Renormalization of Symmetry Improved 3PIEA gap equations at 2 loops

Supplement to "Symmetry improvement of 3PI effective actions for O(N) scalar field theory" by Michael J. Brown and Ian B. Whittingham.

Mathematica notebook to compute couter-terms for two loop truncations of the effective action as described in Section IV of the paper.

Hartree-Fock

```
ClearAll[geom, neom, intrules, msbarrules, mg2soln, cteq, cts, δm, δλ];
```

Hartree-Fock gap equations with counterterms

Goldstone equation of motion. Quantities in reference to the paper are:
p is the four-momentum flowing through the propagators $\Delta_G{}^{-1}$ and $\Delta_N{}^{-1}$,
mg2 is the Goldstone mass squared $m_G{}^2$,
Z and ZΔ are the wavefunction a propagator renormalization constants,
m^2 is the (renormalized) Lagrangian mass parameter, $\delta m_1{}^2$ is its counter-term,
λ is the (renormalized) four point coupling,
$\delta\lambda_{1\,a}$, $\delta\lambda_{2\,a}$, $\delta\lambda_{2\,b}$ are the independent coupling counter-terms,
v is the scalar field vacuum expectation value,
$\hbar$ is the reduced Planck constant,
n is the number of fields in the O(n) symmetry group,
t∞g, t∞n are the divergent tadpole integrals for the Goldstone, Higgs resp.,
tfing, tfinn are the finite parts of the tadpoles for the Goldstone, Higgs resp.

$$\text{geom} = p^2 - \text{mg2} == Z\,Z\Delta\,p^2 - m^2 - \delta m_1{}^2 - Z\Delta\,\frac{\lambda + \delta\lambda_{1\,a}}{6}\,v^2 - \frac{\hbar}{6}\left(\left(n+1\right)\lambda + \left(n-1\right)\delta\lambda_{2\,a} + 2\,\delta\lambda_{2\,b}\right) Z\Delta^2\,(\text{t}\infty\text{g} + \text{tfing}) - \frac{\hbar}{6}\,(\lambda + \delta\lambda_{2\,a})\,Z\Delta^2\,(\text{t}\infty\text{n} + \text{tfinn})$$

$$-\text{mg2} + p^2 == -m^2 + p^2\,Z\,Z\Delta - \delta m_1^2 - \frac{1}{6}\,v^2\,Z\Delta\,(\lambda + \delta\lambda_a) - \frac{1}{6}\,(\text{tfinn} + \text{t}\infty\text{n})\,Z\Delta^2\,\hbar\,(\lambda + \delta\lambda_{2\,a}) - \frac{1}{6}\,(\text{tfing} + \text{t}\infty\text{g})\,Z\Delta^2\,\hbar\,((1+n)\,\lambda + (-1+n)\,\delta\lambda_{2\,a} + 2\,\delta\lambda_{2\,b})$$

Higgs equation of motion

```
neom = p^2 - mn2 == (-λ v^2)/3 ZΔ + p^2 - mg2
```

$$-\text{mn2} + p^2 == -\text{mg2} + p^2 - \frac{1}{3} v^2\, Z\Delta\, \lambda$$

Infinite parts of tadpoles in MSbar

MSbar rules for 4 - 2 ϵ dimensions

```
msbarrules =
  {t∞g → (- mg2)/(16 π^2) (1/ϵ - EulerGamma + 1 + Log[4 π]), t∞n → (- mn2)/(16 π^2) (1/ϵ - EulerGamma + 1 + Log[4 π])}
```

$$\left\{\text{t}\infty\text{g} \to -\frac{\text{mg2}\left(1 - \text{EulerGamma} + \frac{1}{\epsilon} + \text{Log}[4\,\pi]\right)}{16\,\pi^2},\ \text{t}\infty\text{n} \to -\frac{\text{mn2}\left(1 - \text{EulerGamma} + \frac{1}{\epsilon} + \text{Log}[4\,\pi]\right)}{16\,\pi^2}\right\}$$

Sub in tadpole expressions, eliminate mn2 and solve for mg2

```
mg2soln = mg2 /. (geom /. msbarrules /. Solve[neom, mn2][[1]] // Solve[#, mg2][[1]] &)
```

$$\left(-m^2 - p^2 + p^2\, Z\, Z\Delta - \delta m_1^2 - \frac{1}{6} v^2\, Z\Delta\, (\lambda + \delta\lambda_a) - \frac{1}{6}\,\text{tfinn}\, Z\Delta^2\, \hbar\, (\lambda + \delta\lambda_{2\,a}) + \frac{v^2\, Z\Delta^3\, \lambda\, \hbar \left(1 - \text{EulerGamma} + \frac{1}{\epsilon} + \text{Log}[4\,\pi]\right) (\lambda + \delta\lambda_{2\,a})}{288\,\pi^2} - \frac{1}{6}\,\text{tfing}\, Z\Delta^2\, \hbar\, ((1+n)\,\lambda + (-1+n)\,\delta\lambda_{2\,a} + 2\,\delta\lambda_{2\,b})\right) \Big/ \left(-1 - \frac{Z\Delta^2\, \hbar \left(1 - \text{EulerGamma} + \frac{1}{\epsilon} + \text{Log}[4\,\pi]\right) (\lambda + \delta\lambda_{2\,a})}{96\,\pi^2} - \frac{1}{96\,\pi^2} Z\Delta^2\, \hbar \left(1 - \text{EulerGamma} + \frac{1}{\epsilon} + \text{Log}[4\,\pi]\right) ((1+n)\,\lambda + (-1+n)\,\delta\lambda_{2\,a} + 2\,\delta\lambda_{2\,b})\right)$$

