Renormalization of Symmetry Improved 3PIEA gap equations at 2 loops

Supplement to thesis Chapter 4.

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Mathematica notebook to compute couter-terms for two loop truncations of the SI-3PIEA.

Hartree-Fock

```
ClearAll[geom, neom, intrules, regularisedtadpoles, mg2soln, cteq, cts, δm, δλ];
```

Hartree-Fock gap equations with counterterms

Goldstone equation of motion. Quantities in reference to the paper are:
p is the four-momentum flowing through the propagators $\Delta_G{}^{-1}$ and $\Delta_N{}^{-1}$,
mg2 is the Goldstone mass squared $m_G{}^2$,
Z and ZΔ are the wavefunction a propagator renormalization constants,
m^2 is the (renormalized) Lagrangian mass parameter, $\delta m_1{}^2$ is its counter-term,
λ is the (renormalized) four point coupling,
$\delta\lambda_{1a}$, $\delta\lambda_{2a}$, $\delta\lambda_{2b}$ are the independent coupling counter-terms,
v is the scalar field vacuum expectation value,
$\hbar$ is the reduced Planck constant,
n is the number of fields in the O(n) symmetry group,
t∞g, t∞n are the divergent tadpole integrals for the Goldstone, Higgs resp.,
tfing, tfinn are the finite parts of the tadpoles for the Goldstone, Higgs resp.

$$\mathbf{geom} = p^2 - mg2 == Z\,Z\Delta\,p^2 - m^2 - \delta m_1{}^2 - Z\Delta\,\frac{\lambda + \delta\lambda_{1a}}{6}\,v^2 - \frac{\hbar}{6}\left(\left(n+1\right)\lambda + \left(n-1\right)\delta\lambda_{2a} + 2\,\delta\lambda_{2b}\right) Z\Delta^2\,(t\infty g + tfing) - \frac{\hbar}{6}\,(\lambda + \delta\lambda_{2a})\,Z\Delta^2\,(t\infty n + tfinn)$$

$$-mg2 + p^2 == -m^2 + p^2\,Z\,Z\Delta - \delta m_1^2 - \frac{1}{6}\,v^2\,Z\Delta\,(\lambda + \delta\lambda_a) - \frac{1}{6}\,(tfinn + t\infty n)\,Z\Delta^2\,\hbar\,(\lambda + \delta\lambda_{2a}) - \frac{1}{6}\,(tfing + t\infty g)\,Z\Delta^2\,\hbar\,((1+n)\,\lambda + (-1+n)\,\delta\lambda_{2a} + 2\,\delta\lambda_{2b})$$

Higgs equation of motion

```
neom = p^2 - mn2 == (-λ v^2)/3 ZΔ + p^2 - mg2
```

$$-\text{mn2} + p^2 == -\text{mg2} + p^2 - \frac{1}{3} v^2\, Z\Delta\, \lambda$$

Infinite parts of tadpoles in MSbar

MSbar rules for 4 - 2 ϵ dimensions

```
regularisedtadpoles = {t∞g → c0 Λ^2 + c1 mg2 Log[Λ^2 / μ^2], t∞n → c0 Λ^2 + c1 mn2 Log[Λ^2 / μ^2]}
```

$$\left\{\text{t}\infty\text{g} \to \text{c0}\, \Lambda^2 + \text{c1 mg2 Log}\left[\frac{\Lambda^2}{\mu^2}\right],\ \text{t}\infty\text{n} \to \text{c0}\, \Lambda^2 + \text{c1 mn2 Log}\left[\frac{\Lambda^2}{\mu^2}\right]\right\}$$

Sub in tadpole expressions, eliminate mn2 and solve for mg2

```
mg2soln =
 mg2 /. (geom /. regularisedtadpoles /. Solve[neom, mn2][[1]] // Solve[#, mg2][[1]] &)
```

$$\Big(-18\, m^2 - 18\, p^2 + 18\, p^2\, Z\, Z\Delta - 3\, v^2\, Z\Delta\, \lambda - 3\, \text{tfing}\, Z\Delta^2\, \lambda\, \hbar - 3\, n\, \text{tfing}\, Z\Delta^2\, \lambda\, \hbar - 3\, \text{tfinn}\, Z\Delta^2\, \lambda\, \hbar - 6\, \text{c0}\, Z\Delta^2\, \lambda\, \Lambda^2\, \hbar - 3\, \text{c0}\, n\, Z\Delta^2\, \lambda\, \Lambda^2\, \hbar - \text{c1}\, v^2\, Z\Delta^3\, \lambda^2\, \hbar\, \text{Log}\left[\frac{\Lambda^2}{\mu^2}\right] - 18\, \delta m_1^2 - 3\, v^2\, Z\Delta\, \delta\lambda_a + 3\, \text{tfing}\, Z\Delta^2\, \hbar\, \delta\lambda_{2\,a} - 3\, n\, \text{tfing}\, Z\Delta^2\, \hbar\, \delta\lambda_{2\,a} - 3\, \text{tfinn}\, Z\Delta^2\, \hbar\, \delta\lambda_{2\,a} - 3\, \text{c0}\, n\, Z\Delta^2\, \Lambda^2\, \hbar\, \delta\lambda_{2\,a} - \text{c1}\, v^2\, Z\Delta^3\, \lambda\, \hbar\, \text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\delta\lambda_{2\,a} - 6\, \text{tfing}\, Z\Delta^2\, \hbar\, \delta\lambda_{2\,b} - 6\, \text{c0}\, Z\Delta^2\, \Lambda^2\, \hbar\, \delta\lambda_{2\,b}\Big) \Big/ \left(3\left(-6 + 2\, \text{c1}\, Z\Delta^2\, \lambda\, \hbar\, \text{Log}\left[\frac{\Lambda^2}{\mu^2}\right] + \text{c1}\, n\, Z\Delta^2\, \lambda\, \hbar\, \text{Log}\left[\frac{\Lambda^2}{\mu^2}\right] + \text{c1}\, n\, Z\Delta^2\, \hbar\, \text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\delta\lambda_{2\,a} + 2\, \text{c1}\, Z\Delta^2\, \hbar\, \text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\delta\lambda_{2\,b}\right)\right)$$

