Renormalization of Symmetry Improved 2PIEA gap equations at 2 loops

Supplement to chapter 4 of thesis by Michael J. Brown.

Mathematica notebook to compute couter-terms for two loop truncations of the effective action as described in Chapter 4.

Sunset

NOTE: this uses some of the same variable names as the Hartree-Fock code! Be careful not to clobber what you need to keep.

```
ClearAll[geom, neom, intrules, regularisedtadpoles, mg2soln, cteq, δm, δλ, δλ, δλ, δλ];
```

Equations of motion

Goldstone equation of motion. Quantities in reference to the paper are:
p is the four-momentum flowing through the propagators $\Delta_G{}^{-1}$ and $\Delta_N{}^{-1}$,
mg2 is the Goldstone mass squared $m_G{}^2$,
Z and ZΔ are the wavefunction a propagator renormalization constants,
m^2 is the (renormalized) Lagrangian mass parameter, $\delta m_1{}^2$ is its counter-term,
λ is the (renormalized) four point coupling,
$\delta\lambda_{1a}$, $\delta\lambda_{2a}$, $\delta\lambda_{2b}$ are the independent coupling counter-terms,
v is the scalar field vacuum expectation value,
$\hbar$ is the reduced Planck constant,
n is the number of fields in the O(n) symmetry group,
t∞g, t∞n are the divergent tadpole integrals for the Goldstone, Higgs resp.,
tfing, tfinn are the finite parts of the tadpoles for the Goldstone, Higgs resp.
Additional variables relative to the Hartree-Fock case:
Ing is the sunset integral $I_{\mathrm{NG}}(p)$
Ifingp is the finite sunset integral $I_{\mathrm{NG}}^{\mathrm{fin}}(p)$,
Ifing0 is $I_{\mathrm{NG}}^{\mathrm{fin}}(m_G)$,
Ifingn is $I_{\mathrm{NG}}^{\mathrm{fin}}(m_N)$,
$\delta\lambda$ is the sunset graph coupling counter-term,
Iμ, tμ and cμ are the auxiliary integrals I_μ, T_μ and c_μ respectively.

$$\text{geom} = p^2 - \text{mg2} + i\,\hbar \left(\frac{(\lambda)\ v}{3}\right)^2 (\text{Ifingp} - \text{Ifing0}) ==$$
$$Z\,Z\Delta\,p^2 - m^2 - \delta m_1{}^2 - Z\Delta\,\frac{\lambda + \delta\lambda_{1\,a}}{6}\,v^2 - \frac{\hbar}{6}\left((n+1)\,\lambda + (n-1)\,\delta\lambda_{2\,a} + 2\,\delta\lambda_{2\,b}\right) Z\Delta^2\,(\text{tg}) -$$
$$\frac{\hbar}{6}\,(\lambda + \delta\lambda_{2\,a})\,Z\Delta^2\,(\text{tn}) + i\,\hbar \left(\frac{(\lambda+\delta\lambda)\ v}{3}\right)^2 Z\Delta^3\,\text{Ing}$$

$$\text{neom} = p^2 - \text{mn2} + \frac{i\,\hbar}{2}\left(\frac{(\lambda)\ v}{3}\right)^2 (n-1)\ (\text{Ifinggp} - \text{Ifinggn}) + \frac{i\,\hbar}{2}\,(\lambda)^2\,v^2\,(\text{Ifinhhp} - \text{Ifinhhn}) ==$$
$$Z\,Z\Delta\,p^2 - m^2 - \delta m_1{}^2 - Z\Delta\,\frac{3\,\lambda + \delta\lambda_{1\,a} + 2\,\delta\lambda_{1\,b}}{6}\,v^2 - \frac{\hbar}{6}\left(3\,\lambda + \delta\lambda_{2\,a} + 2\,\delta\lambda_{2\,b}\right) Z\Delta^2\,\text{tn} -$$
$$\frac{\hbar}{6}\,(\lambda + \delta\lambda_{2\,a})\,Z\Delta^2\,(n-1)\ \text{tg} + \frac{i\,\hbar}{2}\left(\frac{(\lambda+\delta\lambda)\ v}{3}\right)^2 Z\Delta^3\,(n-1)\ \text{Igg} + \frac{i\,\hbar}{2}\,(\lambda+\delta\lambda)^2\,v^2\,Z\Delta^3\,\text{Ihh}$$

