Assumption-Lean Inference for Spectral Differential Network Analysis of High-Dimensional Time Series

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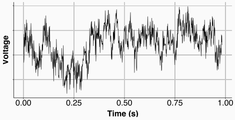
Related work: Hellstern M, Kim B, Harchaoui Z, Shojaie A. *Spectral Differential Network Analysis for High-dimensional Time Series*. AISTATS (2025)

Spectral analysis of time series data

- **Spectral analysis** studies signal at frequency instead of time
- In neuroscience, higher frequency oscillations contain essential brain connectivity information



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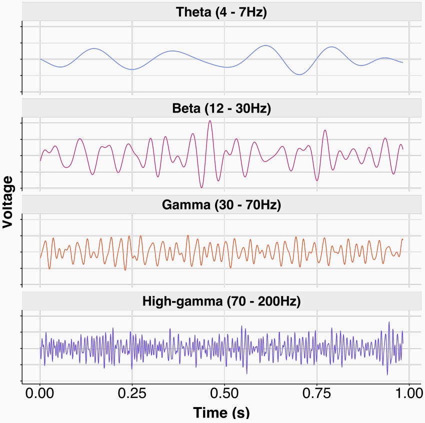
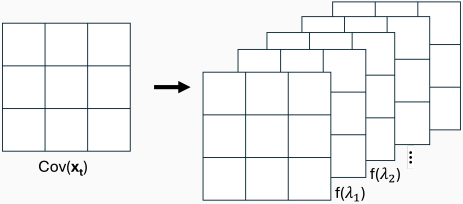


Spectral density

- The **spectral density, $f(\lambda)$,** is frequency domain version of covariance
- **Spectral density** decomposes covariance by frequency



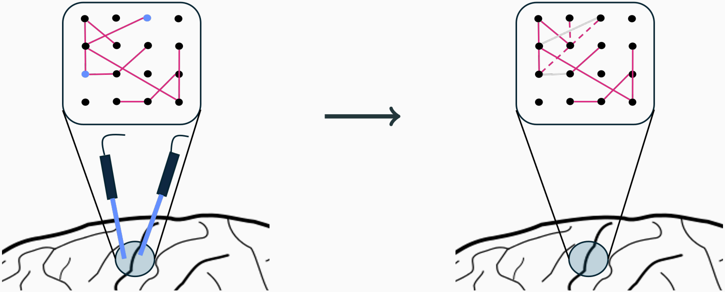
Spectral networks

- **Inverse spectral density**, $f^{-1}(\lambda)$, is a useful network since it encodes conditional relationships[1]
- $f^{-1}(\lambda)_{i,j}$ encodes the (rescaled) coherence between node i and node j after removing the linear effects of all other nodes

[1] Dahlhaus (2000)

Why study a difference in networks?

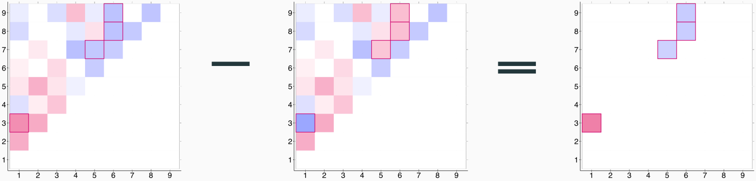
- Studying how external stimuli change the brain network at different frequencies we can hopefully develop treatments



- We are therefore interested in estimating $\mathbf{f}_1^{-1}(\boldsymbol{\lambda}) - \mathbf{f}_2^{-1}(\boldsymbol{\lambda})$

Current methods to estimate a difference in $f^{-1}(\lambda)$

- Only existing method is to estimate $f^{-1}(\lambda)$ in each condition and take difference
- In high-dimensions consistency of $\hat{f}_1^{-1} - \hat{f}_2^{-1}$ requires both $\hat{f}_1^{-1}, \hat{f}_2^{-1}$ to be sparse (unrealistic in practice)



From the complex to real space

- Brief detour to frame problem in real space

$$\Sigma_i = \begin{bmatrix} \mathrm{Re}(f_i) & -\mathrm{Im}(f_i) \\ \mathrm{Im}(f_i) & \mathrm{Re}(f_i) \end{bmatrix}, \quad \Sigma_i^{-1} = \begin{bmatrix} \mathrm{Re}(f_i^{-1}) & -\mathrm{Im}(f_i^{-1}) \\ \mathrm{Im}(f_i^{-1}) & \mathrm{Re}(f_i^{-1}) \end{bmatrix}$$