Gather divergences proportional v, tfing and tfinn and set independently to zero

First we subtract the finite equation of motion, then gather coefficients of the remainder into a list and set each to zero (after some trimming and simplifying).

```
cteq =
 ((CoefficientList[mg2soln + (-m² - λ/6 v² - ℏ/6 ((n + 1) λ) (tfing) - ℏ/6 (λ) (tfinn)), {p, v,
        tfing, tfinn}] // Flatten) //
     DeleteDuplicates // Simplify // FullSimplify) == 0 // Thread
```

$$
\begin{aligned}
\Big\{ & \big(96\,\delta m_1^2\big) \Big/ \Big(96 + \tfrac{1}{\pi^2} Z\Delta^2\,\hbar\,\big(1 - \text{EulerGamma} + \tfrac{1}{\epsilon} + \text{Log}[4] + \text{Log}[\pi]\big)\,(\lambda + \delta\lambda_{2\,a}) + \tfrac{1}{\pi^2} \\
& \quad Z\Delta^2\,\hbar\,\big(1 - \text{EulerGamma} + \tfrac{1}{\epsilon} + \text{Log}[4] + \text{Log}[\pi]\big)\,(\lambda + n\,\lambda + (-1+n)\,\delta\lambda_{2\,a} + 2\,\delta\lambda_{2\,b})\Big) + \\
& \quad m^2\,\Big(-1 + 96 \Big/ \Big(96 + \tfrac{1}{\pi^2} Z\Delta^2\,\hbar\,\big(1 - \text{EulerGamma} + \tfrac{1}{\epsilon} + \text{Log}[4] + \text{Log}[\pi]\big)\,(\lambda + \delta\lambda_{2\,a}) + \tfrac{1}{\pi^2} Z\Delta^2\,\hbar \\
& \qquad \big(1 - \text{EulerGamma} + \tfrac{1}{\epsilon} + \text{Log}[4] + \text{Log}[\pi]\big)\,(\lambda + n\,\lambda + (-1+n)\,\delta\lambda_{2\,a} + 2\,\delta\lambda_{2\,b})\Big)\Big) == 0,\ \tfrac{1}{6}\,\hbar \\
& \quad \Big(-\lambda + \big(96\,Z\Delta^2\,(\lambda + \delta\lambda_{2\,a})\big) \Big/ \Big(96 + \tfrac{1}{\pi^2} Z\Delta^2\,\hbar\,\big(1 - \text{EulerGamma} + \tfrac{1}{\epsilon} + \text{Log}[4] + \text{Log}[\pi]\big)\,(\lambda + \delta\lambda_{2\,a}) + \\
& \qquad \tfrac{1}{\pi^2} Z\Delta^2\,\hbar\,\big(1 - \text{EulerGamma} + \tfrac{1}{\epsilon} + \text{Log}[4] + \text{Log}[\pi]\big)\,(\lambda + n\,\lambda + (-1+n)\,\delta\lambda_{2\,a} + 2\,\delta\lambda_{2\,b})\Big)\Big) == \\
& \quad 0,\ \tfrac{1}{6}\,\hbar\,\Big(-(1+n)\,\lambda + \big(96\,Z\Delta^2\,(\lambda + n\,\lambda + (-1+n)\,\delta\lambda_{2\,a} + 2\,\delta\lambda_{2\,b})\big) \Big/ \\
& \qquad \Big(96 + \tfrac{1}{\pi^2} Z\Delta^2\,\hbar\,\big(1 - \text{EulerGamma} + \tfrac{1}{\epsilon} + \text{Log}[4] + \text{Log}[\pi]\big)\,(\lambda + \delta\lambda_{2\,a}) + \tfrac{1}{\pi^2} Z\Delta^2\,\hbar\,\big(1 - \\
& \qquad \text{EulerGamma} + \tfrac{1}{\epsilon} + \text{Log}[4] + \text{Log}[\pi]\big)\,(\lambda + n\,\lambda + (-1+n)\,\delta\lambda_{2\,a} + 2\,\delta\lambda_{2\,b})\Big)\Big) == 0,\ \text{True}, \\
& \tfrac{1}{6}\,\Big(-\lambda + (96\,Z\Delta\,(\lambda + \delta\lambda_{a})) \Big/ \Big(96 + \tfrac{1}{\pi^2} Z\Delta^2\,\hbar\,\big(1 - \text{EulerGamma} + \tfrac{1}{\epsilon} + \text{Log}[4] + \text{Log}[\pi]\big)\,(\lambda + \delta\lambda_{2\,a}) + \\
& \qquad \tfrac{1}{\pi^2} Z\Delta^2\,\hbar\,\big(1 - \text{EulerGamma} + \tfrac{1}{\epsilon} + \text{Log}[4] + \text{Log}[\pi]\big)\,(\lambda + n\,\lambda + (-1+n)\,\delta\lambda_{2\,a} + 2\,\delta\lambda_{2\,b})\Big) - \\
& \quad \Big(2\,Z\Delta^3\,\lambda\,\hbar\,\big(1 - \text{EulerGamma} + \tfrac{1}{\epsilon} + \text{Log}[4] + \text{Log}[\pi]\big)\,(\lambda + \delta\lambda_{2\,a})\Big) \Big/ \\
& \quad \Big(\pi^2\,\Big(96 + \tfrac{1}{\pi^2} Z\Delta^2\,\hbar\,\big(1 - \text{EulerGamma} + \tfrac{1}{\epsilon} + \text{Log}[4] + \text{Log}[\pi]\big)\,(\lambda + \delta\lambda_{2\,a}) + \tfrac{1}{\pi^2} Z\Delta^2\,\hbar \\
& \qquad \big(1 - \text{EulerGamma} + \tfrac{1}{\epsilon} + \text{Log}[4] + \text{Log}[\pi]\big)\,(\lambda + n\,\lambda + (-1+n)\,\delta\lambda_{2\,a} + 2\,\delta\lambda_{2\,b})\Big)\Big)\Big) == 0, \\
& -\Big(\big(96\,\pi^2\,(-1 + Z\,Z\Delta)\,\epsilon\big) \Big/ \Big(96\,\pi^2\,\epsilon + (2+n)\,Z\Delta^2\,\lambda\,\hbar\,(1 + \epsilon\,(1 - \text{EulerGamma} + \text{Log}[4] + \text{Log}[\pi])) - \\
& \qquad Z\Delta^2\,\hbar\,(-1 + \epsilon\,(-1 + \text{EulerGamma} - \text{Log}[4\,\pi]))\,(n\,\delta\lambda_{2\,a} + 2\,\delta\lambda_{2\,b})\Big)\Big) == 0 \Big\}
\end{aligned}
$$