Gather divergences proportional v, tfing and tfinn and set independently to zero

First we subtract the finite equation of motion, then gather coefficients of the remainder into a list and set each to zero (after some trimming and simplifying).

```
cteq =
 ((CoefficientList[mg2soln + (-m² - λ/6 v² - ℏ/6 ((n + 1) λ) (tfing) - ℏ/6 (λ) (tfinn)), {p, v,
        tfing, tfinn}] // Flatten) //
     DeleteDuplicates // Simplify // FullSimplify) == 0 // Thread
```

$$\Big\{-\Big(\Big(6\,\delta m_1^2 + Z\Delta^2\,\hbar\left(c0\,\Lambda^2 + c1\,m^2\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\right)\left((2+n)\,\lambda + n\,\delta\lambda_{2\,a} + 2\,\delta\lambda_{2\,b}\right)\Big)\Big/$$
$$\left(-6 + c1\,(2+n)\,Z\Delta^2\,\lambda\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right] + c1\,Z\Delta^2\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\left(n\,\delta\lambda_{2\,a} + 2\,\delta\lambda_{2\,b}\right)\right)\Big) == 0,\ -\frac{\lambda\,\hbar}{6} -$$
$$\left(Z\Delta^2\,\hbar\,(\lambda + \delta\lambda_{2\,a})\right)\Big/\left(-6 + c1\,(2+n)\,Z\Delta^2\,\lambda\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right] + c1\,Z\Delta^2\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\left(n\,\delta\lambda_{2\,a} + 2\,\delta\lambda_{2\,b}\right)\right) ==$$
$$0,\ \Big(\hbar\,\Big((1+n)\,\lambda\left(6 - 6\,Z\Delta^2 - c1\,(2+n)\,Z\Delta^2\,\lambda\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\right) +$$
$$Z\Delta^2\Big(-\left(6\,(-1+n) + c1\,n\,(1+n)\,\lambda\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\right)\delta\lambda_{2\,a} -$$
$$2\left(6 + c1\,(1+n)\,\lambda\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\right)\delta\lambda_{2\,b}\Big)\Big)\Big)\Big/$$
$$\left(6\left(-6 + c1\,(2+n)\,Z\Delta^2\,\lambda\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right] + c1\,Z\Delta^2\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\left(n\,\delta\lambda_{2\,a} + 2\,\delta\lambda_{2\,b}\right)\right)\right) == 0,$$
$$\mathrm{True},\ -\Big(\Big(6\,Z\Delta\,\delta\lambda_a + \lambda\left(6\,(-1+Z\Delta) + c1\,Z\Delta^2\,(2+n+2\,Z\Delta)\,\lambda\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right] +\right.$$
$$\left.c1\,Z\Delta^2\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\left((n + 2\,Z\Delta)\,\delta\lambda_{2\,a} + 2\,\delta\lambda_{2\,b}\right)\right)\Big)\Big/$$
$$\left(6\left(-6 + c1\,(2+n)\,Z\Delta^2\,\lambda\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right] + c1\,Z\Delta^2\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\left(n\,\delta\lambda_{2\,a} + 2\,\delta\lambda_{2\,b}\right)\right)\right)\Big) == 0,$$
$$(-6 + 6\,Z\,Z\Delta)\Big/\left(-6 + c1\,(2+n)\,Z\Delta^2\,\lambda\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right] + c1\,Z\Delta^2\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\left(n\,\delta\lambda_{2\,a} + 2\,\delta\lambda_{2\,b}\right)\right) ==$$
$$0\Big\}$$

Solve for counterterms

```
cts = {δm₁², δλ1 a, δλ2 a, δλ2 b, Z, ZΔ} /. Solve[cteq, {δm₁, δλ1 a, δλ2 a, δλ2 b, Z, ZΔ}] //
   FullSimplify // DeleteDuplicates
```

