Snell's Law from an Elementary Particle Viewpoint

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Snell's law of light deflection between media with different indices of refraction is usually discussed in terms of the Maxwell electromagnetic wave theory. Snell's law may also be derived from a photon beam theory of light rays. This latter particle physics view is by far the most simple one for understanding the laws of refraction.

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I. INTRODUCTION

Snell's law of refraction is usually discussed in elementary physics courses wherein the derivations[1] depend on the electromagnetic wave theory of light. The purpose of this note is to show how the laws of refraction may be derived from the particle (i.e. photon) view of light rays. In particular, we show how the photon Hamiltonian $H(\mathbf{p}, \mathbf{r})$ must be derived. In this work, \mathbf{p} and \mathbf{r} denote, respectively, the momentum and position of a photon as it moves along the light ray.

The refraction of a light ray is shown in Fig.1. In terms of the indices of refraction, Snell's law of refraction asserts[2, 3] that

$$n_1 \sin \theta_1 = n_2 \sin \theta_2$$
 (Snell's Law). (1)

The derivation of Eq.(1) from the energy and momentum conservation laws associated with photon deflection is exhibited Sec.II. The index of refraction is then *defined* in terms of photon energy and momentum. The relationship between the index of refraction and the photon velocity is discussed in Sec.III and from a more general geometric optics limit in Sec.IV. As an application of the particle viewpoint, we consider in Sec.V the gravitational lens, i.e. the bending of light rays in a gravitational field.

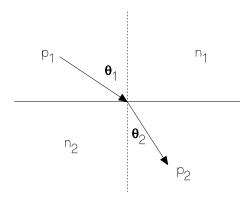


FIG. 1: A light ray moves from a medium with index of refraction n_1 into a medium with index of refraction n_2 . Rays are considered to be made up of photons with momenta \mathbf{p}_1 and \mathbf{p}_2 respectively.

II. CONSERVATION LAWS

We assume that the two rays in Fig.1 are made up of photons with momenta \mathbf{p}_1 and \mathbf{p}_2 respectively. We also assume that the photon energies are E_1 and E_2 respectively. Since there is translational invariance in directions parallel to the plane separating the two media, the photon momentum components parallel to the plane are conserved; i.e.

$$p_1 \sin \theta_1 = p_2 \sin \theta_2. \tag{2}$$

The energies of the photons are also conserved; i.e.

$$E_1 = E_2. (3)$$

In terms of *physical* photon energy and momentum, the indices of refraction are defined by

$$n_1 = \frac{cp_1}{E_1}$$
 and $n_2 = \frac{cp_2}{E_2}$. (4)

Eqs.(2)-(4) imply

$$n_1 \sin \theta_1 = n_2 \sin \theta_2 \ , \tag{5}$$

which constitutes a simple yet rigorous derivation of Snell's law.

Although similar to other treatments which discuss a formal energy and a formal momentum [4, 5], we stress that in our treatment we only refer to the physical photon energy and momentum. For example, E has conventional units of Joules and momentum \mathbf{p} has conventional units of Joule-sec/meter. In other more formal treatments, the so-called "energy" and "momentum" do not have the usual physical dimensions of energy and momentum.

III. PHOTON VELOCITY

The definitions of the indices of refraction in Eq.(4) are equivalent to the more usual definitions

$$n_1 = \frac{c}{u_1}$$
 and $n_2 = \frac{c}{u_2}$. (6)

if one does *not in general* identify u with the physical velocity of the photons. The velocity of a photon in a ray is determined by

$$\mathbf{v} = \frac{\partial E}{\partial \mathbf{p}} \ . \tag{7}$$

In general for light rays moving through continuous media[6]

$$u \neq |\mathbf{v}| \quad \text{since} \quad \frac{E}{p} \neq \left| \frac{\partial E}{\partial \mathbf{p}} \right|.$$
 (8)

In the Maxwell electromagnetic wave theory [7, 8, 9] of light, \mathbf{u} is the *phase* velocity while \mathbf{v} is the *group* velocity. Let us consider this in more detail from a purely particle physics viewpoint.

IV. GEOMETRICAL HAMILTONIAN OPTICS

In inhomogeneous continuous media, the index of refraction in general depends on the photon energy as well as position. Eq.(4) implies the particle energy restriction

$$E = \frac{c|\mathbf{p}|}{n(\mathbf{r}, E)} \ . \tag{9}$$

In principle, the implicit Eq.(9) can be solved for the energy in the form

$$E = H(\mathbf{p}, \mathbf{r}). \tag{10}$$

The particle of light, i.e. the photon in the ray, obeys Hamilton's equations[10, 11, 12, 13]; They are

$$\dot{\mathbf{r}} = \frac{\partial H(\mathbf{p}, \mathbf{r})}{\partial \mathbf{p}}$$
 and $\dot{\mathbf{p}} = -\frac{\partial H(\mathbf{p}, \mathbf{r})}{\partial \mathbf{r}}$. (11)

