Renormalization of Symmetry Improved 3PIEA gap equations at 2 loops

Supplement to "Symmetry improvement of 3PI effective actions for O(N) scalar field theory" by Michael J. Brown and Ian B. Whittingham.

Mathematica notebook to compute couter-terms for two loop truncations of the effective action as described in Section IV of the paper.

Hartree-Fock

In[65]:=
```
ClearAll[geom, neom, intrules, regularisedtadpoles, mg2soln, cteq, cts, δm, δλ];
```

Hartree-Fock gap equations with counterterms

Goldstone equation of motion. Quantities in reference to the paper are:
p is the four-momentum flowing through the propagators $\Delta_G{}^{-1}$ and $\Delta_N{}^{-1}$,
mg2 is the Goldstone mass squared $m_G{}^2$,
Z and ZΔ are the wavefunction a propagator renormalization constants,
m^2 is the (renormalized) Lagrangian mass parameter, $\delta m_1{}^2$ is its counter-term,
λ is the (renormalized) four point coupling,
$\delta\lambda_{1\,a}$, $\delta\lambda_{2\,a}$, $\delta\lambda_{2\,b}$ are the independent coupling counter-terms,
v is the scalar field vacuum expectation value,
$\hbar$ is the reduced Planck constant,
n is the number of fields in the O(n) symmetry group,
t∞g, t∞n are the divergent tadpole integrals for the Goldstone, Higgs resp.,
tfing, tfinn are the finite parts of the tadpoles for the Goldstone, Higgs resp.

In[66]:=
$$\text{geom} = p^2 - \text{mg2} == Z\, Z\Delta\, p^2 - m^2 - \delta m_1{}^2 - Z\Delta\, \frac{\lambda + \delta\lambda_{1\,a}}{6}\, v^2 - \frac{\hbar}{6}\left((n+1)\,\lambda + (n-1)\,\delta\lambda_{2\,a} + 2\,\delta\lambda_{2\,b}\right) Z\Delta^2\, (t\infty g + \text{tfing}) - \frac{\hbar}{6}\,(\lambda + \delta\lambda_{2\,a})\, Z\Delta^2\, (t\infty n + \text{tfinn})$$

Out[66]=
$$-\text{mg2} + p^2 == -m^2 + p^2\, Z\, Z\Delta - \delta m_1^2 - \frac{1}{6}\, v^2\, Z\Delta\, (\lambda + \delta\lambda_a) - \frac{1}{6}\,(\text{tfinn} + t\infty n)\, Z\Delta^2\, \hbar\, (\lambda + \delta\lambda_{2\,a}) - \frac{1}{6}\,(\text{tfing} + t\infty g)\, Z\Delta^2\, \hbar\, ((1+n)\,\lambda + (-1+n)\,\delta\lambda_{2\,a} + 2\,\delta\lambda_{2\,b})$$

Higgs equation of motion

In[67]:= $\mathbf{neom = p^2 - mn2 == \frac{-\lambda v^2}{3} Z\Delta + p^2 - mg2}$

Out[67]= $-\mathrm{mn2} + p^2 == -\mathrm{mg2} + p^2 - \frac{1}{3} v^2 Z\Delta\, \lambda$

Infinite parts of tadpoles in MSbar

MSbar rules for 4 - 2 ϵ dimensions

In[68]:= $\mathbf{regularisedtadpoles = \{t\infty g \to c0\, \Lambda^2 + c1\, mg2\, Log[\Lambda^2 / \mu^2],\ t\infty n \to c0\, \Lambda^2 + c1\, mn2\, Log[\Lambda^2 / \mu^2]\}}$

Out[68]= $\left\{t\infty g \to c0\, \Lambda^2 + c1\, mg2\, \mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right],\ t\infty n \to c0\, \Lambda^2 + c1\, mn2\, \mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\right\}$

Sub in tadpole expressions, eliminate mn2 and solve for mg2

In[69]:=
```
mg2soln =
  mg2 /. (geom /. regularisedtadpoles /. Solve[neom, mn2][[1]] // Solve[#, mg2][[1]] &)
```