Solve::svars : Equations may not give solutions for all "solve" variables. ≫

$$\Big\{\Big\{-\frac{(2+n)\,\lambda\,\hbar\left(c0\,\Lambda^2 + c1\,m^2\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\right)}{6 + c1\,(2+n)\,\lambda\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]},\ \frac{\lambda\left(6 - 6\,Z\Delta - c1\,(4+n)\,Z\Delta\,\lambda\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\right)}{Z\Delta\left(6 + c1\,(2+n)\,\lambda\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\right)},$$
$$\lambda\left(-1 + \frac{6}{Z\Delta^2\left(6 + c1\,(2+n)\,\lambda\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\right)}\right),\ \lambda\left(-1 + \frac{6}{Z\Delta^2\left(6 + c1\,(2+n)\,\lambda\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\right)}\right),\ \frac{1}{Z\Delta},\ Z\Delta\Big\}\Big\}$$

ZΔ is redundant in this truncation, can remove it :

```
cts /. ZΔ → 1 // FullSimplify
```

$$\left\{\left\{-\frac{(2+n)\,\lambda\,\hbar\,\left(c0\,\Lambda^2+c1\,m^2\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\right)}{6+c1\,(2+n)\,\lambda\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]},\ -\frac{c1\,(4+n)\,\lambda^2\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]}{6+c1\,(2+n)\,\lambda\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]},\right.\right.$$
$$\left.\left.\lambda\left(-1+\frac{6}{6+c1\,(2+n)\,\lambda\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]}\right),\ \lambda\left(-1+\frac{6}{6+c1\,(2+n)\,\lambda\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]}\right),\ 1,\ 1\right\}\right\}$$

$$\delta\lambda_{2\,a} == \frac{n+2}{n+4}\,\delta\lambda_{1\,a}\ /.\ \mathrm{Solve}[\mathrm{cteq},\{\delta m_1,\delta\lambda_{1\,a},\delta\lambda_{2\,a},\delta\lambda_{2\,b},Z,Z_\Delta\}]\ /.\ \{Z\Delta\to 1\}\ //$$
$$\mathrm{FullSimplify}\ //\ \mathrm{DeleteDuplicates}$$

Solve::svars : Equations may not give solutions for all "solve" variables. >>

```
{True}
```

Sunset

NOTE: this uses some of the same variable names as the Hartree-Fock code! Be careful not to clobber what you need to keep.

```
ClearAll[geom, neom, intrules, regularisedtadpoles, mg2soln, cteq, δm, δλ, δλ, δλ, δλ];
```

Equations of motion

Additional variables relative to the Hartree-Fock case:
Ing is the sunset integral $I_{\mathrm{NG}}(p)$
Ifingp is the finite sunset integral $I_{\mathrm{NG}}^{\mathrm{fin}}(p)$,
Ifing0 is $I_{\mathrm{NG}}^{\mathrm{fin}}(m_G)$,
Ifingn is $I_{\mathrm{NG}}^{\mathrm{fin}}(m_N)$,
$\delta\lambda$ is the sunset graph coupling counter-term,
Iμ, tμ and cμ are the auxiliary integrals I_μ, T_μ and c_μ respectively.

$$\mathrm{geom} = p^2 - \mathrm{mg2} + i\,\hbar\left(\frac{(\lambda)\,v}{3}\right)^2(\mathrm{Ifingp}-\mathrm{Ifing0}) ==$$
$$Z\,Z\Delta\,p^2 - m^2 - \delta m_1{}^2 - Z\Delta\,\frac{\lambda+\delta\lambda_{1\,a}}{6}\,v^2 - \frac{\hbar}{6}\left((n+1)\,\lambda + (n-1)\,\delta\lambda_{2\,a} + 2\,\delta\lambda_{2\,b}\right)Z\Delta^2\,(\mathrm{tg}) -$$
$$\frac{\hbar}{6}\,(\lambda+\delta\lambda_{2\,a})\,Z\Delta^2\,(\mathrm{tn}) + i\,\hbar\left(\frac{(\lambda+\delta\lambda)\,v}{3}\right)^2 Z\Delta^3\,\mathrm{Ing}$$

$$-\mathrm{mg2}+p^2+\frac{1}{9}\,i\,(-\mathrm{Ifing0}+\mathrm{Ifingp})\,v^2\,\lambda^2\,\hbar == -m^2+p^2\,Z\,Z\Delta+\frac{1}{9}\,i\,\mathrm{Ing}\,v^2\,Z\Delta^3\,(\delta\lambda+\lambda)^2\,\hbar-\delta m_1^2-$$
$$\frac{1}{6}\,v^2\,Z\Delta\,(\lambda+\delta\lambda_a)-\frac{1}{6}\,\mathrm{tn}\,Z\Delta^2\,\hbar\,(\lambda+\delta\lambda_{2\,a})-\frac{1}{6}\,\mathrm{tg}\,Z\Delta^2\,\hbar\,((1+n)\,\lambda+(-1+n)\,\delta\lambda_{2\,a}+2\,\delta\lambda_{2\,b})$$

$$\text{neom} = p^2 - \text{mn2} + i\,\hbar\,\left(\frac{(\lambda)\,v}{3}\right)^2 (\text{Ifingp} - \text{Ifingn}) ==$$
$$\frac{-Z\Delta\,(\lambda + \delta\lambda)\,v^2}{3} + p^2 - \text{mg2} + i\,\hbar\,\left(\frac{(\lambda)\,v}{3}\right)^2 (\text{Ifingp} - \text{Ifing0})$$

$$-\text{mn2} + p^2 + \frac{1}{9}\,i\,(-\text{Ifingn} + \text{Ifingp})\,v^2\,\lambda^2\,\hbar ==$$
$$-\text{mg2} + p^2 - \frac{1}{3}\,v^2\,Z\Delta\,(\delta\lambda + \lambda) + \frac{1}{9}\,i\,(-\text{Ifing0} + \text{Ifingp})\,v^2\,\lambda^2\,\hbar$$

