Opacities in Stellar Atmospheres

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1 A Personal Introduction

Opacities were originally taught to me in 2 contexts: a perturbed Hydrogen atom Hamiltonian (without a mention of quantum field theory) and in an astrophysics context where the Einstein coefficients were somewhat *magical*. I use the word *magic* to describe any physical phenomena that doesn't have a quantitative explanation. The relationships between quantum mechanics, Einstein coefficients and cross sections were something I used but never fully understood.

2 Spontaneous Emission

The first place to begin in understanding opacities is spontaneous emission, i.e. the Einstein A coefficient. A derivation using Fermi's golden rule:

$$A_{fi} = \Gamma_{i \to f} = \frac{2\pi}{\hbar} |\langle f H' i \rangle|^2 \rho(E_f), \tag{1}$$

where the perturbation Hamiltonian for dipole radiation polarized in the z direction is proportional to qz/ϵ_0. The three terms in this equation seem to have SI units of $J^{-1}Hz$, J^2 and $J^{-1}m^{-3}$, but this is not quite correct as the final state f also contains the continuum photon wavefunction. Rather than going through the complex derivation, we can simply write the result in different ways when decaying into a vacuum:

$$\begin{align} A_{21} &= \frac{\omega^3 e^2 |\langle 1|\mathbf{r}|2\rangle|^2}{3\pi\epsilon_0\hbar c^3} \tag{2} \\ &= \frac{8\pi^2 e^2 |\langle 1|\mathbf{r}|2\rangle|^2}{3\hbar\epsilon_0\lambda^3} \tag{3} \\ &\lesssim \frac{4}{3}\alpha \left(\frac{2\pi r_A}{\lambda}\right)^2 \omega \tag{4} \\ &= 2.4 \left(\frac{r_A}{\lambda}\right)^2 \nu, \tag{5} \end{align}$$

where ν is the radiation frequency and r_A is the Bohr radius. The expression is also given in terms of the fine structure constant α and angular frequency ω, which shows a little better why the final constant of 2.4 is close to order unity. This is closer to an equality if the two states are approximately related by, for example, $\langle 2| \approx \langle 1|x|$. For example, for atomic Calcium which has a singlet transition from the first excited state to the ground state and an often listed atomic radius of 194 pm, this relationship gives an Einstein A of no more than 3.6×10^8 Hz, which is very close to the true value of 2.2×10^8 Hz. In terms of fundamental units, the atomic radius is of order the Bohr radius.

3 Einstein B and Cross Section

Linking a forward and reverse process to a single matrix element is a core prediction of quantum mechanics, going beyond Einstein A and B coefficients. The Einstein A and B coefficients are directly linked is directly linked via:

$$B_{12} = A_{21} \frac{\lambda^3 g_2}{2h g_1}, \tag{6}$$

and in turn the cross section is:

$$\begin{aligned} \sigma_{12} &= \frac{h}{\lambda} B_{12} \phi_\nu && (7) \\ &= \frac{\lambda^2 g_2}{2 g_1} A_{21} \phi_\nu && (8) \\ &\lesssim 1.2 r_A^2 \frac{g_2}{g_1} \nu \phi_\nu && (9) \\ &\approx r_A^2 \frac{g_2}{g_1} \frac{\lambda}{\Delta\lambda}, && (10) \end{aligned}$$

for a transition that covers a fractional bandwidth $\Delta\lambda$. This is a remarkably simple result, which will prove invaluable when determining the approximate relative strengths of different opacity sources.

4 Strong Atomic Lines

For L Dwarfs, strong atomic lines can be pressure broadened and cover the full spectrum. Key atomic line databases are NIST and VALD. Important lines from the ground or near-ground state, only counting wavelengths longer than 2700$\AA$ are:

Element	Abundance	Terms	Wavelength
Na I	-5.68	$^2P \rightarrow ^2 S$	5889
K I	-6.9	$^2P \rightarrow ^2 S$	7665,7698
Mg I	-4.43	$^1P \rightarrow ^1 S$	2852
Ca I	-5.68	$^1P \rightarrow ^1 S$	4227
Al I	-5.6	$^2S \rightarrow ^2 P$	3944/3961

For a round upper M dwarf atmosphere column density of $1\,\mathrm{g\,cm^{-3}}$, a relative abundance of 10^{-6} gives a number column density of $6 \times 10^{17}\,\mathrm{cm^{-2}}$, and an optically-thick cross-section of $(12\,\mathrm{pm})^2$.