Out[69]= $$\Big(-18\, m^2 - 18\, p^2 + 18\, p^2\, Z\, Z\Delta - 3\, v^2\, Z\Delta\, \lambda - 3\, tfing\, Z\Delta^2\, \lambda\, \hbar - 3\, n\, tfing\, Z\Delta^2\, \lambda\, \hbar - 3\, tfinn\, Z\Delta^2\, \lambda\, \hbar - 6\, c0\, Z\Delta^2\, \lambda\, \Lambda^2\, \hbar - 3\, c0\, n\, Z\Delta^2\, \lambda\, \Lambda^2\, \hbar - c1\, v^2\, Z\Delta^3\, \lambda^2\, \hbar\, \mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right] - 18\, \delta m_1^2 - 3\, v^2\, Z\Delta\, \delta\lambda_a + 3\, tfing\, Z\Delta^2\, \hbar\, \delta\lambda_{2\,a} - 3\, n\, tfing\, Z\Delta^2\, \hbar\, \delta\lambda_{2\,a} - 3\, tfinn\, Z\Delta^2\, \hbar\, \delta\lambda_{2\,a} - 3\, c0\, n\, Z\Delta^2\, \Lambda^2\, \hbar\, \delta\lambda_{2\,a} - c1\, v^2\, Z\Delta^3\, \lambda\, \hbar\, \mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right] \delta\lambda_{2\,a} - 6\, tfing\, Z\Delta^2\, \hbar\, \delta\lambda_{2\,b} - 6\, c0\, Z\Delta^2\, \Lambda^2\, \hbar\, \delta\lambda_{2\,b}\Big) \Big/ \left(3 \left(-6 + 2\, c1\, Z\Delta^2\, \lambda\, \hbar\, \mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right] + c1\, n\, Z\Delta^2\, \lambda\, \hbar\, \mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right] + c1\, n\, Z\Delta^2\, \hbar\, \mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right] \delta\lambda_{2\,a} + 2\, c1\, Z\Delta^2\, \hbar\, \mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right] \delta\lambda_{2\,b}\right)\right)$$

Gather divergences proportional v, tfing and tfinn and set independently to zero

First we subtract the finite equation of motion, then gather coefficients of the remainder into a list and set each to zero (after some trimming and simplifying).

In[70]:=
```
cteq =
 ((CoefficientList[mg2soln + (-m^2 - λ/6 v^2 - ℏ/6 ((n + 1) λ) (tfing) - ℏ/6 (λ) (tfinn)), {p, v,
        tfing, tfinn}] // Flatten) //
      DeleteDuplicates // Simplify // FullSimplify) == 0 // Thread
```

Out[70]=

$$\Bigg\{-\left(\left(6\,\delta m_1^2 + Z\Delta^2\,\hbar\left(c0\,\Lambda^2 + c1\,m^2\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\right)\left((2+n)\,\lambda + n\,\delta\lambda_{2\,a} + 2\,\delta\lambda_{2\,b}\right)\right)\Big/\left(-6 + c1\,(2+n)\,Z\Delta^2\,\lambda\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right] + c1\,Z\Delta^2\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\left(n\,\delta\lambda_{2\,a} + 2\,\delta\lambda_{2\,b}\right)\right)\right) == 0,\ -\frac{\lambda\,\hbar}{6} - \left(Z\Delta^2\,\hbar\,(\lambda + \delta\lambda_{2\,a})\right)\Big/\left(-6 + c1\,(2+n)\,Z\Delta^2\,\lambda\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right] + c1\,Z\Delta^2\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\left(n\,\delta\lambda_{2\,a} + 2\,\delta\lambda_{2\,b}\right)\right) == 0,$$

$$\left(\hbar\left((1+n)\,\lambda\left(6 - 6\,Z\Delta^2 - c1\,(2+n)\,Z\Delta^2\,\lambda\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\right) + Z\Delta^2\left(-\left(6\,(-1+n) + c1\,n\,(1+n)\,\lambda\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\right)\delta\lambda_{2\,a} - 2\left(6 + c1\,(1+n)\,\lambda\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\right)\delta\lambda_{2\,b}\right)\right)\right)\Big/\left(6\left(-6 + c1\,(2+n)\,Z\Delta^2\,\lambda\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right] + c1\,Z\Delta^2\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\left(n\,\delta\lambda_{2\,a} + 2\,\delta\lambda_{2\,b}\right)\right)\right) == 0,$$

$$\mathrm{True},\ -\left(\left(6\,Z\Delta\,\delta\lambda_a + \lambda\left(6\,(-1+Z\Delta) + c1\,Z\Delta^2\,(2+n+2\,Z\Delta)\,\lambda\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\right) + c1\,Z\Delta^2\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\left((n+2\,Z\Delta)\,\delta\lambda_{2\,a} + 2\,\delta\lambda_{2\,b}\right)\right)\Big/\left(6\left(-6 + c1\,(2+n)\,Z\Delta^2\,\lambda\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right] + c1\,Z\Delta^2\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\left(n\,\delta\lambda_{2\,a} + 2\,\delta\lambda_{2\,b}\right)\right)\right)\right) == 0,$$

$$(-6 + 6\,Z\,Z\Delta)\Big/\left(-6 + c1\,(2+n)\,Z\Delta^2\,\lambda\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right] + c1\,Z\Delta^2\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\left(n\,\delta\lambda_{2\,a} + 2\,\delta\lambda_{2\,b}\right)\right) == 0\Bigg\}$$