- $f_1^{-1} - f_2^{-1}$ can easily be recovered from $\Sigma_1^{-1} - \Sigma_2^{-1}$

Spectral D-trace Difference (SDD) Estimator

- Note
$$\frac{\partial L_D}{\partial \Delta}(\Delta) = \left(\Sigma_2 \Delta \Sigma_1 + \Sigma_1 \Delta \Sigma_2\right)/2 - \left(\Sigma_2 - \Sigma_1\right), \tag{1}$$
is estimating equation for $\Delta^* = \Sigma_1^{-1} - \Sigma_2^{-1}$
- We use the integral of $\frac{\partial L_D}{\partial \Delta}(\Delta)^2$ and include a penalty on $\|\Delta\|_1$ to induce sparsity

$$\hat{\Delta} = \underset{\Delta}{\operatorname{argmin}} \frac{1}{4}\left(\langle \hat{\Sigma}_2 \Delta, \Delta \hat{\Sigma}_1 \rangle + \langle \hat{\Sigma}_1 \Delta, \Delta, \hat{\Sigma}_2 \rangle\right) - \langle \Delta, \hat{\Sigma}_2 - \hat{\Sigma}_1 \rangle + \tau \|\Delta\|_1$$

- $\hat{\Sigma}_i$ are sample estimates of Σ_i - smoothed periodograms with a smoothing bandwidth of B

[2] This is referred to as the D-trace loss and was first introduced in Yuan et al. (2017)

Inference using de-biased estimating equations

- $\hat{\Delta}$ is consistent for Δ under mild data dependence conditions
- Inference can be performed using de-biasing methods for high-dimensional estimating equations[3]
    - De-biases by projecting estimating equation on sparse direction
- Require estimate of $\frac{\partial^2 L_D}{\partial \Delta^2} = \left(\Sigma_1 \otimes \Sigma_2 + \Sigma_2 \otimes \Sigma_1\right)^{-1}$

[3] Neykov et al. (2018)

Inference approach

- Instead of $\text{symm} = (\Sigma_2 \Delta \Sigma_1 + \Sigma_1 \Delta \Sigma_2)/2 - (\Sigma_2 - \Sigma_1)$ we can use either

$$\text{s1Right} = \Sigma_2 \Delta \Sigma_1 - (\Sigma_2 - \Sigma_1) \qquad \text{or}$$
$$\text{s1Left} = \Sigma_1 \Delta \Sigma_2 - (\Sigma_2 - \Sigma_1)$$

- Then inference requires estimates of

$$(\Sigma_1 \otimes \Sigma_2)^{-1} = \Sigma_1^{-1} \otimes \Sigma_2^{-1} \qquad \text{or}$$
$$(\Sigma_2 \otimes \Sigma_1)^{-1} = \Sigma_2^{-1} \otimes \Sigma_1^{-1}$$

Theoretical results

Theorem 1

If the dependence in the data is not too strong[4], for appropriate choices of penalty parameters we have

$$\frac{1}{\hat{\sigma}_{\text{symm}}}\sqrt{\frac{T}{B}}\left(\tilde{\theta}_{\text{symm}}-\theta^*\right)\xrightarrow[T\to\infty]{d}N(0,1)$$
$$\frac{1}{\hat{\sigma}_{\text{s1Left}}}\sqrt{\frac{T}{B}}\left(\tilde{\theta}_{\text{s1Left}}-\theta^*\right)\xrightarrow[T\to\infty]{d}N(0,1)$$
$$\frac{1}{\hat{\sigma}_{\text{s1Right}}}\sqrt{\frac{T}{B}}\left(\tilde{\theta}_{\text{s1Right}}-\theta^*\right)\xrightarrow[T\to\infty]{d}N(0,1)\,,$$

where $\tilde{\theta}$ is the de-biased estimate. B is the bandwidth used to generate sample estimates of spectral density.

