Renormalization of Symmetry Improved 3PIEA gap equations at 2 loops

Supplement to "Symmetry improvement of 3PI effective actions for O(N) scalar field theory" by Michael J. Brown and Ian B. Whittingham.

Mathematica notebook to compute couter-terms for two loop truncations of the effective action as described in Section IV of the paper.

Hartree-Fock

```
ClearAll[geom, neom, intrules, regularisedtadpoles, mg2soln, cteq, cts, δm, δλ];
```

Hartree-Fock gap equations with counterterms

Goldstone equation of motion. Quantities in reference to the paper are:
p is the four-momentum flowing through the propagators $\Delta_G{}^{-1}$ and $\Delta_N{}^{-1}$,
mg2 is the Goldstone mass squared $m_G{}^2$,
Z and ZΔ are the wavefunction a propagator renormalization constants,
m^2 is the (renormalized) Lagrangian mass parameter, $\delta m_1{}^2$ is its counter-term,
λ is the (renormalized) four point coupling,
$\delta\lambda_{1\,a}$, $\delta\lambda_{2\,a}$, $\delta\lambda_{2\,b}$ are the independent coupling counter-terms,
v is the scalar field vacuum expectation value,
$\hbar$ is the reduced Planck constant,
n is the number of fields in the O(n) symmetry group,
t∞g, t∞n are the divergent tadpole integrals for the Goldstone, Higgs resp.,
tfing, tfinn are the finite parts of the tadpoles for the Goldstone, Higgs resp.

In[41]:= $$\text{geom} = p^2 - \text{mg2} == Z\,Z\Delta\,p^2 - m^2 - \delta m_1{}^2 - Z\Delta\,\frac{\lambda + \delta\lambda_{1\,a}}{6}\,v^2 - \frac{\hbar}{6}\left((n+1)\,\lambda + (n-1)\,\delta\lambda_{2\,a} + 2\,\delta\lambda_{2\,b}\right) Z\Delta^2\,(\text{t}\infty\text{g} + \text{tfing}) - \frac{\hbar}{6}\,(\lambda + \delta\lambda_{2\,a})\,Z\Delta^2\,(\text{t}\infty\text{n} + \text{tfinn})$$

Higgs equation of motion

In[42]:= $$\text{neom} = p^2 - \text{mn2} == \frac{-\lambda\,v^2}{3}\,Z\Delta + p^2 - \text{mg2}$$

Infinite parts of tadpoles in MSbar

MSbar rules for 4 - 2 ϵ dimensions

```
regularisedtadpoles = {t∞g → c0 Λ^2 + c1 mg2 Log[Λ^2 / μ^2], t∞n → c0 Λ^2 + c1 mn2 Log[Λ^2 / μ^2]}
```

Sub in tadpole expressions, eliminate mn2 and solve for mg2

```
mg2soln =
 mg2 /. (geom /. regularisedtadpoles /. Solve[neom, mn2][[1]] // Solve[#, mg2][[1]] &)
```

Gather divergences proportional v, tfing and tfinn and set independently to zero

First we subtract the finite equation of motion, then gather coefficients of the remainder into a list and set each to zero (after some trimming and simplifying).

```
In[45]:= cteq =
  ((CoefficientList[mg2soln + (-m^2 - λ/6 v^2 - ℏ/6 ((n + 1) λ) (tfing) - ℏ/6 (λ) (tfinn)), {p, v,
           tfing, tfinn}] // Flatten) //
       DeleteDuplicates // Simplify // FullSimplify) == 0 // Thread
```

Solve for counterterms

```
In[46]:= cts = {δm1^2, δλ1a, δλ2a, δλ2b, Z, ZΔ} /. Solve[cteq, {δm1, δλ1a, δλ2a, δλ2b, Z, ZΔ}] //
    FullSimplify // DeleteDuplicates
```

ZΔ is redundant in this truncation, can remove it :

```
In[47]:= cts /. ZΔ → 1 // FullSimplify
```

```
In[48]:= δλ2a == (n + 2)/(n + 4) δλ1a /. Solve[cteq, {δm1, δλ1a, δλ2a, δλ2b, Z, ZΔ}] /. {ZΔ → 1} //
   FullSimplify // DeleteDuplicates
```

Sunset

NOTE: this uses some of the same variable names as the Hartree-Fock code! Be careful not to clobber what you need to keep.

```
ClearAll[geom, neom, intrules, regularisedtadpoles, mg2soln, cteq, δm, δλ, δλ, δλ, δλ];
```

Equations of motion

Additional variables relative to the Hartree-Fock case:
Ing is the sunset integral $I_{\mathrm{NG}}(p)$

Ifingp is the finite sunset integral $I_{\mathrm{NG}}^{\mathrm{fin}}(p)$,
Ifing0 is $I_{\mathrm{NG}}^{\mathrm{fin}}(m_G)$,
Ifingn is $I_{\mathrm{NG}}^{\mathrm{fin}}(m_N)$,
$\delta\lambda$ is the sunset graph coupling counter-term,
Iμ, tμ and cμ are the auxiliary integrals I_μ, T_μ and c_μ respectively.