The velocity of the photon $\mathbf{v} = \dot{\mathbf{r}}$ obeys[14]

$$\mathbf{v} = \frac{n\mathbf{u}}{[n + E(\partial n/\partial E)]}$$
 wherein $\mathbf{u} = \frac{E\mathbf{p}}{p^2} = \frac{c\mathbf{p}}{np}$. (12)

The force on the photon $\mathbf{f} = \dot{\mathbf{p}}$ obeys

$$\mathbf{f} = \frac{E \ \mathbf{grad}n}{[n + E(\partial n/\partial E)]} \ . \tag{13}$$

For a discontinuity in the index of refraction, such as pictured in Fig.1, the impulsive photon force is normal to the surface of the discontinuity. That there is no force parallel to the surface of the discontinuity leads directly to Snell's law as in Sec.II. We note in passing that Hamilton's Eq.(11) follow from minimizing the optical path length[15, 16, 17]

$$L[\text{Path}, E] = \int_{\text{Path}} n(\mathbf{r}, E) |d\mathbf{r}|$$
 (14)

over sufficiently small sections of the path of the ray.

V. PHOTON DEFLECTION DUE TO GRAVITY

Astrophysical gravitational lenses are naturally formed and can be understood on the basis that light rays bend

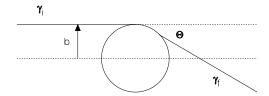


FIG. 2: An incident photon denoted by γ_i with impact parameter b is refracted from a spherical gravitational lens. The refraction index is $n(r) = 1 + (R_s/r)$. The deflection angle Θ describes the final photon denoted by γ_f .

in a gravitational field. Newton's formulation of gravity attributes the weight of an object to a gravitational field

$$\mathbf{g} = -\mathbf{grad}\Phi. \tag{15}$$

The weight of an object of mass m is

$$\mathbf{w} = m\mathbf{g}.\tag{16}$$

The gravitational field in a region of space is weak if for all points within the region the gravitational potential obeys

$$|\Phi(\mathbf{r})| \ll c^2 \ . \tag{17}$$

For a region of space with a weak gravitational field, light rays bend in a manner described by the index of refraction,

$$n(\mathbf{r}) = 1 - \frac{2\Phi(\mathbf{r})}{c^2} + \dots , \qquad (18)$$

independently of the energy E of the photon. For example, a spherical astrophysical object will induce in the neighboring space a potential $\Phi = -(GM/r)$ and thereby a spherical lens with index of refraction

$$n(r) = 1 + \frac{2GM}{c^2 r} + \dots ,$$
 (19)

To a sufficient degree of accuracy, Eqs.(9), (17) and (19) imply

$$|\mathbf{p}|^2 \approx \left(\frac{E}{c}\right)^2 \left(1 + \frac{4GM}{c^2 r}\right).$$
 (20)

The identities

$$r^{2}|\mathbf{p}|^{2} = |\mathbf{r} \times \mathbf{p}|^{2} + |\mathbf{r} \cdot \mathbf{p}|^{2},$$

$$r^{2}|\mathbf{p}|^{2} = J^{2} + r^{2}p_{r}^{2},$$
(21)

and

$$J = \frac{Eb}{c} \quad \text{(angular momentum)},$$

$$R_s = \frac{2GM}{c^2} \quad \text{(gravitational radius)} \tag{22}$$

together with Eq.(20) imply a radial photon momentum

$$p_r = \pm \frac{E}{c} \sqrt{1 + \left(\frac{2R_s}{r}\right) - \left(\frac{b}{r}\right)^2},\tag{23}$$

and its associated scattering angular deflection

$$\Theta = \pi - 2 \int_{r_{min}}^{\infty} \frac{bdr}{r^2 \sqrt{1 + (2R_s/r) - (b/r)^2}}$$
 (24)

as shown in Fig.(2). The impact parameter thereby obeys

$$b = -R_s \cot(\Theta/2). \tag{25}$$

For small angles and large impact parameters[11, 18],

$$\Theta = -\frac{2R_s}{b} = -\frac{4GM}{c^2b} \quad (b \gg R_s), \tag{26}$$

which was used by Einstein to predict the bending of light around the sun[19, 20]. We note in passing, the

geometrical optics refraction cross section

$$\frac{d\sigma}{d\Omega} = \left| \frac{bdb}{\sin \Theta d\Theta} \right| = \frac{R_s^2}{4 \sin^4(\Theta/2)} \ . \tag{27}$$

VI. CONCLUSIONS

The laws of refraction have been shown to follow most easily by employing a classical Hamiltonian, $E = H(\mathbf{p}, \mathbf{r})$, for a photon moving through transparent continuous media. Snell's law of refraction at the boundary of two different media is then a simple consequence of the conservation laws inherent in the Hamiltonian description of the photon. The use of ray optics in the design of lens systems is well known. We have here illustrated the use of the photon Hamiltonian for the case of gravitational lens effects in astronomy.

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