Solve for counterterms

```
cts = {δm1^2, δλ1a, δλ2a, δλ2b, Z, ZΔ} /. Solve[cteq, {δm1, δλ1a, δλ2a, δλ2b, Z, ZΔ}] //
   FullSimplify // DeleteDuplicates
```

Solve::svars : Equations may not give solutions for all "solve" variables. ≫

$$\left\{\left\{\frac{m^2}{-1+\frac{96\pi^2\epsilon}{(2+n)\lambda\hbar(1+\epsilon(1-\text{EulerGamma}+\text{Log}[4]+\text{Log}[\pi]))}},\right.\right.$$
$$-\left(\left(\lambda\left(96\pi^2(-1+Z\Delta)\epsilon+(4+n)Z\Delta\lambda\hbar(-1+\epsilon(-1+\text{EulerGamma}-\text{Log}[4\pi]))\right)\right)\Big/\left(Z\Delta\left(96\pi^2\epsilon+(2+n)\lambda\hbar(-1+\epsilon(-1+\text{EulerGamma}-\text{Log}[4\pi]))\right)\right)\right),$$
$$-\left(\left(\lambda\left(96\pi^2\left(-1+Z\Delta^2\right)\epsilon+(2+n)Z\Delta^2\lambda\hbar(-1+\epsilon(-1+\text{EulerGamma}-\text{Log}[4\pi]))\right)\right)\Big/\left(Z\Delta^2\left(96\pi^2\epsilon+(2+n)\lambda\hbar(-1+\epsilon(-1+\text{EulerGamma}-\text{Log}[4\pi]))\right)\right)\right),$$
$$\left.\left.-\left(\left(\lambda\left(96\pi^2\left(-1+Z\Delta^2\right)\epsilon+(2+n)Z\Delta^2\lambda\hbar(-1+\epsilon(-1+\text{EulerGamma}-\text{Log}[4\pi]))\right)\right)\Big/\left(Z\Delta^2\left(96\pi^2\epsilon+(2+n)\lambda\hbar(-1+\epsilon(-1+\text{EulerGamma}-\text{Log}[4\pi]))\right)\right)\right),\ \frac{1}{Z\Delta},\ Z\Delta\right\}\right\}$$

ZΔ is redundant in this truncation, can remove it :

```
cts /. ZΔ → 1 // FullSimplify
```

$$\left\{\left\{\frac{m^2}{-1+\frac{96\pi^2\epsilon}{(2+n)\lambda\hbar(1+\epsilon(1-\text{EulerGamma}+\text{Log}[4]+\text{Log}[\pi]))}},\ \frac{(4+n)\lambda^2}{-(2+n)\lambda+\frac{96\pi^2\epsilon}{\hbar(1+\epsilon(1-\text{EulerGamma}+\text{Log}[4]+\text{Log}[\pi]))}},\right.\right.$$
$$\left.\left.\frac{\lambda^2}{-\lambda+\frac{96\pi^2\epsilon}{(2+n)\hbar(1+\epsilon(1-\text{EulerGamma}+\text{Log}[4]+\text{Log}[\pi]))}},\ \frac{\lambda^2}{-\lambda+\frac{96\pi^2\epsilon}{(2+n)\hbar(1+\epsilon(1-\text{EulerGamma}+\text{Log}[4]+\text{Log}[\pi]))}},\ 1,\ 1\right\}\right\}$$

```
Series[cts, {ϵ, 0, 1}] /. {ZΔ → 1}
```

$$\left\{\left\{-m^2-\frac{96\left(m^2\pi^2\right)\epsilon}{(2+n)\lambda\hbar}+O[\epsilon]^2,\ -\frac{(4+n)\lambda}{2+n}-\frac{96\left(4\pi^2+n\pi^2\right)\epsilon}{(2+n)^2\hbar}+O[\epsilon]^2,\right.\right.$$
$$\left.\left.-\lambda-\frac{96\pi^2\epsilon}{(2+n)\hbar}+O[\epsilon]^2,\ -\lambda-\frac{96\pi^2\epsilon}{(2+n)\hbar}+O[\epsilon]^2,\ 1,\ 1\right\}\right\}$$

