and (44) for Friedmann-Robertson-Walker metric, and is subluminal even in the presence of higher derivative coupling terms.

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