Renormalization of Symmetry Improved 3PIEA gap equations at 2 loops

Supplement to thesis Chapter 4.

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Mathematica notebook to compute couter-terms for two loop truncations of the SI-3PIEA.

Hartree-Fock

```
ClearAll[geom, neom, intrules, regularisedtadpoles, mg2soln, cteq, cts, δm, δλ];
```

Hartree-Fock gap equations with counterterms

Goldstone equation of motion. Quantities in reference to the paper are:
p is the four-momentum flowing through the propagators $\Delta_G{}^{-1}$ and $\Delta_N{}^{-1}$,
mg2 is the Goldstone mass squared $m_G{}^2$,
Z and ZΔ are the wavefunction a propagator renormalization constants,
m^2 is the (renormalized) Lagrangian mass parameter, $\delta m_1{}^2$ is its counter-term,
λ is the (renormalized) four point coupling,
$\delta\lambda_{1a}$, $\delta\lambda_{2a}$, $\delta\lambda_{2b}$ are the independent coupling counter-terms,
v is the scalar field vacuum expectation value,
$\hbar$ is the reduced Planck constant,
n is the number of fields in the O(n) symmetry group,
t∞g, t∞n are the divergent tadpole integrals for the Goldstone, Higgs resp.,
tfing, tfinn are the finite parts of the tadpoles for the Goldstone, Higgs resp.

$$\text{geom} = p^2 - \text{mg2} == Z\,Z\Delta\,p^2 - m^2 - \delta m_1{}^2 - Z\Delta\,\frac{\lambda + \delta\lambda_{1a}}{6}\,v^2 - \frac{\hbar}{6}\left((n+1)\,\lambda + (n-1)\,\delta\lambda_{2a} + 2\,\delta\lambda_{2b}\right) Z\Delta^2\,(\text{t}\infty\text{g} + \text{tfing}) - \frac{\hbar}{6}\,(\lambda + \delta\lambda_{2a})\,Z\Delta^2\,(\text{t}\infty\text{n} + \text{tfinn})$$

Higgs equation of motion

$$\text{neom} = p^2 - \text{mn2} == \frac{-\lambda\,v^2}{3}\,Z\Delta + p^2 - \text{mg2}$$

Infinite parts of tadpoles in MSbar

MSbar rules for 4 - 2 ϵ dimensions

```
regularisedtadpoles = {t∞g → c0 Λ^2 + c1 mg2 Log[Λ^2 / μ^2], t∞n → c0 Λ^2 + c1 mn2 Log[Λ^2 / μ^2]}
```

Sub in tadpole expressions, eliminate mn2 and solve for mg2

```
mg2soln =
  mg2 /. (geom /. regularisedtadpoles /. Solve[neom, mn2][[1]] // Solve[#, mg2][[1]] &)
```

Gather divergences proportional v, tfing and tfinn and set independently to zero

First we subtract the finite equation of motion, then gather coefficients of the remainder into a list and set each to zero (after some trimming and simplifying).

```
cteq =
 ((CoefficientList[mg2soln + (-m^2 - λ/6 v^2 - ℏ/6 ((n + 1) λ) (tfing) - ℏ/6 (λ) (tfinn)), {p, v,
          tfing, tfinn}] // Flatten) //
       DeleteDuplicates // Simplify // FullSimplify) == 0 // Thread
```

Solve for counterterms

```
cts = {δm1^2, δλ1a, δλ2a, δλ2b, Z, ZΔ} /. Solve[cteq, {δm1, δλ1a, δλ2a, δλ2b, Z, ZΔ}] //
   FullSimplify // DeleteDuplicates
```

ZΔ is redundant in this truncation, can remove it :

```
cts /. ZΔ → 1 // FullSimplify
```

```
δλ2a == (n + 2)/(n + 4) δλ1a /. Solve[cteq, {δm1, δλ1a, δλ2a, δλ2b, Z, ZΔ}] /. {ZΔ → 1} //
  FullSimplify // DeleteDuplicates
```

Sunset

NOTE: this uses some of the same variable names as the Hartree-Fock code! Be careful not to clobber what you need to keep.

```
ClearAll[geom, neom, intrules, regularisedtadpoles, mg2soln, cteq, δm, δλ, δλ, δλ, δλ];
```

Equations of motion

Additional variables relative to the Hartree-Fock case:
Ing is the sunset integral $I_{\mathrm{NG}}(p)$

Ifingp is the finite sunset integral $I_{\mathrm{NG}}^{\mathrm{fin}}(p)$,
Ifing0 is $I_{\mathrm{NG}}^{\mathrm{fin}}(m_G)$,
Ifingn is $I_{\mathrm{NG}}^{\mathrm{fin}}(m_N)$,
$\delta\lambda$ is the sunset graph coupling counter-term,
Iμ, tμ and cμ are the auxiliary integrals I_μ, T_μ and c_μ respectively.