Sunset

NOTE: this uses some of the same variable names as the Hartree-Fock code! Be careful not to clobber what you need to keep.

```
ClearAll[geom, neom, intrules, msbarrules, mg2soln, cteq, δm, δλ, δλ, δλ, δλ];
```

Equations of motion

Additional variables relative to the Hartree-Fock case:

Ing is the sunset integral $I_{\mathrm{NG}}(p)$

Ifingp is the finite sunset integral $I^{\mathrm{fin}}_{\mathrm{NG}}(p)$,

Ifing0 is $I^{\mathrm{fin}}_{\mathrm{NG}}(m_G)$,

Ifingn is $I^{\mathrm{fin}}_{\mathrm{NG}}(m_N)$,

$\delta\lambda$ is the sunset graph coupling counter-term,
Iμ, tμ and cμ are the auxiliary integrals I_μ, T_μ and c_μ respectively.

$$\text{geom} = p^2 - \text{mg2} + i\,\hbar \left(\frac{(\lambda)\,v}{3}\right)^2 (\text{Ifingp} - \text{Ifing0}) == \\ Z\,Z\Delta\,p^2 - m^2 - \delta m_1{}^2 - Z\Delta\,\frac{\lambda + \delta\lambda_{1\,a}}{6}\,v^2 - \frac{\hbar}{6}\left((n+1)\,\lambda + (n-1)\,\delta\lambda_{2\,a} + 2\,\delta\lambda_{2\,b}\right) Z\Delta^2\,(\text{tg}) - \\ \frac{\hbar}{6}\,(\lambda + \delta\lambda_{2\,a})\,Z\Delta^2\,(\text{tn}) + i\,\hbar \left(\frac{(\lambda + \delta\lambda)\,v}{3}\right)^2 Z\Delta^3\,\text{Ing}$$

$$-\text{mg2} + p^2 + \frac{1}{9}\,i\,(-\text{Ifing0} + \text{Ifingp})\,v^2\,\lambda^2\,\hbar == -m^2 + p^2\,Z\,Z\Delta + \frac{1}{9}\,i\,\text{Ing}\,v^2\,Z\Delta^3\,(\delta\lambda + \lambda)^2\,\hbar - \delta m_1^2 - \\ \frac{1}{6}\,v^2\,Z\Delta\,(\lambda + \delta\lambda_a) - \frac{1}{6}\,\text{tn}\,Z\Delta^2\,\hbar\,(\lambda + \delta\lambda_{2\,a}) - \frac{1}{6}\,\text{tg}\,Z\Delta^2\,\hbar\,((1+n)\,\lambda + (-1+n)\,\delta\lambda_{2\,a} + 2\,\delta\lambda_{2\,b})$$

$$\text{neom} = p^2 - \text{mn2} + i\,\hbar \left(\frac{(\lambda)\,v}{3}\right)^2 (\text{Ifingp} - \text{Ifingn}) == \\ \frac{-Z\Delta\,(\lambda + \delta\lambda)\,v^2}{3} + p^2 - \text{mg2} + i\,\hbar \left(\frac{(\lambda)\,v}{3}\right)^2 (\text{Ifingp} - \text{Ifing0})$$

$$-\text{mn2} + p^2 + \frac{1}{9}\,i\,(-\text{Ifingn} + \text{Ifingp})\,v^2\,\lambda^2\,\hbar == \\ -\text{mg2} + p^2 - \frac{1}{3}\,v^2\,Z\Delta\,(\delta\lambda + \lambda) + \frac{1}{9}\,i\,(-\text{Ifing0} + \text{Ifingp})\,v^2\,\lambda^2\,\hbar$$

Divergent parts subtracted with auxiliary integrals and MSbar

$$\text{intrules} = \{\text{Ing} \to \text{I}\mu + \text{Ifingp} + \text{Ifing0}, \\ \text{tg} \to \text{t}\mu - i\,(\text{mg2} - \mu^2)\,\text{I}\mu + \hbar \left(\frac{(\lambda + \delta\lambda)\,v}{3}\right)^2 \text{c}\mu + \text{tfing}, \\ \text{tn} \to \text{t}\mu - i\,(\text{mn2} - \mu^2)\,\text{I}\mu + \hbar \left(\frac{(\lambda + \delta\lambda)\,v}{3}\right)^2 \text{c}\mu + \text{tfinn}\}$$

$$\{\text{Ing} \to \text{Ifing0} + \text{Ifingp} + \text{I}\mu,\ \text{tg} \to \text{tfing} + \text{t}\mu - i\,\text{I}\mu\,(\text{mg2} - \mu^2) + \frac{1}{9}\,\text{c}\mu\,v^2\,(\delta\lambda + \lambda)^2\,\hbar, \\ \text{tn} \to \text{tfinn} + \text{t}\mu - i\,\text{I}\mu\,(\text{mn2} - \mu^2) + \frac{1}{9}\,\text{c}\mu\,v^2\,(\delta\lambda + \lambda)^2\,\hbar\}$$