Divergent parts subtracted with auxiliary integrals and MSbar

$$\text{intrules} = \{\text{Ing} \to \text{I}\mu + \text{Ifingp} + \text{Ifing0},$$
$$\text{tg} \to \text{t}\mu - i\,(\text{mg2} - \mu^2)\,\text{I}\mu + \hbar\,\left(\frac{(\lambda + \delta\lambda)\,v}{3}\right)^2 \text{c}\mu + \text{tfing},$$
$$\text{tn} \to \text{t}\mu - i\,(\text{mn2} - \mu^2)\,\text{I}\mu + \hbar\,\left(\frac{(\lambda + \delta\lambda)\,v}{3}\right)^2 \text{c}\mu + \text{tfinn}\}$$

$$\{\text{Ing} \to \text{Ifing0} + \text{Ifingp} + \text{I}\mu,\ \text{tg} \to \text{tfing} + \text{t}\mu - i\,\text{I}\mu\,(\text{mg2} - \mu^2) + \frac{1}{9}\,\text{c}\mu\,v^2\,(\delta\lambda + \lambda)^2\,\hbar,$$
$$\text{tn} \to \text{tfinn} + \text{t}\mu - i\,\text{I}\mu\,(\text{mn2} - \mu^2) + \frac{1}{9}\,\text{c}\mu\,v^2\,(\delta\lambda + \lambda)^2\,\hbar\}$$

$$\text{regularisedtadpoles} =$$
$$\left\{\text{I}\mu \to \text{c2}\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right],\ \text{t}\mu \to \text{c0}\,\Lambda^2 + \text{c1}\,\mu^2\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right],\ \text{c}\mu \to \text{a0}\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]^2 + \text{a1}\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\right\}$$

$$\left\{\text{I}\mu \to \text{c2}\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right],\ \text{t}\mu \to \text{c0}\,\Lambda^2 + \text{c1}\,\mu^2\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right],\ \text{c}\mu \to \text{a1}\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right] + \text{a0}\,\text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]^2\right\}$$

Sub everything in, eliminate mn2 and solve for mg2

```
mg2soln = ((geom /. intrules(*/.regularisedtadpoles*) /. Solve[neom, mn2][[1]]) //
    Solve[#, mg2] &)[[1]]
```

$$\left\{\text{mg2} \to \left(-m^2 - p^2 + p^2\, Z\, Z\Delta - \frac{1}{9}\, i\, (-\text{Ifing0} + \text{Ifingp})\, v^2 \lambda^2 \hbar + \frac{1}{9}\, i\, (\text{Ifing0} + \text{Ifingp} + \text{I}\mu)\, v^2\, Z\Delta^3\, (\delta\lambda + \lambda)^2 \hbar - \delta m_1^2 - \frac{1}{6} v^2\, Z\Delta\, (\lambda + \delta\lambda_a) - \frac{1}{6}\, \text{tfinn}\, Z\Delta^2 \hbar\, (\lambda + \delta\lambda_{2\,a}) - \frac{1}{6}\, \text{t}\mu\, Z\Delta^2 \hbar\, (\lambda + \delta\lambda_{2\,a}) + \frac{1}{18}\, i\, \text{I}\mu\, v^2\, Z\Delta^3\, \delta\lambda\, \hbar\, (\lambda + \delta\lambda_{2\,a}) + \frac{1}{18}\, i\, \text{I}\mu\, v^2\, Z\Delta^3\, \lambda\, \hbar\, (\lambda + \delta\lambda_{2\,a}) - \frac{1}{6}\, i\, \text{I}\mu\, Z\Delta^2 \mu^2 \hbar\, (\lambda + \delta\lambda_{2\,a}) - \frac{1}{54}\, \text{c}\mu\, v^2\, Z\Delta^2\, (\delta\lambda + \lambda)^2 \hbar^2\, (\lambda + \delta\lambda_{2\,a}) - \frac{1}{54}\, \text{I}\mu\, Z\Delta^2 \hbar \left(\text{Ifing0}\, v^2 \lambda^2 \hbar - \text{Ifingn}\, v^2 \lambda^2 \hbar\right) (\lambda + \delta\lambda_{2\,a}) - \frac{1}{6}\, \text{tfing}\, Z\Delta^2 \hbar\, ((1+n)\,\lambda + (-1+n)\,\delta\lambda_{2\,a} + 2\,\delta\lambda_{2\,b}) - \frac{1}{6}\, \text{t}\mu\, Z\Delta^2 \hbar\, ((1+n)\,\lambda + (-1+n)\,\delta\lambda_{2\,a} + 2\,\delta\lambda_{2\,b}) - \frac{1}{6}\, i\, \text{I}\mu\, Z\Delta^2 \mu^2 \hbar\, ((1+n)\,\lambda + (-1+n)\,\delta\lambda_{2\,a} + 2\,\delta\lambda_{2\,b}) - \frac{1}{54}\, \text{c}\mu\, v^2\, Z\Delta^2\, (\delta\lambda + \lambda)^2 \hbar^2\, ((1+n)\,\lambda + (-1+n)\,\delta\lambda_{2\,a} + 2\,\delta\lambda_{2\,b})\right) \Big/ \left(-1 - \frac{1}{6}\, i\, \text{I}\mu\, Z\Delta^2 \hbar\, (\lambda + \delta\lambda_{2\,a}) - \frac{1}{6}\, i\, \text{I}\mu\, Z\Delta^2 \hbar\, ((1+n)\,\lambda + (-1+n)\,\delta\lambda_{2\,a} + 2\,\delta\lambda_{2\,b})\right)\right\}$$