[4]details in Appendix

Inference simulations

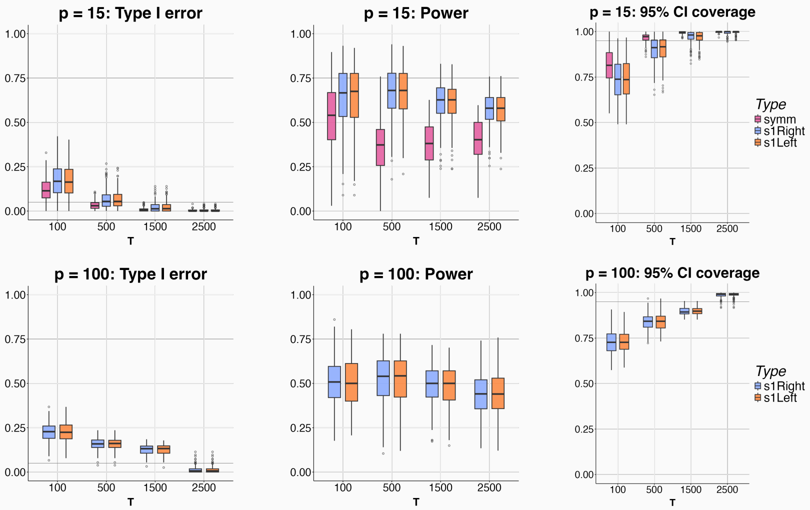


Figure 1: Top row: $p = 15$. Bottom row: $p = 100$

Application to EEG data

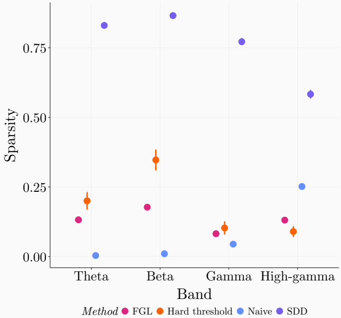
- 64 channel EEG recording of 111 healthy subjects [5]
- 4 minutes recording while subjects at rest with eyes closed
- 42 subjects have follow-up session 2-3 months later

- Subjects with follow-up (across session analysis)
  - Estimated differential network across sessions
- Subjects without follow-up (within session analysis)
  - Estimated differential network from 0-60s to 120-180s

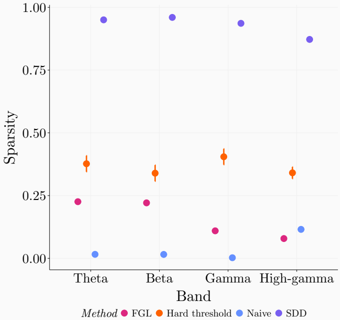
[5] This data is from Hatlestad-Hall et al. (2022)

Application to EEG data

Within session analysis has sparser differential network



Across session analysis



Within session analysis

Questions?

References

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Appendix

Spectral density definition

Let $\mathbf{x}_t \in \mathbb{R}^{p\times 1}$ then autocovariance at lag $h \in \mathbb{Z}$ is denoted as $\Gamma(h) = \mathbb{E}\left(\mathbf{x}_t \mathbf{x}_{+h}^T\right)$. **Spectral density, $\mathbf{f}(\lambda)$,** at frequency λ is then defined as

$$f(\lambda) = \frac{1}{2\pi}\sum_{h=-\infty}^{\infty} e^{-i\lambda h}\Gamma(h), \qquad -\pi \leq \lambda \leq \pi.$$

Periodogram and smoothing

For condition l, the periodogram at Fourier frequency $\{\lambda_j = 2\pi j/n_l, -\lfloor (n_l - 1)/2 \rfloor \leq j \leq \lfloor n_l/2 \rfloor\}$ is defined as

$$P_l(\lambda_j) = \frac{1}{2\pi n_l} \left(\sum_{t=1}^{n_l} \mathbf{x}_{l,t} \exp(-i\lambda_j t) \right) \left(\sum_{t=1}^{n_l} \mathbf{x}_{l,t} \exp(-i\lambda_j t) \right)^H .$$