$$\text{msbarrules} = \left\{\text{I}\mu \to \frac{-i}{16\,\pi^2}\left(\frac{1}{\epsilon} - \text{EulerGamma} + \text{Log}[4\,\pi]\right), \\ \text{t}\mu \to \frac{-\mu^2}{16\,\pi^2}\left(\frac{1}{\epsilon} - \text{EulerGamma} + 1 + \text{Log}[4\,\pi]\right),\ \text{c}\mu \to \left(\frac{\text{a0}}{\epsilon^2} + \frac{\text{a1}}{\epsilon} + \text{a2}\right)\right\}$$

$$\left\{\text{I}\mu \to -\frac{i\left(-\text{EulerGamma} + \frac{1}{\epsilon} + \text{Log}[4\,\pi]\right)}{16\,\pi^2}, \\ \text{t}\mu \to -\frac{\mu^2\left(1 - \text{EulerGamma} + \frac{1}{\epsilon} + \text{Log}[4\,\pi]\right)}{16\,\pi^2},\ \text{c}\mu \to \text{a2} + \frac{\text{a0}}{\epsilon^2} + \frac{\text{a1}}{\epsilon}\right\}$$

Sub everything in, eliminate mn2 and solve for mg2

```
mg2soln =
  ((geom /. intrules(*/.msbarrules*) /. Solve[neom, mn2][[1]]) // Solve[#, mg2] &)[[
   1]]
```

$$\left\{mg2 \to \left(-m^2 - p^2 + p^2\, Z\, Z\Delta - \frac{1}{9}\, i\, (-Ifing0 + Ifingp)\, v^2\, \lambda^2\, \hbar + \frac{1}{9}\, i\, (Ifing0 + Ifingp + I\mu)\, v^2\, Z\Delta^3\, (\delta\lambda + \lambda)^2\, \hbar - \delta m_1^2 - \frac{1}{6}\, v^2\, Z\Delta\, (\lambda + \delta\lambda_a) - \frac{1}{6}\, tfinn\, Z\Delta^2\, \hbar\, (\lambda + \delta\lambda_{2\,a}) - \frac{1}{6}\, t\mu\, Z\Delta^2\, \hbar\, (\lambda + \delta\lambda_{2\,a}) + \frac{1}{18}\, i\, I\mu\, v^2\, Z\Delta^3\, \delta\lambda\, \hbar\, (\lambda + \delta\lambda_{2\,a}) + \frac{1}{18}\, i\, I\mu\, v^2\, Z\Delta^3\, \lambda\, \hbar\, (\lambda + \delta\lambda_{2\,a}) - \frac{1}{6}\, i\, I\mu\, Z\Delta^2\, \mu^2\, \hbar\, (\lambda + \delta\lambda_{2\,a}) - \frac{1}{54}\, c\mu\, v^2\, Z\Delta^2\, (\delta\lambda + \lambda)^2\, \hbar^2\, (\lambda + \delta\lambda_{2\,a}) - \frac{1}{54}\, I\mu\, Z\Delta^2\, \hbar\, \left(Ifing0\, v^2\, \lambda^2\, \hbar - Ifingn\, v^2\, \lambda^2\, \hbar\right) (\lambda + \delta\lambda_{2\,a}) - \frac{1}{6}\, tfing\, Z\Delta^2\, \hbar\, ((1 + n)\, \lambda + (-1 + n)\, \delta\lambda_{2\,a} + 2\, \delta\lambda_{2\,b}) - \frac{1}{6}\, t\mu\, Z\Delta^2\, \hbar\, ((1 + n)\, \lambda + (-1 + n)\, \delta\lambda_{2\,a} + 2\, \delta\lambda_{2\,b}) - \frac{1}{6}\, i\, I\mu\, Z\Delta^2\, \mu^2\, \hbar\, ((1 + n)\, \lambda + (-1 + n)\, \delta\lambda_{2\,a} + 2\, \delta\lambda_{2\,b}) - \frac{1}{54}\, c\mu\, v^2\, Z\Delta^2\, (\delta\lambda + \lambda)^2\, \hbar^2\, ((1 + n)\, \lambda + (-1 + n)\, \delta\lambda_{2\,a} + 2\, \delta\lambda_{2\,b})\right) \Big/ \left(-1 - \frac{1}{6}\, i\, I\mu\, Z\Delta^2\, \hbar\, (\lambda + \delta\lambda_{2\,a}) - \frac{1}{6}\, i\, I\mu\, Z\Delta^2\, \hbar\, ((1 + n)\, \lambda + (-1 + n)\, \delta\lambda_{2\,a} + 2\, \delta\lambda_{2\,b})\right)\right\}$$

Gather kinematically distinct divergences for Goldstone EOM

```
cteq = ((mg2 - m^2 - λ/6 v^2 - ħ/6 ((n + 1) λ) (tfing) - ħ/6 (λ) (tfinn) /. mg2soln) //
      CoefficientList[#, {p, v, tfing, tfinn, Ifingp}] & // Flatten //
    Simplify // DeleteDuplicates) == 0 // Thread
```

$$\Big\{\left(-6\,i\,\delta m_1^2 + Z\Delta^2\left(-i\,t\mu + I\mu\left(-m^2+\mu^2\right)\right)\hbar\left((2+n)\,\lambda + n\,\delta\lambda_{2a} + 2\,\delta\lambda_{2b}\right)\right)\Big/\left(-6\,i + 2\,I\mu\,Z\Delta^2\,\lambda\,\hbar + I\mu\,n\,Z\Delta^2\,\lambda\,\hbar + I\mu\,n\,Z\Delta^2\,\hbar\,\delta\lambda_{2a} + 2\,I\mu\,Z\Delta^2\,\hbar\,\delta\lambda_{2b}\right) = 0,$$