Gather kinematically distinct divergences for Goldstone EOM

$$\text{cteq} = \left(\left(\text{mg2} - \text{m}^2 - \frac{\lambda}{6}\,\text{v}^2 - \frac{\hbar}{6}\left(\left(\text{n}+1\right)\lambda\right)(\text{tfing}) - \frac{\hbar}{6}(\lambda)(\text{tfinn}) \;/.\; \text{mg2soln}\right) \;//$$

```
      CoefficientList[#, {p, v, tfing, tfinn, Ifingp}] & // Flatten //
    Simplify // DeleteDuplicates) == 0 // Thread
```

$$\Big\{\left(-6\,\mathrm{i}\,\delta\text{m}_1^2 + \text{Z}\Delta^2\left(-\mathrm{i}\,\text{t}\mu + \text{I}\mu\left(-\text{m}^2+\mu^2\right)\right)\hbar\left((2+\text{n})\,\lambda + \text{n}\,\delta\lambda_{2\,\text{a}} + 2\,\delta\lambda_{2\,\text{b}}\right)\right) \Big/$$
$$\left(-6\,\mathrm{i} + 2\,\text{I}\mu\,\text{Z}\Delta^2\,\lambda\,\hbar + \text{I}\mu\,\text{n}\,\text{Z}\Delta^2\,\lambda\,\hbar + \text{I}\mu\,\text{n}\,\text{Z}\Delta^2\,\hbar\,\delta\lambda_{2\,\text{a}} + 2\,\text{I}\mu\,\text{Z}\Delta^2\,\hbar\,\delta\lambda_{2\,\text{b}}\right) = 0,$$
$$\text{True},\; \frac{1}{6}\,\hbar\left(-\lambda - \left(6\,\mathrm{i}\,\text{Z}\Delta^2\left(\lambda+\delta\lambda_{2\,\text{a}}\right)\right) \Big/\right.$$
$$\left.\left(-6\,\mathrm{i} + 2\,\text{I}\mu\,\text{Z}\Delta^2\,\lambda\,\hbar + \text{I}\mu\,\text{n}\,\text{Z}\Delta^2\,\lambda\,\hbar + \text{I}\mu\,\text{n}\,\text{Z}\Delta^2\,\hbar\,\delta\lambda_{2\,\text{a}} + 2\,\text{I}\mu\,\text{Z}\Delta^2\,\hbar\,\delta\lambda_{2\,\text{b}}\right)\right) = 0,$$
$$\frac{1}{6}\,\hbar\left(-(1+\text{n})\,\lambda - \left(6\,\mathrm{i}\,\text{Z}\Delta^2\left(\lambda + \text{n}\,\lambda + (-1+\text{n})\,\delta\lambda_{2\,\text{a}} + 2\,\delta\lambda_{2\,\text{b}}\right)\right) \Big/\right.$$
$$\left.\left(-6\,\mathrm{i} + 2\,\text{I}\mu\,\text{Z}\Delta^2\,\lambda\,\hbar + \text{I}\mu\,\text{n}\,\text{Z}\Delta^2\,\lambda\,\hbar + \text{I}\mu\,\text{n}\,\text{Z}\Delta^2\,\hbar\,\delta\lambda_{2\,\text{a}} + 2\,\text{I}\mu\,\text{Z}\Delta^2\,\hbar\,\delta\lambda_{2\,\text{b}}\right)\right) = 0,$$
$$-\Big(\Big(\mathrm{i}\,\Big(-18\,\lambda + 18\,\text{Z}\Delta\,\lambda - 12\,\mathrm{i}\,\text{Ifing0}\,\text{Z}\Delta^3\,\delta\lambda^2\,\hbar - 12\,\mathrm{i}\,\text{I}\mu\,\text{Z}\Delta^3\,\delta\lambda^2\,\hbar - 24\,\mathrm{i}\,\text{Ifing0}\,\text{Z}\Delta^3\,\delta\lambda\,\lambda\,\hbar -$$
$$30\,\mathrm{i}\,\text{I}\mu\,\text{Z}\Delta^3\,\delta\lambda\,\lambda\,\hbar - 12\,\mathrm{i}\,\text{Ifing0}\,\lambda^2\,\hbar - 6\,\mathrm{i}\,\text{I}\mu\,\text{Z}\Delta^2\,\lambda^2\,\hbar - 3\,\mathrm{i}\,\text{I}\mu\,\text{n}\,\text{Z}\Delta^2\,\lambda^2\,\hbar -$$