$$\mathtt{geom} = p^2 - \mathtt{mg2} + i\,\hbar\left(\frac{(\lambda)\,v}{3}\right)^2 (\mathtt{Ifingp} - \mathtt{Ifing0}) == \mathtt{Z}\,\mathtt{Z\Delta}\,p^2 - m^2 - \delta m_1{}^2 - \mathtt{Z\Delta}\,\frac{\lambda + \delta\lambda_{1\,a}}{6}\,v^2 - \frac{\hbar}{6}\left((n+1)\,\lambda + (n-1)\,\delta\lambda_{2\,a} + 2\,\delta\lambda_{2\,b}\right)\mathtt{Z\Delta}^2\,(\mathtt{tg}) - \frac{\hbar}{6}\,(\lambda + \delta\lambda_{2\,a})\,\mathtt{Z\Delta}^2\,(\mathtt{tn}) + i\,\hbar\left(\frac{(\lambda+\delta\lambda)\,v}{3}\right)^2 \mathtt{Z\Delta}^3\,\mathtt{Ing}$$

$$\mathtt{neom} = p^2 - \mathtt{mn2} + i\,\hbar\left(\frac{(\lambda)\,v}{3}\right)^2 (\mathtt{Ifingp} - \mathtt{Ifingn}) == \frac{-\mathtt{Z\Delta}\,(\lambda+\delta\lambda)\,v^2}{3} + p^2 - \mathtt{mg2} + i\,\hbar\left(\frac{(\lambda)\,v}{3}\right)^2 (\mathtt{Ifingp} - \mathtt{Ifing0})$$

Divergent parts subtracted with auxiliary integrals and MSbar

$$\mathtt{intrules} = \left\{\mathtt{Ing} \to \mathtt{I}\mu + \mathtt{Ifingp} + \mathtt{Ifing0},\ \mathtt{tg} \to \mathtt{t}\mu - i\,(\mathtt{mg2} - \mu^2)\,\mathtt{I}\mu + \hbar\left(\frac{(\lambda+\delta\lambda)\,v}{3}\right)^2 \mathtt{c}\mu + \mathtt{tfing},\ \mathtt{tn} \to \mathtt{t}\mu - i\,(\mathtt{mn2} - \mu^2)\,\mathtt{I}\mu + \hbar\left(\frac{(\lambda+\delta\lambda)\,v}{3}\right)^2 \mathtt{c}\mu + \mathtt{tfinn}\right\}$$

$$\mathtt{regularisedtadpoles} = \left\{\mathtt{I}\mu \to \mathtt{c2}\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right],\ \mathtt{t}\mu \to \mathtt{c0}\,\Lambda^2 + \mathtt{c1}\,\mu^2\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right],\ \mathtt{c}\mu \to \mathtt{a0}\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]^2 + \mathtt{a1}\,\mathrm{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\right\}$$

Sub everything in, eliminate mn2 and solve for mg2

```
mg2soln = ((geom /. intrules(*/.regularisedtadpoles*) /. Solve[neom, mn2][[1]]) //
    Solve[#, mg2] &)[[1]]
```

Gather kinematically distinct divergences for Goldstone EOM

$$\mathtt{cteq} = \left(\left(\mathtt{mg2} - m^2 - \frac{\lambda}{6}\,v^2 - \frac{\hbar}{6}\left((n+1)\,\lambda\right)(\mathtt{tfing}) - \frac{\hbar}{6}\,(\lambda)\,(\mathtt{tfinn})\ /.\ \mathtt{mg2soln}\right) // \mathtt{CoefficientList}[\#, \{p, v, \mathtt{tfing}, \mathtt{tfinn}, \mathtt{Ifingp}\}]\ \&\ // \mathtt{Flatten} // \mathtt{Simplify} // \mathtt{DeleteDuplicates}\right) == 0\ //\ \mathtt{Thread}$$

Solve for counter-terms from Goldstone EOM

Note there are two solutions differing by a sign for $\delta\lambda$.

```
cts =
  Solve[cteq, {δm₁, δλ₁ₐ, δλ₂ₐ, δλ₂ᵦ, δλ, Z, ZΔ}] // FullSimplify // DeleteDuplicates;
```

```
{δm₁², δλ1 a, δλ2 a, δλ2 b, δλ, Z, ZΔ} /. cts // DeleteDuplicates
```

Gather kinematically distinct divergences for Higgs EOM

```
cteq2 =
 (((((mn2 - (λ v^2/3) - m^2 - λ/6 v^2 - ħ/6 ((n + 1) λ) (tfing) - ħ/6 (λ) (tfinn) /. mg2soln) /. Solve[
            neom, mn2][[1]] /. mg2soln) /. cts // FullSimplify //
         DeleteDuplicates) /. {tfing → 0, tfinn → 0} // Expand) //
      CoefficientList[#, {p, v, tfing, tfinn, Ifingp}] & // Flatten //
    Simplify // DeleteDuplicates) == 0 // Thread
```

Solve for counter-terms from Higgs EOM

```
cts2 = Solve[cteq2[[2]], {ZΔ}]
```

Both equations should have the same solution:

```
(ZΔ /. Solve[cteq2[[3]], {ZΔ}][[1]]) - (ZΔ /. cts2[[1]]) == 0
```

Final Counterterms

```
({δm₁², δλ1 a, δλ2 a, δλ2 b, δλ, Z, ZΔ} /. cts /. cts2 // Simplify)[[1]] //
  DeleteDuplicates;
```

```
counterterms = Thread[{δm₁², δλ1 a, δλ2 a, δλ2 b, δλ, Z, ZΔ} → %[[1]]]
```

The should be momentum independent :

```
({δm₁², δλ1 a, δλ2 a, δλ2 b, δλ, Z, ZΔ} /. counterterms // DeleteDuplicates // D[#, p] &)[[
   1]] == 0 // Thread
({δm₁², δλ1 a, δλ2 a, δλ2 b, δλ, Z, ZΔ} /. counterterms // DeleteDuplicates //
     D[#, Ifingp] &)[[1]] == 0 // Thread
```