Renormalization of Symmetry Improved 2PIEA gap equations at 2 loops

Supplement to chapter 4 of thesis by Michael J. Brown.

Mathematica notebook to compute couter-terms for two loop truncations of the effective action as described in Chapter 4.

Sunset

NOTE: this uses some of the same variable names as the Hartree-Fock code! Be careful not to clobber what you need to keep.

```
ClearAll[geom, neom, intrules, msbarrules, mg2soln, cteq, δm, δλ, δλ, δλ, δλ];
```

Equations of motion

Goldstone equation of motion. Quantities in reference to the paper are:
p is the four-momentum flowing through the propagators $\Delta_G{}^{-1}$ and $\Delta_N{}^{-1}$,
mg2 is the Goldstone mass squared $m_G{}^2$,
Z and ZΔ are the wavefunction a propagator renormalization constants,
m^2 is the (renormalized) Lagrangian mass parameter, $\delta m_1{}^2$ is its counter-term,
λ is the (renormalized) four point coupling,
$\delta\lambda_{1a}$, $\delta\lambda_{2a}$, $\delta\lambda_{2b}$ are the independent coupling counter-terms,
v is the scalar field vacuum expectation value,
$\hbar$ is the reduced Planck constant,
n is the number of fields in the O(n) symmetry group,
t∞g, t∞n are the divergent tadpole integrals for the Goldstone, Higgs resp.,
tfing, tfinn are the finite parts of the tadpoles for the Goldstone, Higgs resp.
Additional variables relative to the Hartree-Fock case:
Ing is the sunset integral $I_{\mathrm{NG}}(p)$
Ifingp is the finite sunset integral $I^{\mathrm{fin}}_{\mathrm{NG}}(p)$,
Ifing0 is $I^{\mathrm{fin}}_{\mathrm{NG}}(m_G)$,
Ifingn is $I^{\mathrm{fin}}_{\mathrm{NG}}(m_N)$,
$\delta\lambda$ is the sunset graph coupling counter-term,
Iμ, tμ and cμ are the auxiliary integrals I_μ, T_μ and c_μ respectively.

$$\texttt{geom} = p^2 - \texttt{mg2} + i\,\hbar\left(\frac{(\lambda)\,v}{3}\right)^2(\texttt{Ifingp} - \texttt{Ifing0}) ==$$
$$Z\,Z\Delta\,p^2 - m^2 - \delta m_1{}^2 - Z\Delta\,\frac{\lambda + \delta\lambda_{1\,a}}{6}\,v^2 - \frac{\hbar}{6}\left((n+1)\,\lambda + (n-1)\,\delta\lambda_{2\,a} + 2\,\delta\lambda_{2\,b}\right)Z\Delta^2\,(\texttt{tg}) -$$
$$\frac{\hbar}{6}\,(\lambda + \delta\lambda_{2\,a})\,Z\Delta^2\,(\texttt{tn}) + i\,\hbar\left(\frac{(\lambda + \delta\lambda)\,v}{3}\right)^2 Z\Delta^3\,\texttt{Ing}$$

$$\texttt{neom} = p^2 - \texttt{mn2} + \frac{i\,\hbar}{2}\left(\frac{(\lambda)\,v}{3}\right)^2(n-1)\,(\texttt{Ifinggp} - \texttt{Ifinggn}) + \frac{i\,\hbar}{2}\,(\lambda)^2\,v^2\,(\texttt{Ifinhhp} - \texttt{Ifinhhn}) ==$$
$$Z\,Z\Delta\,p^2 - m^2 - \delta m_1{}^2 - Z\Delta\,\frac{3\,\lambda + \delta\lambda_{1\,a} + 2\,\delta\lambda_{1\,b}}{6}\,v^2 - \frac{\hbar}{6}\left(3\,\lambda + \delta\lambda_{2\,a} + 2\,\delta\lambda_{2\,b}\right)Z\Delta^2\,\texttt{tn} -$$
$$\frac{\hbar}{6}\,(\lambda + \delta\lambda_{2\,a})\,Z\Delta^2\,(n-1)\,\texttt{tg} + \frac{i\,\hbar}{2}\left(\frac{(\lambda + \delta\lambda)\,v}{3}\right)^2 Z\Delta^3\,(n-1)\,\texttt{Igg} + \frac{i\,\hbar}{2}\,(\lambda + \delta\lambda)^2\,v^2\,Z\Delta^3\,\texttt{Ihh}$$