Solve for counterterms

In[71]:=
```
cts = {δm₁², δλ₁ₐ, δλ₂ₐ, δλ₂ᵦ, Z, ZΔ} /. Solve[cteq, {δm₁, δλ₁ₐ, δλ₂ₐ, δλ₂ᵦ, Z, Z_Δ}] //
    FullSimplify // DeleteDuplicates
```

Solve::svars : Equations may not give solutions for all "solve" variables. ≫

Out[71]=

$$\left\{\left\{-\frac{(2+n)\,\lambda\,\hbar\left(c0\,\Lambda^2 + c1\,m^2\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\right)}{6 + c1\,(2+n)\,\lambda\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]},\ \frac{\lambda\left(6 - 6\,Z\Delta - c1\,(4+n)\,Z\Delta\,\lambda\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\right)}{Z\Delta\left(6 + c1\,(2+n)\,\lambda\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\right)},\ \lambda\left(-1 + \frac{6}{Z\Delta^2\left(6 + c1\,(2+n)\,\lambda\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\right)}\right),\ \lambda\left(-1 + \frac{6}{Z\Delta^2\left(6 + c1\,(2+n)\,\lambda\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\right)}\right),\ \frac{1}{Z\Delta},\ Z\Delta\right\}\right\}$$

ZΔ is redundant in this truncation, can remove it :

In[72]:=
```
cts /. ZΔ → 1 // FullSimplify
```

Out[72]=
$$\left\{\left\{-\frac{(2+n)\,\lambda\,\hbar\,\left(c0\,\Lambda^2+c1\,m^2\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\right)}{6+c1\,(2+n)\,\lambda\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]},\ -\frac{c1\,(4+n)\,\lambda^2\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]}{6+c1\,(2+n)\,\lambda\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]},\ \lambda\left(-1+\frac{6}{6+c1\,(2+n)\,\lambda\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]}\right),\ \lambda\left(-1+\frac{6}{6+c1\,(2+n)\,\lambda\,\hbar\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]}\right),\ 1,\ 1\right\}\right\}$$

In[73]:=
$$\delta\lambda_{2\,a}==\frac{n+2}{n+4}\,\delta\lambda_{1\,a}\ \text{/. Solve[cteq, }\{\delta m_1,\ \delta\lambda_{1\,a},\ \delta\lambda_{2\,a},\ \delta\lambda_{2\,b},\ Z,\ Z_\Delta\}]\ \text{/. }\{Z\Delta\to 1\}\ \text{// FullSimplify // DeleteDuplicates}$$

Solve::svars : Equations may not give solutions for all "solve" variables. ≫

Out[73]= {True}

Sunset

NOTE: this uses some of the same variable names as the Hartree-Fock code! Be careful not to clobber what you need to keep.

In[74]:=
```
ClearAll[geom, neom, intrules, regularisedtadpoles, mg2soln, cteq, δm, δλ, δλ, δλ, δλ];
```

Equations of motion

Additional variables relative to the Hartree-Fock case:
Ing is the sunset integral $I_{\mathrm{NG}}(p)$
Ifingp is the finite sunset integral $I^{\mathrm{fin}}_{\mathrm{NG}}(p)$,
Ifing0 is $I^{\mathrm{fin}}_{\mathrm{NG}}(m_G)$,
Ifingn is $I^{\mathrm{fin}}_{\mathrm{NG}}(m_N)$,
$\delta\lambda$ is the sunset graph coupling counter-term,
Iμ, tμ and cμ are the auxiliary integrals I_μ, T_μ and c_μ respectively.

In[75]:=
$$\text{geom} = p^2 - \text{mg2} + i\,\hbar\left(\frac{(\lambda)\,v}{3}\right)^2(\text{Ifingp}-\text{Ifing0}) == Z\,Z\Delta\,p^2 - m^2 - \delta m_1{}^2 - Z\Delta\,\frac{\lambda+\delta\lambda_{1\,a}}{6}\,v^2 - \frac{\hbar}{6}\left((n+1)\,\lambda+(n-1)\,\delta\lambda_{2\,a}+2\,\delta\lambda_{2\,b}\right)Z\Delta^2\,(\text{tg}) - \frac{\hbar}{6}\,(\lambda+\delta\lambda_{2\,a})\,Z\Delta^2\,(\text{tn}) + i\,\hbar\left(\frac{(\lambda+\delta\lambda)\,v}{3}\right)^2 Z\Delta^3\,\text{Ing}$$