$$12\,\mathrm{i}\,\text{Ifing0}\,\text{Z}\Delta^3\,\lambda^2\,\hbar - 18\,\mathrm{i}\,\text{I}\mu\,\text{Z}\Delta^3\,\lambda^2\,\hbar + 4\,\text{c}\mu\,\text{Z}\Delta^2\,\delta\lambda^2\,\lambda\,\hbar^2 + 2\,\text{c}\mu\,\text{n}\,\text{Z}\Delta^2\,\delta\lambda^2\,\lambda\,\hbar^2 +$$
$$8\,\text{c}\mu\,\text{Z}\Delta^2\,\delta\lambda\,\lambda^2\,\hbar^2 + 4\,\text{c}\mu\,\text{n}\,\text{Z}\Delta^2\,\delta\lambda\,\lambda^2\,\hbar^2 + 4\,\text{c}\mu\,\text{Z}\Delta^2\,\lambda^3\,\hbar^2 + 2\,\text{Ifing0}\,\text{I}\mu\,\text{Z}\Delta^2\,\lambda^3\,\hbar^2 -$$
$$2\,\text{Ifingn}\,\text{I}\mu\,\text{Z}\Delta^2\,\lambda^3\,\hbar^2 + 2\,\text{c}\mu\,\text{n}\,\text{Z}\Delta^2\,\lambda^3\,\hbar^2 + 18\,\text{Z}\Delta\,\delta\lambda_{\text{a}} + \text{Z}\Delta^2\,\hbar\,\Big(2\,\text{c}\mu\,\text{n}\,(\delta\lambda+\lambda)^2\,\hbar +$$
$$\text{I}\mu\left(-6\,\mathrm{i}\,\text{Z}\Delta\,(\delta\lambda+\lambda) + \lambda\left(-3\,\mathrm{i}\,\text{n} + 2\,(\text{Ifing0}-\text{Ifingn})\,\lambda\,\hbar\right)\right)\Big)\,\delta\lambda_{2\,\text{a}} -$$
$$6\,\mathrm{i}\,\text{I}\mu\,\text{Z}\Delta^2\,\lambda\,\hbar\,\delta\lambda_{2\,\text{b}} + 4\,\text{c}\mu\,\text{Z}\Delta^2\,\delta\lambda^2\,\hbar^2\,\delta\lambda_{2\,\text{b}} + 8\,\text{c}\mu\,\text{Z}\Delta^2\,\delta\lambda\,\lambda\,\hbar^2\,\delta\lambda_{2\,\text{b}} + 4\,\text{c}\mu\,\text{Z}\Delta^2\,\lambda^2\,\hbar^2\,\delta\lambda_{2\,\text{b}}\Big)\Big) \Big/$$
$$\left(18\left(-6\,\mathrm{i} + 2\,\text{I}\mu\,\text{Z}\Delta^2\,\lambda\,\hbar + \text{I}\mu\,\text{n}\,\text{Z}\Delta^2\,\lambda\,\hbar + \text{I}\mu\,\text{n}\,\text{Z}\Delta^2\,\hbar\,\delta\lambda_{2\,\text{a}} + 2\,\text{I}\mu\,\text{Z}\Delta^2\,\hbar\,\delta\lambda_{2\,\text{b}}\right)\right)\Big) =$$
$$0,\; -\left(\left(2\left(-\lambda^2 + \text{Z}\Delta^3\,(\delta\lambda+\lambda)^2\right)\hbar\right) \Big/\right.$$
$$\left.\left(3\left(-6\,\mathrm{i} + 2\,\text{I}\mu\,\text{Z}\Delta^2\,\lambda\,\hbar + \text{I}\mu\,\text{n}\,\text{Z}\Delta^2\,\lambda\,\hbar + \text{I}\mu\,\text{n}\,\text{Z}\Delta^2\,\hbar\,\delta\lambda_{2\,\text{a}} + 2\,\text{I}\mu\,\text{Z}\Delta^2\,\hbar\,\delta\lambda_{2\,\text{b}}\right)\right)\right) = 0,$$
$$\left(6\,\mathrm{i}\,(-1+\text{Z}\,\text{Z}\Delta)\right) \Big/ \left(-6\,\mathrm{i} + 2\,\text{I}\mu\,\text{Z}\Delta^2\,\lambda\,\hbar + \text{I}\mu\,\text{n}\,\text{Z}\Delta^2\,\lambda\,\hbar + \text{I}\mu\,\text{n}\,\text{Z}\Delta^2\,\hbar\,\delta\lambda_{2\,\text{a}} + 2\,\text{I}\mu\,\text{Z}\Delta^2\,\hbar\,\delta\lambda_{2\,\text{b}}\right) =$$
$$0\Big\}$$