$$\text{True},\ \frac{1}{6}\hbar\left(-\lambda - \left(6\,i\,Z\Delta^2\left(\lambda+\delta\lambda_{2a}\right)\right)\Big/\left(-6\,i + 2\,I\mu\,Z\Delta^2\,\lambda\,\hbar + I\mu\,n\,Z\Delta^2\,\lambda\,\hbar + I\mu\,n\,Z\Delta^2\,\hbar\,\delta\lambda_{2a} + 2\,I\mu\,Z\Delta^2\,\hbar\,\delta\lambda_{2b}\right)\right) = 0,$$

$$\frac{1}{6}\hbar\left(-(1+n)\,\lambda - \left(6\,i\,Z\Delta^2\left(\lambda + n\,\lambda + (-1+n)\,\delta\lambda_{2a} + 2\,\delta\lambda_{2b}\right)\right)\Big/\left(-6\,i + 2\,I\mu\,Z\Delta^2\,\lambda\,\hbar + I\mu\,n\,Z\Delta^2\,\lambda\,\hbar + I\mu\,n\,Z\Delta^2\,\hbar\,\delta\lambda_{2a} + 2\,I\mu\,Z\Delta^2\,\hbar\,\delta\lambda_{2b}\right)\right) = 0,$$

$$\begin{aligned}-\Big(\Big(i\,\Big(&-18\,\lambda + 18\,Z\Delta\,\lambda - 12\,i\,\text{Ifing0}\,Z\Delta^3\,\delta\lambda^2\,\hbar - 12\,i\,I\mu\,Z\Delta^3\,\delta\lambda^2\,\hbar - 24\,i\,\text{Ifing0}\,Z\Delta^3\,\delta\lambda\,\lambda\,\hbar -\\ &30\,i\,I\mu\,Z\Delta^3\,\delta\lambda\,\lambda\,\hbar - 12\,i\,\text{Ifing0}\,\lambda^2\,\hbar - 6\,i\,I\mu\,Z\Delta^2\,\lambda^2\,\hbar - 3\,i\,I\mu\,n\,Z\Delta^2\,\lambda^2\,\hbar -\\ &12\,i\,\text{Ifing0}\,Z\Delta^3\,\lambda^2\,\hbar - 18\,i\,I\mu\,Z\Delta^3\,\lambda^2\,\hbar + 4\,c\mu\,Z\Delta^2\,\delta\lambda^2\,\lambda\,\hbar^2 + 2\,c\mu\,n\,Z\Delta^2\,\delta\lambda^2\,\lambda\,\hbar^2 +\\ &8\,c\mu\,Z\Delta^2\,\delta\lambda\,\lambda^2\,\hbar^2 + 4\,c\mu\,n\,Z\Delta^2\,\delta\lambda\,\lambda^2\,\hbar^2 + 4\,c\mu\,Z\Delta^2\,\lambda^3\,\hbar^2 + 2\,\text{Ifing0}\,I\mu\,Z\Delta^2\,\lambda^3\,\hbar^2 -\\ &2\,\text{Ifingn}\,I\mu\,Z\Delta^2\,\lambda^3\,\hbar^2 + 2\,c\mu\,n\,Z\Delta^2\,\lambda^3\,\hbar^2 + 18\,Z\Delta\,\delta\lambda_a + Z\Delta^2\,\hbar\,\Big(2\,c\mu\,n\,(\delta\lambda+\lambda)^2\,\hbar +\\ &\quad I\mu\left(-6\,i\,Z\Delta\,(\delta\lambda+\lambda) + \lambda\left(-3\,i\,n + 2\,(\text{Ifing0} - \text{Ifingn})\,\lambda\,\hbar\right)\right)\Big)\,\delta\lambda_{2a} -\\ &6\,i\,I\mu\,Z\Delta^2\,\lambda\,\hbar\,\delta\lambda_{2b} + 4\,c\mu\,Z\Delta^2\,\delta\lambda^2\,\hbar^2\,\delta\lambda_{2b} + 8\,c\mu\,Z\Delta^2\,\delta\lambda\,\lambda\,\hbar^2\,\delta\lambda_{2b} + 4\,c\mu\,Z\Delta^2\,\lambda^2\,\hbar^2\,\delta\lambda_{2b}\Big)\Big)\Big/\\ &\Big(18\left(-6\,i + 2\,I\mu\,Z\Delta^2\,\lambda\,\hbar + I\mu\,n\,Z\Delta^2\,\lambda\,\hbar + I\mu\,n\,Z\Delta^2\,\hbar\,\delta\lambda_{2a} + 2\,I\mu\,Z\Delta^2\,\hbar\,\delta\lambda_{2b}\right)\Big)\Big) = 0,\end{aligned}$$

$$-\left(\left(2\left(-\lambda^2 + Z\Delta^3\,(\delta\lambda+\lambda)^2\right)\hbar\right)\Big/\left(3\left(-6\,i + 2\,I\mu\,Z\Delta^2\,\lambda\,\hbar + I\mu\,n\,Z\Delta^2\,\lambda\,\hbar + I\mu\,n\,Z\Delta^2\,\hbar\,\delta\lambda_{2a} + 2\,I\mu\,Z\Delta^2\,\hbar\,\delta\lambda_{2b}\right)\right)\right) = 0,$$

$$\left(6\,i\,(-1 + Z\,Z\Delta)\right)\Big/\left(-6\,i + 2\,I\mu\,Z\Delta^2\,\lambda\,\hbar + I\mu\,n\,Z\Delta^2\,\lambda\,\hbar + I\mu\,n\,Z\Delta^2\,\hbar\,\delta\lambda_{2a} + 2\,I\mu\,Z\Delta^2\,\hbar\,\delta\lambda_{2b}\right) = 0\Big\}$$