Divergent parts subtracted with auxiliary integrals and MSbar

$$\texttt{intrules} = \{\texttt{Ing} \to \texttt{I}\mu + \texttt{Ifingp},\ \texttt{Igg} \to \texttt{I}\mu + \texttt{Ifinggp},\ \texttt{Ihh} \to \texttt{I}\mu + \texttt{Ifinhhp},$$
$$\texttt{tg} \to \texttt{t}\mu - i\,(\texttt{mg2} - \mu^2)\,\texttt{I}\mu + \hbar\left(\frac{(\lambda + \delta\lambda)\,v}{3}\right)^2 \texttt{c}\mu + \texttt{tfing},$$
$$\texttt{tn} \to \texttt{t}\mu - i\,(\texttt{mn2} - \mu^2)\,\texttt{I}\mu + \hbar\left(\frac{(\lambda + \delta\lambda)\,v}{3}\right)^2 \texttt{c}\mu + \texttt{tfinn}\}$$

$$\texttt{msbarrules} = \left\{\texttt{I}\mu \to \texttt{c2}\,\texttt{Log}\left[\frac{\Lambda^2}{\mu^2}\right],\ \texttt{t}\mu \to \texttt{c0}\,\Lambda^2 + \texttt{c1}\,\mu^2\,\texttt{Log}\left[\frac{\Lambda^2}{\mu^2}\right],\ \texttt{c}\mu \to \texttt{a0}\,\texttt{Log}\left[\frac{\Lambda^2}{\mu^2}\right]^2 + \texttt{a1}\,\texttt{Log}\left[\frac{\Lambda^2}{\mu^2}\right]\right\}$$

Sub everything in, eliminate mn2 and solve for mg2

```
{mg2soln, mn2soln} =
 (Solve[{geom, neom} /. intrules, {mg2, mn2}] // ExpandAll // Simplify)[[1]]
```

Check solutions

$$\left(p^2 - \texttt{mg2} + i\,\hbar\left(\frac{(\lambda)\,v}{3}\right)^2(\texttt{Ifingp} - \texttt{Ifing0}) - \left(Z\,Z\Delta\,p^2 - m^2 - \delta m_1{}^2 - Z\Delta\,\frac{\lambda + \delta\lambda_{1\,a}}{6}\,v^2 - \frac{\hbar}{6}\left((n+1)\,\lambda + (n-1)\,\delta\lambda_{2\,a} + 2\,\delta\lambda_{2\,b}\right)Z\Delta^2\,(\texttt{tg}) - \frac{\hbar}{6}\,(\lambda + \delta\lambda_{2\,a})\,Z\Delta^2\,(\texttt{tn}) + i\,\hbar\left(\frac{(\lambda + \delta\lambda)\,v}{3}\right)^2 Z\Delta^3\,\texttt{Ing}\right)\right) /.$$

```
intrules /. mn2soln /. mg2soln /. msbarrules // Simplify
```

$$\left(p^2 - \texttt{mn2} + \frac{i\,\hbar}{2}\left(\frac{(\lambda)\,v}{3}\right)^2(n-1)\,(\texttt{Ifinggp} - \texttt{Ifinggn}) + \frac{i\,\hbar}{2}\,(\lambda)^2\,v^2\,(\texttt{Ifinhhp} - \texttt{Ifinhhn}) - \left(Z\,Z\Delta\,p^2 - m^2 - \delta m_1{}^2 - Z\Delta\,\frac{3\,\lambda + \delta\lambda_{1\,a} + 2\,\delta\lambda_{1\,b}}{6}\,v^2 - \frac{\hbar}{6}\left(3\,\lambda + \delta\lambda_{2\,a} + 2\,\delta\lambda_{2\,b}\right)Z\Delta^2\,\texttt{tn} - \frac{\hbar}{6}\,(\lambda + \delta\lambda_{2\,a})\,Z\Delta^2\,(n-1)\,\texttt{tg} + \frac{i\,\hbar}{2}\left(\frac{(\lambda + \delta\lambda)\,v}{3}\right)^2 Z\Delta^3\,(n-1)\,\texttt{Igg} + \frac{i\,\hbar}{2}\,(\lambda + \delta\lambda)^2\,v^2\,Z\Delta^3\,\texttt{Ihh}\right)\right) /.$$

```
intrules /. mn2soln /. mg2soln /. msbarrules // Simplify
```

