Renormalization of Symmetry Improved 3PIEA gap equations at 2 loops

Supplement to "Symmetry improvement of 3PI effective actions for O(N) scalar field theory" by Michael J. Brown and Ian B. Whittingham.

Mathematica notebook to compute couter-terms for two loop truncations of the effective action as described in Section IV of the paper.

Hartree-Fock

```
ClearAll[geom, neom, intrules, msbarrules, mg2soln, cteq, cts, δm, δλ];
```

Hartree-Fock gap equations with counterterms

Goldstone equation of motion. Quantities in reference to the paper are:
p is the four-momentum flowing through the propagators $\Delta_G{}^{-1}$ and $\Delta_N{}^{-1}$,
mg2 is the Goldstone mass squared $m_G{}^2$,
Z and ZΔ are the wavefunction a propagator renormalization constants,
m^2 is the (renormalized) Lagrangian mass parameter, $\delta m_1{}^2$ is its counter-term,
λ is the (renormalized) four point coupling,
$\delta\lambda_{1\,a}$, $\delta\lambda_{2\,a}$, $\delta\lambda_{2\,b}$ are the independent coupling counter-terms,
v is the scalar field vacuum expectation value,
$\hbar$ is the reduced Planck constant,
n is the number of fields in the O(n) symmetry group,
t∞g, t∞n are the divergent tadpole integrals for the Goldstone, Higgs resp.,
tfing, tfinn are the finite parts of the tadpoles for the Goldstone, Higgs resp.

$$\mathrm{geom} = p^2 - \mathrm{mg2} == Z\, Z\Delta\, p^2 - m^2 - \delta m_1{}^2 - Z\Delta\, \frac{\lambda + \delta\lambda_{1\,a}}{6}\, v^2 - \frac{\hbar}{6}\left(\left(n+1\right)\lambda + \left(n-1\right)\delta\lambda_{2\,a} + 2\,\delta\lambda_{2\,b}\right) Z\Delta^2\, (\mathrm{t\infty g} + \mathrm{tfing}) - \frac{\hbar}{6}\,(\lambda + \delta\lambda_{2\,a})\, Z\Delta^2\, (\mathrm{t\infty n} + \mathrm{tfinn})$$

Higgs equation of motion

$$\mathrm{neom} = p^2 - \mathrm{mn2} == \frac{-\lambda\, v^2}{3}\, Z\Delta + p^2 - \mathrm{mg2}$$

Infinite parts of tadpoles in MSbar

MSbar rules for 4 - 2 ϵ dimensions

$$\text{msbarrules} = \left\{t\infty g \to \frac{-\,\text{mg2}}{16\,\pi^2}\left(\frac{1}{\epsilon} - \text{EulerGamma} + 1 + \text{Log}[4\,\pi]\right),\ t\infty n \to \frac{-\,\text{mn2}}{16\,\pi^2}\left(\frac{1}{\epsilon} - \text{EulerGamma} + 1 + \text{Log}[4\,\pi]\right)\right\}$$

Sub in tadpole expressions, eliminate mn2 and solve for mg2

```
mg2soln = mg2 /. (geom /. msbarrules /. Solve[neom, mn2][[1]] // Solve[#, mg2][[1]] &)
```

Gather divergences proportional v, tfing and tfinn and set independently to zero

First we subtract the finite equation of motion, then gather coefficients of the remainder into a list and set each to zero (after some trimming and simplifying).

$$\text{cteq} = \left(\left(\text{CoefficientList}\left[\text{mg2soln} + \left(-m^2 - \frac{\lambda}{6}v^2 - \frac{\hbar}{6}\left(\left(n+1\right)\lambda\right)(\text{tfing}) - \frac{\hbar}{6}(\lambda)(\text{tfinn})\right), \{p, v, \text{tfing}, \text{tfinn}\}\right] // \text{Flatten}\right) // \text{DeleteDuplicates} // \text{Simplify} // \text{FullSimplify}\right) == 0 // \text{Thread}$$