Solve for counter-terms from Goldstone EOM

Note there are two solutions differing by a sign for $\delta\lambda$.

```
cts =
  Solve[cteq, {δm₁, δλ1a, δλ2a, δλ2b, δλ, Z, ZΔ}] // FullSimplify // DeleteDuplicates;
```

Solve::svars : Equations may not give solutions for all "solve" variables. >>

```
{δm1^2, δλ1a, δλ2a, δλ2b, δλ, Z, ZΔ} /. cts // DeleteDuplicates
```

$$\left\{\left\{-\frac{(2+n)\,\lambda\,\left(i\,t\mu+I\mu\,(m-\mu)\,(m+\mu)\right)\,\hbar}{6\,i+I\mu\,(2+n)\,\lambda\,\hbar},\right.\right.$$
$$\frac{1}{3\,Z\Delta^4\,\left(6\,i+I\mu\,(2+n)\,\lambda\,\hbar\right)}\lambda\,\left(6\,I\mu\,Z\Delta^{5/2}\,\lambda\,\hbar-2\,i\,c\mu\,(2+n)\,\lambda^2\,\hbar^2-3\,Z\Delta^4\,\left(6\,i+I\mu\,(2+n)\,\lambda\,\hbar\right)+\right.$$
$$\left.2\,Z\Delta^3\,\left(9\,i+\lambda\,\hbar\,\left(-6\,(2\,\mathrm{Ifing0}+I\mu)+i\,I\mu\,(\mathrm{Ifingn}+I\mu\,(2+n)+\mathrm{Ifing0}\,(3+2\,n))\,\lambda\,\hbar\right)\right)\right),$$
$$\lambda\,\left(-1+\frac{6\,i}{Z\Delta^2\,\left(6\,i+I\mu\,(2+n)\,\lambda\,\hbar\right)}\right),\ \lambda\,\left(-1+\frac{6\,i}{Z\Delta^2\,\left(6\,i+I\mu\,(2+n)\,\lambda\,\hbar\right)}\right),$$
$$\left.\left(-1-\frac{1}{Z\Delta^{3/2}}\right)\lambda,\ \frac{1}{Z\Delta},\ Z\Delta\right\},\ \left\{-\frac{(2+n)\,\lambda\,\left(i\,t\mu+I\mu\,(m-\mu)\,(m+\mu)\right)\,\hbar}{6\,i+I\mu\,(2+n)\,\lambda\,\hbar},\right.$$
$$\frac{1}{3\,Z\Delta^4\,\left(6\,i+I\mu\,(2+n)\,\lambda\,\hbar\right)}\lambda\,\left(-6\,I\mu\,Z\Delta^{5/2}\,\lambda\,\hbar-2\,i\,c\mu\,(2+n)\,\lambda^2\,\hbar^2-3\,Z\Delta^4\,\left(6\,i+I\mu\,(2+n)\,\lambda\,\hbar\right)+\right.$$
$$\left.2\,Z\Delta^3\,\left(9\,i+\lambda\,\hbar\,\left(-6\,(2\,\mathrm{Ifing0}+I\mu)+i\,I\mu\,(\mathrm{Ifingn}+I\mu\,(2+n)+\mathrm{Ifing0}\,(3+2\,n))\,\lambda\,\hbar\right)\right)\right),$$
$$\lambda\,\left(-1+\frac{6\,i}{Z\Delta^2\,\left(6\,i+I\mu\,(2+n)\,\lambda\,\hbar\right)}\right),\ \lambda\,\left(-1+\frac{6\,i}{Z\Delta^2\,\left(6\,i+I\mu\,(2+n)\,\lambda\,\hbar\right)}\right),$$
$$\left.\left.\left(-1+\frac{1}{Z\Delta^{3/2}}\right)\lambda,\ \frac{1}{Z\Delta},\ Z\Delta\right\}\right\}$$

Gather kinematically distinct divergences for Higgs EOM

```
cteq2 =
 (((((mn2 - (λ v^2/3) - m^2 - λ/6 v^2 - ħ/6 ((n + 1) λ) (tfing) - ħ/6 (λ) (tfinn) /. mg2soln) /. Solve[
          neom, mn2][[1]] /. mg2soln) /. cts // FullSimplify //
       DeleteDuplicates) /. {tfing → 0, tfinn → 0} // Expand) //
    CoefficientList[#, {p, v, tfing, tfinn, Ifingp}] & // Flatten //
   Simplify // DeleteDuplicates) == 0 // Thread
```

$$\left\{\mathrm{True},\ \frac{i\,\lambda\,\left(3\,i+\sqrt{Z\Delta}\,\left(3\,i+\mathrm{Ifing0}\,\lambda\,\hbar-\mathrm{Ifingn}\,\lambda\,\hbar\right)\right)}{9\,\sqrt{Z\Delta}}==0,\right.$$
$$\left.\frac{\lambda\,\left(3+i\,\sqrt{Z\Delta}\,\left(3\,i+\mathrm{Ifing0}\,\lambda\,\hbar-\mathrm{Ifingn}\,\lambda\,\hbar\right)\right)}{9\,\sqrt{Z\Delta}}==0\right\}$$