Out[75]=
$$-\text{mg2}+p^2+\frac{1}{9}\,i\,(-\text{Ifing0}+\text{Ifingp})\,v^2\,\lambda^2\,\hbar == -m^2+p^2\,Z\,Z\Delta+\frac{1}{9}\,i\,\text{Ing}\,v^2\,Z\Delta^3\,(\delta\lambda+\lambda)^2\,\hbar-\delta m_1^2-\frac{1}{6}\,v^2\,Z\Delta\,(\lambda+\delta\lambda_a)-\frac{1}{6}\,\text{tn}\,Z\Delta^2\,\hbar\,(\lambda+\delta\lambda_{2\,a})-\frac{1}{6}\,\text{tg}\,Z\Delta^2\,\hbar\,((1+n)\,\lambda+(-1+n)\,\delta\lambda_{2\,a}+2\,\delta\lambda_{2\,b})$$

In[76]:= neom = $p^2 - \text{mn2} + i\,\hbar \left(\frac{(\lambda)\ v}{3}\right)^2 (\text{Ifingp} - \text{Ifingn}) ==$

$\frac{-Z\Delta\ (\lambda + \delta\lambda)\ v^2}{3} + p^2 - \text{mg2} + i\,\hbar \left(\frac{(\lambda)\ v}{3}\right)^2 (\text{Ifingp} - \text{Ifing0})$

Out[76]= $-\text{mn2} + p^2 + \frac{1}{9}\, i\ (-\text{Ifingn} + \text{Ifingp})\ v^2\ \lambda^2\ \hbar ==$

$-\text{mg2} + p^2 - \frac{1}{3}\, v^2\ Z\Delta\ (\delta\lambda + \lambda) + \frac{1}{9}\, i\ (-\text{Ifing0} + \text{Ifingp})\ v^2\ \lambda^2\ \hbar$

Divergent parts subtracted with auxiliary integrals and MSbar

In[77]:= intrules = $\{\text{Ing} \to \text{I}\mu + \text{Ifingp} + \text{Ifing0},$

$\text{tg} \to \text{t}\mu - i\ (\text{mg2} - \mu^2)\ \text{I}\mu + \hbar \left(\frac{(\lambda + \delta\lambda)\ v}{3}\right)^2 \text{c}\mu + \text{tfing},$

$\text{tn} \to \text{t}\mu - i\ (\text{mn2} - \mu^2)\ \text{I}\mu + \hbar \left(\frac{(\lambda + \delta\lambda)\ v}{3}\right)^2 \text{c}\mu + \text{tfinn}\}$

Out[77]= $\{\text{Ing} \to \text{Ifing0} + \text{Ifingp} + \text{I}\mu,\ \text{tg} \to \text{tfing} + \text{t}\mu - i\ \text{I}\mu\ (\text{mg2} - \mu^2) + \frac{1}{9}\, \text{c}\mu\ v^2\ (\delta\lambda + \lambda)^2\ \hbar,$

$\text{tn} \to \text{tfinn} + \text{t}\mu - i\ \text{I}\mu\ (\text{mn2} - \mu^2) + \frac{1}{9}\, \text{c}\mu\ v^2\ (\delta\lambda + \lambda)^2\ \hbar\}$

In[78]:= regularisedtadpoles =

$\left\{\text{I}\mu \to \text{c2}\ \text{Log}\left[\frac{\Lambda^2}{\mu^2}\right],\ \text{t}\mu \to \text{c0}\ \Lambda^2 + \text{c1}\ \mu^2\ \text{Log}\left[\frac{\Lambda^2}{\mu^2}\right],\ \text{c}\mu \to \text{a0}\ \text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]^2 + \text{a1}\ \text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\right\}$

Out[78]= $\left\{\text{I}\mu \to \text{c2}\ \text{Log}\left[\frac{\Lambda^2}{\mu^2}\right],\ \text{t}\mu \to \text{c0}\ \Lambda^2 + \text{c1}\ \mu^2\ \text{Log}\left[\frac{\Lambda^2}{\mu^2}\right],\ \text{c}\mu \to \text{a1}\ \text{Log}\left[\frac{\Lambda^2}{\mu^2}\right] + \text{a0}\ \text{Log}\left[\frac{\Lambda^2}{\mu^2}\right]^2\right\}$

Sub everything in, eliminate mn2 and solve for mg2

In[79]:=
```
mg2soln = ((geom /. intrules(*/.regularisedtadpoles*) /. Solve[neom, mn2][[1]]) //
    Solve[#, mg2] &)[[1]]
```