Divergent parts subtracted with auxiliary integrals and MSbar

$$\text{intrules} = \{\text{Ing} \to \text{I}\mu + \text{Ifingp},\ \text{Igg} \to \text{I}\mu + \text{Ifinggp},\ \text{Ihh} \to \text{I}\mu + \text{Ifinhhp},$$
$$\text{tg} \to \text{t}\mu - i\,(\text{mg2} - \mu^2)\ \text{I}\mu + \hbar \left(\frac{(\lambda+\ \delta\lambda)\ v}{3}\right)^2 \text{c}\mu + \text{tfing},$$
$$\text{tn} \to \text{t}\mu - i\,(\text{mn2} - \mu^2)\ \text{I}\mu + \hbar \left(\frac{(\lambda+\ \delta\lambda)\ v}{3}\right)^2 \text{c}\mu + \text{tfinn}\}$$

$$\text{regularisedtadpoles} =$$
$$\left\{\text{I}\mu \to \text{c2 Log}\left[\frac{\Lambda^2}{\mu^2}\right],\ \text{t}\mu \to \text{c0}\ \Lambda^2 + \text{c1}\ \mu^2\ \text{Log}\left[\frac{\Lambda^2}{\mu^2}\right],\ \text{c}\mu \to \text{a0 Log}\left[\frac{\Lambda^2}{\mu^2}\right]^2 + \text{a1 Log}\left[\frac{\Lambda^2}{\mu^2}\right]\right\}$$

Sub everything in, eliminate mn2 and solve for mg2

```
{mg2soln, mn2soln} =
  (Solve[{geom, neom} /. intrules, {mg2, mn2}] // ExpandAll // Simplify)[[1]]
```

Check solutions

$$\left(p^2 - \text{mg2} + i\,\hbar \left(\frac{(\lambda)\ v}{3}\right)^2 (\text{Ifingp} - \text{Ifing0}) -\right.$$
$$\left(Z\,Z\Delta\,p^2 - m^2 - \delta m_1{}^2 - Z\Delta\,\frac{\lambda + \delta\lambda_{1\,a}}{6}\,v^2 - \frac{\hbar}{6}\left((n+1)\,\lambda + (n-1)\,\delta\lambda_{2\,a} + 2\,\delta\lambda_{2\,b}\right) Z\Delta^2\,(\text{tg}) -\right.$$
$$\left.\left.\frac{\hbar}{6}\,(\lambda + \delta\lambda_{2\,a})\,Z\Delta^2\,(\text{tn}) + i\,\hbar \left(\frac{(\lambda+\delta\lambda)\ v}{3}\right)^2 Z\Delta^3\,\text{Ing}\right)\right) \text{ /. intrules /.}$$
$$\text{mn2soln /. mg2soln /. regularisedtadpoles // Simplify}$$

$$\left(p^2 - \text{mn2} + \frac{i\hbar}{2}\left(\frac{(\lambda)\,v}{3}\right)^2 (n-1)\,(\text{Ifinggp} - \text{Ifinggn}) + \frac{i\hbar}{2}(\lambda)^2 v^2\,(\text{Ifinhhp} - \text{Ifinhhn}) - \left(Z\,Z\Delta\,p^2 - m^2 - \delta m_1{}^2 - Z\Delta\,\frac{3\lambda + \delta\lambda_{1a} + 2\,\delta\lambda_{1b}}{6}\,v^2 - \frac{\hbar}{6}\left(3\lambda + \delta\lambda_{2a} + 2\,\delta\lambda_{2b}\right) Z\Delta^2\,\text{tn} - \frac{\hbar}{6}(\lambda + \delta\lambda_{2a})\,Z\Delta^2\,(n-1)\,\text{tg} + \frac{i\hbar}{2}\left(\frac{(\lambda+\delta\lambda)\,v}{3}\right)^2 Z\Delta^3\,(n-1)\,\text{Igg} + \frac{i\hbar}{2}(\lambda+\delta\lambda)^2\,v^2\,Z\Delta^3\,\text{Ihh}\right)\right)\ /.\ \text{intrules}\ /.\ \text{mn2soln}\ /.\ \text{mg2soln}\ /.\ \text{regularisedtadpoles}\ //\ \text{Simplify}$$