In[50]:=
$$\text{geom} = p^2 - \text{mg2} + i\hbar\left(\frac{(\lambda)\,v}{3}\right)^2(\text{Ifingp} - \text{Ifing0}) == \text{Z}\,\text{Z}\Delta\,p^2 - m^2 - \delta m_1{}^2 - \text{Z}\Delta\frac{\lambda + \delta\lambda_{1\,a}}{6}v^2 - \frac{\hbar}{6}\left((n+1)\,\lambda + (n-1)\,\delta\lambda_{2\,a} + 2\,\delta\lambda_{2\,b}\right)\text{Z}\Delta^2\,(\text{tg}) - \frac{\hbar}{6}(\lambda + \delta\lambda_{2\,a})\,\text{Z}\Delta^2\,(\text{tn}) + i\hbar\left(\frac{(\lambda+\delta\lambda)\,v}{3}\right)^2\text{Z}\Delta^3\,\text{Ing}$$

In[51]:=
$$\text{neom} = p^2 - \text{mn2} + i\hbar\left(\frac{(\lambda)\,v}{3}\right)^2(\text{Ifingp} - \text{Ifingn}) == \frac{-\text{Z}\Delta\,(\lambda+\delta\lambda)\,v^2}{3} + p^2 - \text{mg2} + i\hbar\left(\frac{(\lambda)\,v}{3}\right)^2(\text{Ifingp} - \text{Ifing0})$$

Divergent parts subtracted with auxiliary integrals and MSbar

In[52]:=
$$\text{intrules} = \left\{\text{Ing} \to \text{I}\mu + \text{Ifingp} + \text{Ifing0},\; \text{tg} \to \text{t}\mu - i\,(\text{mg2} - \mu^2)\,\text{I}\mu + \hbar\left(\frac{(\lambda+\delta\lambda)\,v}{3}\right)^2\text{c}\mu + \text{tfing},\; \text{tn} \to \text{t}\mu - i\,(\text{mn2} - \mu^2)\,\text{I}\mu + \hbar\left(\frac{(\lambda+\delta\lambda)\,v}{3}\right)^2\text{c}\mu + \text{tfinn}\right\}$$

$$\text{regularisedtadpoles} = \left\{\text{I}\mu \to \text{c2}\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right],\; \text{t}\mu \to \text{c0}\,\Lambda^2 + \text{c1}\,\mu^2\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right],\; \text{c}\mu \to \text{a0}\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]^2 + \text{a1}\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\right\}$$

Sub everything in, eliminate mn2 and solve for mg2

```
mg2soln = ((geom /. intrules(*/.regularisedtadpoles*) /. Solve[neom, mn2][[1]]) //
    Solve[#, mg2] &)[[1]]
```

Gather kinematically distinct divergences for Goldstone EOM

In[55]:=
$$\text{cteq} = \left(\left(\text{mg2} - m^2 - \frac{\lambda}{6}v^2 - \frac{\hbar}{6}\left((n+1)\,\lambda\right)(\text{tfing}) - \frac{\hbar}{6}(\lambda)\,(\text{tfinn}) \;/.\; \text{mg2soln}\right) \;//\; \text{CoefficientList}[\#, \{p, v, \text{tfing}, \text{tfinn}, \text{Ifingp}\}]\,\&\; // \;\text{Flatten}\; //\; \text{Simplify}\; //\; \text{DeleteDuplicates}\right) == 0 \;//\; \text{Thread}$$

Solve for counter-terms from Goldstone EOM

Note there are two solutions differing by a sign for $\delta\lambda$.

In[56]:=
$$\text{cts} = \text{Solve}[\text{cteq}, \{\delta m_1, \delta\lambda_{1\,a}, \delta\lambda_{2\,a}, \delta\lambda_{2\,b}, \delta\lambda, \text{Z}, \text{Z}\Delta\}] \;//\; \text{FullSimplify} \;//\; \text{DeleteDuplicates};$$

In[57]:=
```
{δm₁², δλ1a, δλ2a, δλ2b, δλ, Z, ZΔ} /. cts // DeleteDuplicates
```

Gather kinematically distinct divergences for Higgs EOM

In[58]:=
```
cteq2 =
  (((((mn2 - (λ v²/3) - m² - λ/6 v² - ℏ/6 ((n + 1) λ) (tfing) - ℏ/6 (λ) (tfinn) /. mg2soln) /. Solve[
                neom, mn2][[1]] /. mg2soln) /. cts // FullSimplify //
            DeleteDuplicates) /. {tfing → 0, tfinn → 0} // Expand) //
        CoefficientList[#, {p, v, tfing, tfinn, Ifingp}] & // Flatten //
      Simplify // DeleteDuplicates) == 0 // Thread
```

Solve for counter-terms from Higgs EOM

In[59]:=
```
cts2 = Solve[cteq2[[2]], {ZΔ}]
```

Both equations should have the same solution:

In[60]:=
```
(ZΔ /. Solve[cteq2[[3]], {ZΔ}][[1]]) - (ZΔ /. cts2[[1]]) == 0
```

Final Counterterms

In[61]:=
```
({δm₁², δλ1a, δλ2a, δλ2b, δλ, Z, ZΔ} /. cts /. cts2 // Simplify)[[1]] //
  DeleteDuplicates;
```

In[62]:=
```
counterterms = Thread[{δm₁², δλ1a, δλ2a, δλ2b, δλ, Z, ZΔ} → %[[1]]]
```

The should be momentum independent :

In[63]:=
```
({δm₁², δλ1a, δλ2a, δλ2b, δλ, Z, ZΔ} /. counterterms // DeleteDuplicates // D[#, p] &)[[
   1]] == 0 // Thread
({δm₁², δλ1a, δλ2a, δλ2b, δλ, Z, ZΔ} /. counterterms // DeleteDuplicates //
     D[#, Ifingp] &)[[1]] == 0 // Thread
```