Out[79]=

$$\Big\{mg2 \to \Big(-m^2 - p^2 + p^2\, Z\, Z\Delta - \frac{1}{9}\, i\, (-Ifing0 + Ifingp)\, v^2\, \lambda^2\, \hbar + \frac{1}{9}\, i\, (Ifing0 + Ifingp + I\mu)\, v^2\, Z\Delta^3\, (\delta\lambda + \lambda)^2\, \hbar - \delta m_1^2 - \frac{1}{6}\, v^2\, Z\Delta\, (\lambda + \delta\lambda_a) - \frac{1}{6}\, tfinn\, Z\Delta^2\, \hbar\, (\lambda + \delta\lambda_{2\,a}) - \frac{1}{6}\, t\mu\, Z\Delta^2\, \hbar\, (\lambda + \delta\lambda_{2\,a}) + \frac{1}{18}\, i\, I\mu\, v^2\, Z\Delta^3\, \delta\lambda\, \hbar\, (\lambda + \delta\lambda_{2\,a}) + \frac{1}{18}\, i\, I\mu\, v^2\, Z\Delta^3\, \lambda\, \hbar\, (\lambda + \delta\lambda_{2\,a}) - \frac{1}{6}\, i\, I\mu\, Z\Delta^2\, \mu^2\, \hbar\, (\lambda + \delta\lambda_{2\,a}) - \frac{1}{54}\, c\mu\, v^2\, Z\Delta^2\, (\delta\lambda + \lambda)^2\, \hbar^2\, (\lambda + \delta\lambda_{2\,a}) - \frac{1}{54}\, I\mu\, Z\Delta^2\, \hbar\, \big(Ifing0\, v^2\, \lambda^2\, \hbar - Ifingn\, v^2\, \lambda^2\, \hbar\big)\, (\lambda + \delta\lambda_{2\,a}) - \frac{1}{6}\, tfing\, Z\Delta^2\, \hbar\, ((1+n)\, \lambda + (-1+n)\, \delta\lambda_{2\,a} + 2\, \delta\lambda_{2\,b}) - \frac{1}{6}\, t\mu\, Z\Delta^2\, \hbar\, ((1+n)\, \lambda + (-1+n)\, \delta\lambda_{2\,a} + 2\, \delta\lambda_{2\,b}) - \frac{1}{6}\, i\, I\mu\, Z\Delta^2\, \mu^2\, \hbar\, ((1+n)\, \lambda + (-1+n)\, \delta\lambda_{2\,a} + 2\, \delta\lambda_{2\,b}) - \frac{1}{54}\, c\mu\, v^2\, Z\Delta^2\, (\delta\lambda + \lambda)^2\, \hbar^2\, ((1+n)\, \lambda + (-1+n)\, \delta\lambda_{2\,a} + 2\, \delta\lambda_{2\,b})\Big) \Big/ \Big(-1 - \frac{1}{6}\, i\, I\mu\, Z\Delta^2\, \hbar\, (\lambda + \delta\lambda_{2\,a}) - \frac{1}{6}\, i\, I\mu\, Z\Delta^2\, \hbar\, ((1+n)\, \lambda + (-1+n)\, \delta\lambda_{2\,a} + 2\, \delta\lambda_{2\,b})\Big)\Big\}$$

Gather kinematically distinct divergences for Goldstone EOM

In[80]:=
$$\text{cteq} = \left(\left(\text{mg2} - \text{m}^2 - \frac{\lambda}{6}\text{v}^2 - \frac{\hbar}{6}\left(\left(\text{n}+1\right)\lambda\right)(\text{tfing}) - \frac{\hbar}{6}(\lambda)(\text{tfinn}) \text{ /. mg2soln}\right) \text{ // CoefficientList[\#, \{p, v, tfing, tfinn, Ifingp\}] \& // Flatten // Simplify // DeleteDuplicates}\right) == 0 \text{ // Thread}$$