Gather kinematically distinct divergences for Goldstone EOM

$$\left(\left(p^2 - \text{mg2} + i\hbar\left(\frac{(\lambda)\,v}{3}\right)^2 (\text{Ifingp} - \text{Ifing0}) - \left(p^2 - m^2 - \frac{\lambda}{6}v^2 - \frac{\hbar}{6}\left((n+1)\,\lambda\right)(\text{tfing}) - \frac{\hbar}{6}(\lambda)\,(\text{tfinn}) + i\hbar\left(\frac{(\lambda)\,v}{3}\right)^2(\text{Ifingp})\right)\right) \text{/. intrules /. mn2soln /. mg2soln /. msbarrules // Simplify}\right) \text{//}$$

```
  CoefficientList[#, {p, v, tfing, tfinn, Ifingp, Ifinggp, Ifinhhp}] & //
 Flatten // Simplify // DeleteDuplicates
```

$$\text{cteq} = \left(\left(\text{mg2} - m^2 - \frac{\lambda}{6}v^2 - \frac{\hbar}{6}\left((n+1)\,\lambda\right)(\text{tfing}) - \frac{\hbar}{6}(\lambda)\,(\text{tfinn}) \text{ /. mg2soln}\right) \text{//}$$

```
     CoefficientList[#, {p, v, tfing, tfinn, Ifingp, Ifinggp, Ifinhhp}] & //
    Flatten // Simplify // DeleteDuplicates) == 0 // Thread
```

```
cteq = (cteq /. msbarrules // Simplify // DeleteDuplicates)
```

Solve for counter-terms from Goldstone EOM

Note there are two solutions differing by a sign for $\delta\lambda$.

cts = Solve[cteq, {δm_1, $\delta\lambda_{1\,a}$, $\delta\lambda_{1\,b}$, $\delta\lambda_{2\,a}$, $\delta\lambda_{2\,b}$, $\delta\lambda$, Z, ZΔ}] // DeleteDuplicates;

{$\delta m_1{}^2$, $\delta\lambda_{1\,a}$, $\delta\lambda_{1\,b}$, $\delta\lambda_{2\,a}$, $\delta\lambda_{2\,b}$, $\delta\lambda$, Z, ZΔ} /. cts // DeleteDuplicates

Gather kinematically distinct divergences for Higgs EOM

cteq2 =

$$\left(\left(\left(\left(\left(\text{mn2} - \left(\frac{\lambda\,v^2}{3}\right) - m^2 - \frac{\lambda}{6}v^2 - \frac{\hbar}{6}\left((n+1)\,\lambda\right)(\text{tfing}) - \frac{\hbar}{6}(\lambda)\,(\text{tfinn}) \text{ /. mg2soln}\right) \text{/. Solve[neom, mn2][[1]] /. mg2soln}\right) \text{/. cts // FullSimplify // DeleteDuplicates}\right) \text{/. \{tfing} \to 0\text{, tfinn} \to 0\text{\} // Expand}\right) \text{//}$$

```
     CoefficientList[#, {p, v, tfing, tfinn, Ifingp}] & // Flatten //
   Simplify // DeleteDuplicates) == 0 // Thread
```

Solve for counter-terms from Higgs EOM

cts2 = Solve[cteq2[[2]], {ZΔ}]

Both equations should have the same solution:

(ZΔ /. Solve[cteq2[[3]], {ZΔ}][[1]]) - (ZΔ /. cts2[[1]]) == 0

Final Counterterms

```
({δm₁², δλ₁ₐ, δλ₂ₐ, δλ₂ᵦ, δλ, Z, ZΔ} /. cts /. cts2 // Simplify)[[1]] //
  DeleteDuplicates;
```

```
counterterms = Thread[{δm₁², δλ₁ₐ, δλ₂ₐ, δλ₂ᵦ, δλ, Z, ZΔ} → %[[1]]]
```

The should be momentum independent :

```
({δm₁², δλ₁ₐ, δλ₂ₐ, δλ₂ᵦ, δλ, Z, ZΔ} /. counterterms // DeleteDuplicates // D[#, p] &)[[
   1]] == 0 // Thread
({δm₁², δλ₁ₐ, δλ₂ₐ, δλ₂ᵦ, δλ, Z, ZΔ} /. counterterms // DeleteDuplicates //
     D[#, Ifingp] &)[[1]] == 0 // Thread
```