Solve for counter-terms from Goldstone EOM

Note there are two solutions differing by a sign for $\delta\lambda$.

```
cts =
  Solve[cteq, {δm₁, δλ1 a, δλ2 a, δλ2 b, δλ, Z, ZΔ}] // FullSimplify // DeleteDuplicates;
```

Solve::svars : Equations may not give solutions for all "solve" variables. >>

```
{δm₁², δλ1a, δλ2a, δλ2b, δλ, Z, ZΔ} /. cts // DeleteDuplicates
```

$$\Big\{\Big\{-\frac{(2+n)\,\lambda\,\left(\mathrm{i}\,t\mu+I\mu\,(m-\mu)\,(m+\mu)\right)\hbar}{6\,\mathrm{i}+I\mu\,(2+n)\,\lambda\,\hbar},$$
$$\left(\lambda\left(6\,I\mu\,Z\Delta^{5/2}\,\lambda\,\hbar-2\,\mathrm{i}\,c\mu\,(2+n)\,\lambda^2\,\hbar^2-3\,Z\Delta^4\left(6\,\mathrm{i}+I\mu\,(2+n)\,\lambda\,\hbar\right)+2\,Z\Delta^3\left(9\,\mathrm{i}+\lambda\,\hbar\left(-6\,(2\,\mathrm{Ifing0}+I\mu)+\mathrm{i}\,I\mu\,(\mathrm{Ifingn}+I\mu\,(2+n)+\mathrm{Ifing0}\,(3+2\,n))\,\lambda\,\hbar\right)\right)\right)\right)\Big/$$
$$\left(3\,Z\Delta^4\left(6\,\mathrm{i}+I\mu\,(2+n)\,\lambda\,\hbar\right)\right),\ \lambda\left(-1+\frac{6\,\mathrm{i}}{Z\Delta^2\left(6\,\mathrm{i}+I\mu\,(2+n)\,\lambda\,\hbar\right)}\right),$$
$$\lambda\left(-1+\frac{6\,\mathrm{i}}{Z\Delta^2\left(6\,\mathrm{i}+I\mu\,(2+n)\,\lambda\,\hbar\right)}\right),\ \left(-1-\frac{1}{Z\Delta^{3/2}}\right)\lambda,\ \frac{1}{Z\Delta},\ Z\Delta\Big\},$$
$$\Big\{-\frac{(2+n)\,\lambda\,\left(\mathrm{i}\,t\mu+I\mu\,(m-\mu)\,(m+\mu)\right)\hbar}{6\,\mathrm{i}+I\mu\,(2+n)\,\lambda\,\hbar},$$
$$\left(\lambda\left(-6\,I\mu\,Z\Delta^{5/2}\,\lambda\,\hbar-2\,\mathrm{i}\,c\mu\,(2+n)\,\lambda^2\,\hbar^2-3\,Z\Delta^4\left(6\,\mathrm{i}+I\mu\,(2+n)\,\lambda\,\hbar\right)+2\,Z\Delta^3\left(9\,\mathrm{i}+\lambda\,\hbar\left(-6\,(2\,\mathrm{Ifing0}+I\mu)+\mathrm{i}\,I\mu\,(\mathrm{Ifingn}+I\mu\,(2+n)+\mathrm{Ifing0}\,(3+2\,n))\,\lambda\,\hbar\right)\right)\right)\right)\Big/$$
$$\left(3\,Z\Delta^4\left(6\,\mathrm{i}+I\mu\,(2+n)\,\lambda\,\hbar\right)\right),\ \lambda\left(-1+\frac{6\,\mathrm{i}}{Z\Delta^2\left(6\,\mathrm{i}+I\mu\,(2+n)\,\lambda\,\hbar\right)}\right),$$
$$\lambda\left(-1+\frac{6\,\mathrm{i}}{Z\Delta^2\left(6\,\mathrm{i}+I\mu\,(2+n)\,\lambda\,\hbar\right)}\right),\ \left(-1+\frac{1}{Z\Delta^{3/2}}\right)\lambda,\ \frac{1}{Z\Delta},\ Z\Delta\Big\}\Big\}$$

Gather kinematically distinct divergences for Higgs EOM

```
cteq2 =
  (((((mn2 - (λ v²/3) - m² - λ/6 v² - ħ/6 ((n + 1) λ) (tfing) - ħ/6 (λ) (tfinn) /. mg2soln) /. Solve[
            neom, mn2][[1]] /. mg2soln) /. cts // FullSimplify //
        DeleteDuplicates) /. {tfing → 0, tfinn → 0} // Expand) //
     CoefficientList[#, {p, v, tfing, tfinn, Ifingp}] & // Flatten //
    Simplify // DeleteDuplicates) == 0 // Thread
```

$$\left\{\mathrm{True},\ \frac{\mathrm{i}\,\lambda\left(3\,\mathrm{i}+\sqrt{Z\Delta}\,\left(3\,\mathrm{i}+\mathrm{Ifing0}\,\lambda\,\hbar-\mathrm{Ifingn}\,\lambda\,\hbar\right)\right)}{9\,\sqrt{Z\Delta}}==0,\ \frac{\lambda\left(3+\mathrm{i}\,\sqrt{Z\Delta}\,\left(3\,\mathrm{i}+\mathrm{Ifing0}\,\lambda\,\hbar-\mathrm{Ifingn}\,\lambda\,\hbar\right)\right)}{9\,\sqrt{Z\Delta}}==0\right\}$$