Out[80]=
$$\Big\{\left(-6\,i\,\delta m_1^2 + Z\Delta^2\left(-i\,t\mu + I\mu\left(-m^2+\mu^2\right)\right)\hbar\left((2+n)\lambda + n\,\delta\lambda_{2a} + 2\,\delta\lambda_{2b}\right)\right) \Big/ \left(-6\,i + 2\,I\mu\,Z\Delta^2\lambda\hbar + I\mu\,n\,Z\Delta^2\lambda\hbar + I\mu\,n\,Z\Delta^2\hbar\,\delta\lambda_{2a} + 2\,I\mu\,Z\Delta^2\hbar\,\delta\lambda_{2b}\right) == 0,$$
$$\text{True},\ \frac{1}{6}\hbar\left(-\lambda - \left(6\,i\,Z\Delta^2\left(\lambda+\delta\lambda_{2a}\right)\right) \Big/ \left(-6\,i + 2\,I\mu\,Z\Delta^2\lambda\hbar + I\mu\,n\,Z\Delta^2\lambda\hbar + I\mu\,n\,Z\Delta^2\hbar\,\delta\lambda_{2a} + 2\,I\mu\,Z\Delta^2\hbar\,\delta\lambda_{2b}\right)\right) == 0,$$
$$\frac{1}{6}\hbar\left(-(1+n)\lambda - \left(6\,i\,Z\Delta^2\left(\lambda + n\,\lambda + (-1+n)\,\delta\lambda_{2a} + 2\,\delta\lambda_{2b}\right)\right) \Big/ \left(-6\,i + 2\,I\mu\,Z\Delta^2\lambda\hbar + I\mu\,n\,Z\Delta^2\lambda\hbar + I\mu\,n\,Z\Delta^2\hbar\,\delta\lambda_{2a} + 2\,I\mu\,Z\Delta^2\hbar\,\delta\lambda_{2b}\right)\right) == 0,$$
$$-\Big(\Big(i\,\Big(-18\,\lambda + 18\,Z\Delta\,\lambda - 12\,i\,\text{Ifing0}\,Z\Delta^3\,\delta\lambda^2\,\hbar - 12\,i\,I\mu\,Z\Delta^3\,\delta\lambda^2\,\hbar - 24\,i\,\text{Ifing0}\,Z\Delta^3\,\delta\lambda\,\lambda\,\hbar - 30\,i\,I\mu\,Z\Delta^3\,\delta\lambda\,\lambda\,\hbar - 12\,i\,\text{Ifing0}\,\lambda^2\,\hbar - 6\,i\,I\mu\,Z\Delta^2\,\lambda^2\,\hbar - 3\,i\,I\mu\,n\,Z\Delta^2\,\lambda^2\,\hbar - 12\,i\,\text{Ifing0}\,Z\Delta^3\,\lambda^2\,\hbar - 18\,i\,I\mu\,Z\Delta^3\,\lambda^2\,\hbar + 4\,c\mu\,Z\Delta^2\,\delta\lambda^2\,\lambda\,\hbar^2 + 2\,c\mu\,n\,Z\Delta^2\,\delta\lambda^2\,\lambda\,\hbar^2 + 8\,c\mu\,Z\Delta^2\,\delta\lambda\,\lambda^2\,\hbar^2 + 4\,c\mu\,n\,Z\Delta^2\,\delta\lambda\,\lambda^2\,\hbar^2 + 4\,c\mu\,Z\Delta^2\,\lambda^3\,\hbar^2 + 2\,\text{Ifing0}\,I\mu\,Z\Delta^2\,\lambda^3\,\hbar^2 - 2\,\text{Ifingn}\,I\mu\,Z\Delta^2\,\lambda^3\,\hbar^2 + 2\,c\mu\,n\,Z\Delta^2\,\lambda^3\,\hbar^2 + 18\,Z\Delta\,\delta\lambda_a + Z\Delta^2\,\hbar\,\Big(2\,c\mu\,n\,(\delta\lambda+\lambda)^2\,\hbar + I\mu\left(-6\,i\,Z\Delta\,(\delta\lambda+\lambda) + \lambda\left(-3\,i\,n + 2\,(\text{Ifing0} - \text{Ifingn})\,\lambda\,\hbar\right)\right)\Big)\,\delta\lambda_{2a} - 6\,i\,I\mu\,Z\Delta^2\,\lambda\,\hbar\,\delta\lambda_{2b} + 4\,c\mu\,Z\Delta^2\,\delta\lambda^2\,\hbar^2\,\delta\lambda_{2b} + 8\,c\mu\,Z\Delta^2\,\delta\lambda\,\lambda\,\hbar^2\,\delta\lambda_{2b} + 4\,c\mu\,Z\Delta^2\,\lambda^2\,\hbar^2\,\delta\lambda_{2b}\Big)\Big) \Big/ \left(18\left(-6\,i + 2\,I\mu\,Z\Delta^2\lambda\hbar + I\mu\,n\,Z\Delta^2\lambda\hbar + I\mu\,n\,Z\Delta^2\hbar\,\delta\lambda_{2a} + 2\,I\mu\,Z\Delta^2\hbar\,\delta\lambda_{2b}\right)\right)\Big) == 0,$$
$$-\left(\left(2\left(-\lambda^2 + Z\Delta^3(\delta\lambda+\lambda)^2\right)\hbar\right) \Big/ \left(3\left(-6\,i + 2\,I\mu\,Z\Delta^2\lambda\hbar + I\mu\,n\,Z\Delta^2\lambda\hbar + I\mu\,n\,Z\Delta^2\hbar\,\delta\lambda_{2a} + 2\,I\mu\,Z\Delta^2\hbar\,\delta\lambda_{2b}\right)\right)\right) == 0,$$
$$\left(6\,i\,(-1+Z\,Z\Delta)\right) \Big/ \left(-6\,i + 2\,I\mu\,Z\Delta^2\lambda\hbar + I\mu\,n\,Z\Delta^2\lambda\hbar + I\mu\,n\,Z\Delta^2\hbar\,\delta\lambda_{2a} + 2\,I\mu\,Z\Delta^2\hbar\,\delta\lambda_{2b}\right) == 0\Big\}$$

Solve for counter-terms from Goldstone EOM

Note there are two solutions differing by a sign for $\delta\lambda$.