Solve for counterterms

$$\text{cts} = \{\delta m_1{}^2, \delta\lambda_{1\,a}, \delta\lambda_{2\,a}, \delta\lambda_{2\,b}, Z, Z\Delta\} /. \text{Solve}[\text{cteq}, \{\delta m_1, \delta\lambda_{1\,a}, \delta\lambda_{2\,a}, \delta\lambda_{2\,b}, Z, Z_\Delta\}] // \text{FullSimplify} // \text{DeleteDuplicates}$$

ZΔ is redundant in this truncation, can remove it :

```
cts /. ZΔ → 1 // FullSimplify
```

```
Series[cts, {ϵ, 0, 1}] /. {ZΔ → 1}
```

Sunset

NOTE: this uses some of the same variable names as the Hartree-Fock code! Be careful not to clobber what you need to keep.

```
ClearAll[geom, neom, intrules, msbarrules, mg2soln, cteq, δm, δλ, δλ, δλ, δλ];
```

Equations of motion

Additional variables relative to the Hartree-Fock case:
Ing is the sunset integral $I_{\text{NG}}(p)$
Ifingp is the finite sunset integral $I^{\text{fin}}_{\text{NG}}(p)$,

Ifing0 is $I_{\mathrm{NG}}^{\mathrm{fin}}(m_G)$,
Ifingn is $I_{\mathrm{NG}}^{\mathrm{fin}}(m_N)$,
$\delta\lambda$ is the sunset graph coupling counter-term,
Iμ, tμ and cμ are the auxiliary integrals I_μ, T_μ and c_μ respectively.

$$\texttt{geom} = p^2 - \texttt{mg2} + i\,\hbar\left(\frac{(\lambda)\,v}{3}\right)^2(\texttt{Ifingp} - \texttt{Ifing0}) == \texttt{Z}\,\texttt{Z}\Delta\,p^2 - m^2 - \delta m_1{}^2 - \texttt{Z}\Delta\,\frac{\lambda + \delta\lambda_{1\,a}}{6}\,v^2 - \frac{\hbar}{6}\left((n+1)\,\lambda + (n-1)\,\delta\lambda_{2\,a} + 2\,\delta\lambda_{2\,b}\right)\texttt{Z}\Delta^2\,(\texttt{tg}) - \frac{\hbar}{6}\,(\lambda + \delta\lambda_{2\,a})\,\texttt{Z}\Delta^2\,(\texttt{tn}) + i\,\hbar\left(\frac{(\lambda+\delta\lambda)\,v}{3}\right)^2\texttt{Z}\Delta^3\,\texttt{Ing}$$

$$\texttt{neom} = p^2 - \texttt{mn2} + i\,\hbar\left(\frac{(\lambda)\,v}{3}\right)^2(\texttt{Ifingp} - \texttt{Ifingn}) == \frac{-\texttt{Z}\Delta\,(\lambda+\delta\lambda)\,v^2}{3} + p^2 - \texttt{mg2} + i\,\hbar\left(\frac{(\lambda)\,v}{3}\right)^2(\texttt{Ifingp} - \texttt{Ifing0})$$

Divergent parts subtracted with auxiliary integrals and MSbar

$$\texttt{intrules} = \Big\{\texttt{Ing} \to \texttt{I}\mu + \texttt{Ifingp} + \texttt{Ifing0},\ \texttt{tg} \to \texttt{t}\mu - i\,(\texttt{mg2} - \mu^2)\,\texttt{I}\mu + \hbar\left(\frac{(\lambda+\delta\lambda)\,v}{3}\right)^2\texttt{c}\mu + \texttt{tfing},\ \texttt{tn} \to \texttt{t}\mu - i\,(\texttt{mn2} - \mu^2)\,\texttt{I}\mu + \hbar\left(\frac{(\lambda+\delta\lambda)\,v}{3}\right)^2\texttt{c}\mu + \texttt{tfinn}\Big\}$$