Solve for counter-terms from Higgs EOM

```
cts2 = Solve[cteq2[[2]], {ZΔ}]
```

$$\left\{\left\{Z\Delta\to-\frac{9}{\left(3\,i+\mathrm{Ifing0}\,\lambda\,\hbar-\mathrm{Ifingn}\,\lambda\,\hbar\right)^2}\right\}\right\}$$

Both equations should have the same solution:

```
(ZΔ /. Solve[cteq2[[3]], {ZΔ}][[1]]) - (ZΔ /. cts2[[1]]) == 0
```

```
True
```

Final Counterterms

```
({δm1^2, δλ1a, δλ2a, δλ2b, δλ, Z, ZΔ} /. cts /. cts2 // Simplify)[[1]] //
  DeleteDuplicates;
```

```
counterterms = Thread[{δm1^2, δλ1a, δλ2a, δλ2b, δλ, Z, ZΔ} → %[[1]]]
```

$$\Big\{\delta m_1^2 \to -\frac{(2+n)\,\lambda\,\left(\mathrm{i}\,\mathrm{t}\mu + \mathrm{I}\mu\,(m-\mu)\,(m+\mu)\right)\hbar}{6\,\mathrm{i} + \mathrm{I}\mu\,(2+n)\,\lambda\,\hbar},$$

$$\delta\lambda_a \to \frac{1}{19\,683\,\left(6\,\mathrm{i} + \mathrm{I}\mu\,(2+n)\,\lambda\,\hbar\right)}\,\lambda\,\left(3\,\mathrm{i} + \mathrm{Ifing0}\,\lambda\,\hbar - \mathrm{Ifingn}\,\lambda\,\hbar\right)^8$$
$$\left(-2\,\mathrm{i}\,\mathrm{c}\mu\,(2+n)\,\lambda^2\,\hbar^2 + 1458\,\mathrm{I}\mu\,\lambda\,\hbar\,\left(-\frac{1}{\left(3\,\mathrm{i} + \mathrm{Ifing0}\,\lambda\,\hbar - \mathrm{Ifingn}\,\lambda\,\hbar\right)^2}\right)^{5/2} - \right.$$
$$\frac{19\,683\,\left(6\,\mathrm{i} + \mathrm{I}\mu\,(2+n)\,\lambda\,\hbar\right)}{\left(3\,\mathrm{i} + \mathrm{Ifing0}\,\lambda\,\hbar - \mathrm{Ifingn}\,\lambda\,\hbar\right)^8} - \Big(1458\,\Big(9\,\mathrm{i} + \lambda\,\hbar\,\Big(-6\,(2\,\mathrm{Ifing0} + \mathrm{I}\mu) + \mathrm{i}\,\mathrm{I}\mu\,(\mathrm{Ifingn}\,+$$
$$\left.\mathrm{I}\mu\,(2+n) + \mathrm{Ifing0}\,(3+2\,n))\,\lambda\,\hbar\Big)\Big)\Big) \Big/ \left(3\,\mathrm{i} + \mathrm{Ifing0}\,\lambda\,\hbar - \mathrm{Ifingn}\,\lambda\,\hbar\right)^6\right),$$

$$\delta\lambda_{2a} \to \lambda\,\left(-1 + \frac{2\,\mathrm{i}\,\left(3\,\mathrm{i} + \mathrm{Ifing0}\,\lambda\,\hbar - \mathrm{Ifingn}\,\lambda\,\hbar\right)^4}{27\,\left(6\,\mathrm{i} + \mathrm{I}\mu\,(2+n)\,\lambda\,\hbar\right)}\right),\ \delta\lambda_{2b} \to$$
$$\lambda\,\left(-1 + \frac{2\,\mathrm{i}\,\left(3\,\mathrm{i} + \mathrm{Ifing0}\,\lambda\,\hbar - \mathrm{Ifingn}\,\lambda\,\hbar\right)^4}{27\,\left(6\,\mathrm{i} + \mathrm{I}\mu\,(2+n)\,\lambda\,\hbar\right)}\right),$$

$$\delta\lambda \to \lambda\,\left(-1 - \frac{1}{27\,\left(-\frac{1}{(3\,\mathrm{i}+\mathrm{Ifing0}\,\lambda\,\hbar-\mathrm{Ifingn}\,\lambda\,\hbar)^2}\right)^{3/2}}\right),$$

$$Z \to -\frac{1}{9}\,\left(3\,\mathrm{i} + \mathrm{Ifing0}\,\lambda\,\hbar - \mathrm{Ifingn}\,\lambda\,\hbar\right)^2,$$

$$Z\Delta \to -\frac{9}{\left(3\,\mathrm{i} + \mathrm{Ifing0}\,\lambda\,\hbar - \mathrm{Ifingn}\,\lambda\,\hbar\right)^2}\Big\}$$

The should be momentum independent :

```
({δm1^2, δλ1a, δλ2a, δλ2b, δλ, Z, ZΔ} /. counterterms // DeleteDuplicates // D[#, p] &)[[
   1]] == 0 // Thread
({δm1^2, δλ1a, δλ2a, δλ2b, δλ, Z, ZΔ} /. counterterms // DeleteDuplicates //
     D[#, Ifingp] &)[[1]] == 0 // Thread
```

```
True
```

```
True
```