Gather kinematically distinct divergences for Goldstone EOM

$$\left(\left(p^2 - \text{mg2} + i\hbar\left(\frac{(\lambda)\,v}{3}\right)^2 (\text{Ifingp} - \text{Ifing0}) - \left(p^2 - m^2 - \frac{\lambda}{6}v^2 - \frac{\hbar}{6}\left((n+1)\,\lambda\right)(\text{tfing}) - \frac{\hbar}{6}(\lambda)\,(\text{tfinn}) + i\hbar\left(\frac{(\lambda)\,v}{3}\right)^2 (\text{Ifingp})\right)\right)\ /.\ \text{intrules}\ /.\ \text{mn2soln}\ /.\ \text{mg2soln}\ /.\ \text{regularisedtadpoles}\ //\ \text{Simplify}\right)\ //$$

```
CoefficientList[#, {p, v, tfing, tfinn, Ifingp, Ifinggp, Ifinhhp}] & //
  Flatten // Simplify // DeleteDuplicates
```

$$\text{cteq} = \left(\left(\text{mg2} - m^2 - \frac{\lambda}{6}v^2 - \frac{\hbar}{6}\left((n+1)\,\lambda\right)(\text{tfing}) - \frac{\hbar}{6}(\lambda)\,(\text{tfinn})\ /.\ \text{mg2soln}\right)\ //\ \text{CoefficientList}[\#, \{p, v, \text{tfing}, \text{tfinn}, \text{Ifingp}, \text{Ifinggp}, \text{Ifinhhp}\}]\ \&\ //\ \text{Flatten}\ //\ \text{Simplify}\ //\ \text{DeleteDuplicates}\right) == 0\ //\ \text{Thread}$$

```
cteq = (cteq /. regularisedtadpoles // Simplify // DeleteDuplicates)
```

Solve for counter-terms from Goldstone EOM

Note there are two solutions differing by a sign for $\delta\lambda$.

$$\text{cts} = \text{Solve}[\text{cteq}, \{\delta m_1, \delta\lambda_{1a}, \delta\lambda_{1b}, \delta\lambda_{2a}, \delta\lambda_{2b}, \delta\lambda, Z, Z\Delta\}]\ //\ \text{DeleteDuplicates};$$

$$\{\delta m_1{}^2, \delta\lambda_{1a}, \delta\lambda_{1b}, \delta\lambda_{2a}, \delta\lambda_{2b}, \delta\lambda, Z, Z\Delta\}\ /.\ \text{cts}\ //\ \text{DeleteDuplicates}$$

Gather kinematically distinct divergences for Higgs EOM

```
cteq2 =
 (((((mn2 - (λ v^2/3) - m^2 - λ/6 v^2 - ℏ/6 ((n + 1) λ) (tfing) - ℏ/6 (λ) (tfinn) /. mg2soln) /. Solve[
            neom, mn2][[1]] /. mg2soln) /. cts // FullSimplify //
        DeleteDuplicates) /. {tfing → 0, tfinn → 0} // Expand) //
     CoefficientList[#, {p, v, tfing, tfinn, Ifingp}] & // Flatten //
    Simplify // DeleteDuplicates) == 0 // Thread
```

Solve for counter-terms from Higgs EOM

```
cts2 = Solve[cteq2[[2]], {ZΔ}]
```

Both equations should have the same solution:

```
(ZΔ /. Solve[cteq2[[3]], {ZΔ}][[1]]) - (ZΔ /. cts2[[1]]) == 0
```

Final Counterterms

```
({δm1^2, δλ1a, δλ2a, δλ2b, δλ, Z, ZΔ} /. cts /. cts2 // Simplify)[[1]] //
  DeleteDuplicates;
```

```
counterterms = Thread[{δm1^2, δλ1a, δλ2a, δλ2b, δλ, Z, ZΔ} → %[[1]]]
```

The should be momentum independent :

```
({δm1^2, δλ1a, δλ2a, δλ2b, δλ, Z, ZΔ} /. counterterms // DeleteDuplicates // D[#, p] &)[[
   1]] == 0 // Thread
({δm1^2, δλ1a, δλ2a, δλ2b, δλ, Z, ZΔ} /. counterterms // DeleteDuplicates //
      D[#, Ifingp] &)[[1]] == 0 // Thread
```