$$\texttt{msbarrules} = \Big\{\texttt{I}\mu \to \frac{-i}{16\,\pi^2}\left(\frac{1}{\epsilon} - \texttt{EulerGamma} + \texttt{Log}[4\,\pi]\right),\ \texttt{t}\mu \to \frac{-\mu^2}{16\,\pi^2}\left(\frac{1}{\epsilon} - \texttt{EulerGamma} + 1 + \texttt{Log}[4\,\pi]\right),\ \texttt{c}\mu \to \left(\frac{\texttt{a0}}{\epsilon^2} + \frac{\texttt{a1}}{\epsilon} + \texttt{a2}\right)\Big\}$$

Sub everything in, eliminate mn2 and solve for mg2

```
mg2soln =
  ((geom /. intrules(*/.msbarrules*) /. Solve[neom, mn2][[1]]) // Solve[#, mg2] &)[[
   1]]
```

Gather kinematically distinct divergences for Goldstone EOM

$$\texttt{cteq} = \left(\left(\texttt{mg2} - m^2 - \frac{\lambda}{6}\,v^2 - \frac{\hbar}{6}\left((n+1)\,\lambda\right)(\texttt{tfing}) - \frac{\hbar}{6}\,(\lambda)\,(\texttt{tfinn})\ /.\ \texttt{mg2soln}\right)\ //\ \texttt{CoefficientList}[\#, \{p, v, \texttt{tfing}, \texttt{tfinn}, \texttt{Ifingp}\}]\ \&\ //\ \texttt{Flatten}\ //\ \texttt{Simplify}\ //\ \texttt{DeleteDuplicates}\right) == 0\ //\ \texttt{Thread}$$

Solve for counter-terms from Goldstone EOM

Note there are two solutions differing by a sign for $\delta\lambda$.

```
cts =
  Solve[cteq, {δm1, δλ1a, δλ2a, δλ2b, δλ, Z, ZΔ}] // FullSimplify // DeleteDuplicates;
{δm1^2, δλ1a, δλ2a, δλ2b, δλ, Z, ZΔ} /. cts // DeleteDuplicates
```

Gather kinematically distinct divergences for Higgs EOM

```
cteq2 =
 (((((mn2 - (λ v^2/3) - m^2 - λ/6 v^2 - ℏ/6 ((n + 1) λ) (tfing) - ℏ/6 (λ) (tfinn) /. mg2soln) /. Solve[
             neom, mn2][[1]] /. mg2soln) /. cts // FullSimplify //
         DeleteDuplicates) /. {tfing → 0, tfinn → 0} // Expand) //
      CoefficientList[#, {p, v, tfing, tfinn, Ifingp}] & // Flatten //
     Simplify // DeleteDuplicates) == 0 // Thread
```

Solve for counter-terms from Higgs EOM

```
cts2 = Solve[cteq2[[2]], {ZΔ}]
```

Both equations should have the same solution:

```
(ZΔ /. Solve[cteq2[[3]], {ZΔ}][[1]]) - (ZΔ /. cts2[[1]]) == 0
```

Final Counterterms

```
({δm1^2, δλ1a, δλ2a, δλ2b, δλ, Z, ZΔ} /. cts /. cts2 // Simplify)[[1]] //
  DeleteDuplicates;
counterterms = Thread[{δm1^2, δλ1a, δλ2a, δλ2b, δλ, Z, ZΔ} → %[[1]]]
```

The should be momentum independent :

```
({δm1^2, δλ1a, δλ2a, δλ2b, δλ, Z, ZΔ} /. counterterms // DeleteDuplicates // D[#, p] &)[[
   1]] == 0 // Thread
({δm1^2, δλ1a, δλ2a, δλ2b, δλ, Z, ZΔ} /. counterterms // DeleteDuplicates //
     D[#, Ifingp] &)[[1]] == 0 // Thread
```