In[81]:=
```
cts =
  Solve[cteq, {δm₁, δλ1a, δλ2a, δλ2b, δλ, Z, ZΔ}] // FullSimplify // DeleteDuplicates;
```

Solve::svars : Equations may not give solutions for all "solve" variables. >>

In[82]:=
```
{δm1^2, δλ1a, δλ2a, δλ2b, δλ, Z, ZΔ} /. cts // DeleteDuplicates
```

Out[82]=
$$\left\{\left\{-\frac{(2+n)\,\lambda\left(i\,t\mu+I\mu\,(m-\mu)\,(m+\mu)\right)\hbar}{6\,i+I\mu\,(2+n)\,\lambda\,\hbar},\right.\right.$$
$$\left(\lambda\left(6\,I\mu\,Z\Delta^{5/2}\,\lambda\,\hbar-2\,i\,c\mu\,(2+n)\,\lambda^2\,\hbar^2-3\,Z\Delta^4\left(6\,i+I\mu\,(2+n)\,\lambda\,\hbar\right)+2\,Z\Delta^3\left(9\,i+\lambda\,\hbar\left(-6\,(2\,Ifing0+I\mu)+i\,I\mu\,(Ifingn+I\mu\,(2+n)+Ifing0\,(3+2\,n))\,\lambda\,\hbar\right)\right)\right)\right)/$$
$$\left(3\,Z\Delta^4\left(6\,i+I\mu\,(2+n)\,\lambda\,\hbar\right)\right),\ \lambda\left(-1+\frac{6\,i}{Z\Delta^2\left(6\,i+I\mu\,(2+n)\,\lambda\,\hbar\right)}\right),$$
$$\lambda\left(-1+\frac{6\,i}{Z\Delta^2\left(6\,i+I\mu\,(2+n)\,\lambda\,\hbar\right)}\right),\ \left(-1-\frac{1}{Z\Delta^{3/2}}\right)\lambda,\ \frac{1}{Z\Delta},\ Z\Delta\right\},$$
$$\left\{-\frac{(2+n)\,\lambda\left(i\,t\mu+I\mu\,(m-\mu)\,(m+\mu)\right)\hbar}{6\,i+I\mu\,(2+n)\,\lambda\,\hbar},\right.$$
$$\left(\lambda\left(-6\,I\mu\,Z\Delta^{5/2}\,\lambda\,\hbar-2\,i\,c\mu\,(2+n)\,\lambda^2\,\hbar^2-3\,Z\Delta^4\left(6\,i+I\mu\,(2+n)\,\lambda\,\hbar\right)+2\,Z\Delta^3\left(9\,i+\lambda\,\hbar\left(-6\,(2\,Ifing0+I\mu)+i\,I\mu\,(Ifingn+I\mu\,(2+n)+Ifing0\,(3+2\,n))\,\lambda\,\hbar\right)\right)\right)\right)/$$
$$\left(3\,Z\Delta^4\left(6\,i+I\mu\,(2+n)\,\lambda\,\hbar\right)\right),\ \lambda\left(-1+\frac{6\,i}{Z\Delta^2\left(6\,i+I\mu\,(2+n)\,\lambda\,\hbar\right)}\right),$$
$$\left.\left.\lambda\left(-1+\frac{6\,i}{Z\Delta^2\left(6\,i+I\mu\,(2+n)\,\lambda\,\hbar\right)}\right),\ \left(-1+\frac{1}{Z\Delta^{3/2}}\right)\lambda,\ \frac{1}{Z\Delta},\ Z\Delta\right\}\right\}$$

Gather kinematically distinct divergences for Higgs EOM

In[83]:=
```
cteq2 =
 (((((mn2 - (λ v^2/3) - m^2 - λ/6 v^2 - ℏ/6 ((n + 1) λ) (tfing) - ℏ/6 (λ) (tfinn) /. mg2soln) /. Solve[
        neom, mn2][[1]] /. mg2soln) /. cts // FullSimplify //
     DeleteDuplicates) /. {tfing → 0, tfinn → 0} // Expand) //
   CoefficientList[#, {p, v, tfing, tfinn, Ifingp}] & // Flatten //
  Simplify // DeleteDuplicates) == 0 // Thread
```

Out[83]=
$$\left\{\text{True},\ \frac{i\,\lambda\left(3\,i+\sqrt{Z\Delta}\,\left(3\,i+Ifing0\,\lambda\,\hbar-Ifingn\,\lambda\,\hbar\right)\right)}{9\,\sqrt{Z\Delta}}=0,\right.$$
$$\left.\frac{\lambda\left(3+i\,\sqrt{Z\Delta}\,\left(3\,i+Ifing0\,\lambda\,\hbar-Ifingn\,\lambda\,\hbar\right)\right)}{9\,\sqrt{Z\Delta}}=0\right\}$$