Solve for counter-terms from Higgs EOM

```
cts2 = Solve[cteq2[[2]], {ZΔ}]
```

$$\left\{\left\{Z\Delta\to-\frac{9}{\left(3\,\mathrm{i}+\mathrm{Ifing0}\,\lambda\,\hbar-\mathrm{Ifingn}\,\lambda\,\hbar\right)^2}\right\}\right\}$$

Both equations should have the same solution:

```
(ZΔ /. Solve[cteq2[[3]], {ZΔ}][[1]]) - (ZΔ /. cts2[[1]]) == 0
```

```
True
```

Final Counterterms

```
({δm1^2, δλ1a, δλ2a, δλ2b, δλ, Z, ZΔ} /. cts /. cts2 // Simplify)[[1]] //
  DeleteDuplicates;

counterterms = Thread[{δm1^2, δλ1a, δλ2a, δλ2b, δλ, Z, ZΔ} → %[[1]]]
```

$$\left\{\delta m_1^2 \to -\frac{(2+n)\,\lambda\,\left(i\,t\mu + I\mu\,(m-\mu)\,(m+\mu)\right)\hbar}{6\,i + I\mu\,(2+n)\,\lambda\,\hbar},\right.$$

$$\delta\lambda_a \to \left(\lambda\,\left(3\,i + \text{Ifing0}\,\lambda\,\hbar - \text{Ifingn}\,\lambda\,\hbar\right)^8\left(-2\,i\,c\mu\,(2+n)\,\lambda^2\,\hbar^2 + 1458\,I\mu\,\lambda\,\hbar\left(-\frac{1}{\left(3\,i + \text{Ifing0}\,\lambda\,\hbar - \text{Ifingn}\,\lambda\,\hbar\right)^2}\right)^{5/2} - \frac{19\,683\,\left(6\,i + I\mu\,(2+n)\,\lambda\,\hbar\right)}{\left(3\,i + \text{Ifing0}\,\lambda\,\hbar - \text{Ifingn}\,\lambda\,\hbar\right)^8} - \left(1458\left(9\,i + \lambda\,\hbar\,\left(-6\,(2\,\text{Ifing0} + I\mu) + i\,I\mu\,(\text{Ifingn} + I\mu\,(2+n) + \text{Ifing0}\,(3+2\,n))\,\lambda\,\hbar\right)\right)\right)\Big/\left(3\,i + \text{Ifing0}\,\lambda\,\hbar - \text{Ifingn}\,\lambda\,\hbar\right)^6\right)\right)\Big/\left(19\,683\,\left(6\,i + I\mu\,(2+n)\,\lambda\,\hbar\right)\right),$$

$$\delta\lambda_{2a} \to \lambda\left(-1 + \frac{2\,i\,\left(3\,i + \text{Ifing0}\,\lambda\,\hbar - \text{Ifingn}\,\lambda\,\hbar\right)^4}{27\,\left(6\,i + I\mu\,(2+n)\,\lambda\,\hbar\right)}\right),$$

$$\delta\lambda_{2b} \to \lambda\left(-1 + \frac{2\,i\,\left(3\,i + \text{Ifing0}\,\lambda\,\hbar - \text{Ifingn}\,\lambda\,\hbar\right)^4}{27\,\left(6\,i + I\mu\,(2+n)\,\lambda\,\hbar\right)}\right),$$

$$\delta\lambda \to \lambda\left(-1 - \frac{1}{27\left(-\frac{1}{(3\,i+\text{Ifing0}\,\lambda\,\hbar-\text{Ifingn}\,\lambda\,\hbar)^2}\right)^{3/2}}\right),$$

$$Z \to -\frac{1}{9}\left(3\,i + \text{Ifing0}\,\lambda\,\hbar - \text{Ifingn}\,\lambda\,\hbar\right)^2,$$

$$\left.Z\Delta \to -\frac{9}{\left(3\,i + \text{Ifing0}\,\lambda\,\hbar - \text{Ifingn}\,\lambda\,\hbar\right)^2}\right\}$$

The should be momentum independent :

```
({δm1^2, δλ1a, δλ2a, δλ2b, δλ, Z, ZΔ} /. counterterms // DeleteDuplicates // D[#, p] &)[[
   1]] == 0 // Thread
({δm1^2, δλ1a, δλ2a, δλ2b, δλ, Z, ZΔ} /. counterterms // DeleteDuplicates //
     D[#, Ifingp] &)[[1]] == 0 // Thread
```

```
True
```

```
True
```