But periodogram not consistent, so must smooth over nearby frequencies

$$\hat{f}_l(\lambda_j) = \frac{1}{2B+1} \sum_{k=j-B}^{j+B} P_l(\lambda_k).$$

Functional dependence

Let ϵ_0^* and $\{\epsilon_t\}_{t\in\mathbb{Z}}$ be i.i.d. vectors in $\mathbb{R}^b$ and $\mathcal{F}_t = (\ldots, \epsilon_{t-1}, \epsilon_t)$. We further let $\mathbf{x}_{l,t} = (x_{l,1t}, \ldots x_{l,pt})^T$ be a p-dimensional process in condition l where

$$x_{l,jt} = R_{l,j}(\mathcal{F}_t).$$

To measure the dependence on ϵ_0 we replace ϵ_0 in $\mathcal{F}_t$ with an i.i.d. copy ϵ_0^*. This gives $\mathcal{F}_t' = (\ldots, \epsilon_{-1}, \epsilon_0^*, \epsilon_1, \ldots, \epsilon_t)$ and

$$x_{l,jt}' = R_{l,j}(\mathcal{F}_t').$$

The dependence on ϵ_0 can then be measured by

$$\delta_{l,t,q,j} = \left(\mathbb{E}\left|x_{l,jt} - x_{l,jt}'\right|^q\right)^{1/q}$$

where $\mathbb{E}\,|x|^q$ is the expected value of $|x|^q$.

Assume there exists $\rho \in (0,1)$ such that

$$\|X_{l,\cdot j}\|_{L_q} = \sup_{m\geq 0} \rho^{-m} \Delta_{l,m,q,j} < \infty, \text{where } \Delta_{l,m,q,j} = \sum_{t=m}^{\infty} \delta_{l,t,q,j}\,.$$

Dependence assumption

Assumption 1

There exists some constant $\alpha_l \geq 0$ *such that*

$$\|X_{l,\cdot j}\|_{\psi_{\alpha_l}} := \sup_{q \geq 2} q^{-\alpha_l} \|X_{l,\cdot j}\|_{L_q} < \infty \, .$$

In Assumption 1, α_l represents the dependence in the process for condition l. As noted in Zhang and Zhang (2025), in the iid case, $\alpha_l = 1/2$ corresponds to the classical sub-Gaussian norm while $\alpha_l = 1$ corresponds to the sub-Exponential norm. The larger α_l is, the heavier the tails.

Form of asymptotic variance of de-biased SDD

The full asymptotic distribution of the (symm) de-biased D-trace loss function is

$$\sqrt{T/B}v^{*T}\mathrm{vec}\left[\left(\hat{\Sigma}_2\Delta^*\hat{\Sigma}_1+\hat{\Sigma}_1\Delta^*\hat{\Sigma}_2\right)/2-\left(\hat{\Sigma}_2-\hat{\Sigma}_1\right)\right]\rightsquigarrow N\left(0,v^{*T}\left(M_1V_1M_1^T+M_2V_2M_2^T\right)v^*\right)$$

where

- Δ^* is the true difference
- $v^* = \left(\frac{\partial^2 L_D}{\partial \Delta^2}^{-1}\right)_{i*}$
- $M_1 = \frac{\left(\Sigma_2^T\Delta^{*,T}\otimes I\right)+(I\otimes\Sigma_2\Delta^*)}{2}+I$
- $M_2 = \frac{\left(\Sigma_1^T\Delta^{*,T}\otimes I\right)+(I\otimes\Sigma_1\Delta^*)}{2}-I$
- $V_i = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} V_i' \begin{bmatrix} P_1^T & P_2^T \end{bmatrix}$, P_i are known 0/1 permutation matrices
- $$V_i' = \frac{1}{2}\begin{bmatrix} \left(I_{p^2}+K\right)\left(A_i\otimes A_i+B_i\otimes B_i\right) & \left(I_{p^2}+K\right)\left(B_i\otimes A_i-A_i\otimes B_i\right) \\ \left[\left(I_{p^2}+K\right)\left(B_i\otimes A_i-A_i\otimes B_i\right)\right]^T & \left(I_{p^2}-K\right)\left(A_i\otimes A_i+B_i\otimes B_i\right) \end{bmatrix}$$

Inference challenges

1. Requires new concentration inequality on smoothed spectral density estimates
2. Extended de-biasing methods to arbitrary scaling s_n
3. Solve for form of asymptotic variance of $\mathrm{vec}\left(\hat{\Sigma}_i - \Sigma_i\right)$
4. Inference requires estimate of $\frac{\partial^2 L_D}{\partial \Delta^2} = \left(\Sigma_1 \otimes \Sigma_2 + \Sigma_2 \otimes \Sigma_1\right)^{-1}$, too large to compute even for moderate p
5. Moreover, computing asymptotic variance requires multiplication of matrices too large to compute