Solve for counter-terms from Higgs EOM

In[84]:=
```
cts2 = Solve[cteq2[[2]], {ZΔ}]
```

Out[84]=
$$\left\{\left\{Z\Delta\to-\frac{9}{\left(3\,i+Ifing0\,\lambda\,\hbar-Ifingn\,\lambda\,\hbar\right)^2}\right\}\right\}$$

Both equations should have the same solution:

```
In[85]:= (ZΔ /. Solve[cteq2[[3]], {ZΔ}][[1]]) - (ZΔ /. cts2[[1]]) == 0
Out[85]= True
```

Final Counterterms

```
In[86]:= ({δm1^2, δλ1a, δλ2a, δλ2b, δλ, Z, ZΔ} /. cts /. cts2 // Simplify)[[1]] //
           DeleteDuplicates;

In[87]:= counterterms = Thread[{δm1^2, δλ1a, δλ2a, δλ2b, δλ, Z, ZΔ} → %[[1]]]
```

Out[87]=

$$\Big\{\delta m_1^2 \to -\frac{(2+n)\,\lambda\,(i\,t\mu + I\mu\,(m-\mu)\,(m+\mu))\,\hbar}{6\,i + I\mu\,(2+n)\,\lambda\,\hbar},$$

$$\delta\lambda_a \to \Bigg(\lambda\,(3\,i + \text{Ifing0}\,\lambda\,\hbar - \text{Ifingn}\,\lambda\,\hbar)^8\,\Bigg(-2\,i\,c\mu\,(2+n)\,\lambda^2\,\hbar^2 + 1458\,I\mu\,\lambda\,\hbar\left(-\frac{1}{(3\,i + \text{Ifing0}\,\lambda\,\hbar - \text{Ifingn}\,\lambda\,\hbar)^2}\right)^{5/2} - \frac{19683\,(6\,i + I\mu\,(2+n)\,\lambda\,\hbar)}{(3\,i + \text{Ifing0}\,\lambda\,\hbar - \text{Ifingn}\,\lambda\,\hbar)^8} - \big(1458\,(9\,i + \lambda\,\hbar\,(-6\,(2\,\text{Ifing0} + I\mu) + i\,I\mu\,(\text{Ifingn} + I\mu\,(2+n) + \text{Ifing0}\,(3+2\,n))\,\lambda\,\hbar))\big) \Big/ (3\,i + \text{Ifing0}\,\lambda\,\hbar - \text{Ifingn}\,\lambda\,\hbar)^6\Bigg)\Bigg) \Big/ \big(19683\,(6\,i + I\mu\,(2+n)\,\lambda\,\hbar)\big),$$

$$\delta\lambda_{2a} \to \lambda\left(-1 + \frac{2\,i\,(3\,i + \text{Ifing0}\,\lambda\,\hbar - \text{Ifingn}\,\lambda\,\hbar)^4}{27\,(6\,i + I\mu\,(2+n)\,\lambda\,\hbar)}\right),$$

$$\delta\lambda_{2b} \to \lambda\left(-1 + \frac{2\,i\,(3\,i + \text{Ifing0}\,\lambda\,\hbar - \text{Ifingn}\,\lambda\,\hbar)^4}{27\,(6\,i + I\mu\,(2+n)\,\lambda\,\hbar)}\right),$$

$$\delta\lambda \to \lambda\left(-1 - \frac{1}{27\left(-\frac{1}{(3\,i + \text{Ifing0}\,\lambda\,\hbar - \text{Ifingn}\,\lambda\,\hbar)^2}\right)^{3/2}}\right),$$

$$Z \to -\frac{1}{9}\,(3\,i + \text{Ifing0}\,\lambda\,\hbar - \text{Ifingn}\,\lambda\,\hbar)^2,$$

$$Z\Delta \to -\frac{9}{(3\,i + \text{Ifing0}\,\lambda\,\hbar - \text{Ifingn}\,\lambda\,\hbar)^2}\Big\}$$

The should be momentum independent :

```
In[88]:= ({δm1^2, δλ1a, δλ2a, δλ2b, δλ, Z, ZΔ} /. counterterms // DeleteDuplicates // D[#, p] &)[[
           1]] == 0 // Thread
         ({δm1^2, δλ1a, δλ2a, δλ2b, δλ, Z, ZΔ} /. counterterms // DeleteDuplicates //
              D[#, Ifingp] &)[[1]] == 0 // Thread
Out[88]= True
Out[89]= True
```