2.1 Michael McCormack

$f(x) = 100(x_2 - x_1^2)^2 + (1-x_1)^2$

$\nabla f(x) = \left(\frac{df}{dx_1}, \frac{df}{dx_2}\right)$

$\frac{df}{dx_2} = 200(x_2 - x_1^2)$

$\frac{df}{dx_1} = 200 \cdot -2x_1 \cdot (x_2 - x_1^2) + 2(1-x)(-1)$

$= -400x_1(x_2 - x_1^2) - 2(1-x)$

$\boxed{\nabla f = (-400x_1(x_2 - x_1^2) - 2(1-x), 200(x_2 - x_1^2))} = \nabla f(x)$ Gradient

Hessian $\nabla^2 f(x) = \begin{bmatrix} \frac{d^2 f}{dx_1^2} & \frac{d^2 f}{dx_1 dx_2} \\ \frac{d^2 f}{dx_1 dx_2} & \frac{d^2 f}{dx_2^2} \end{bmatrix}$

$\nabla^2 f(x) = \begin{bmatrix} 1200x_1^2 - 400x_2 + 2 & -400x_1 \\ -400x_1 & 200 \end{bmatrix}$ = Hessian $\nabla^2 f(x)$

2.1 cont

If $\nabla f = 0$ $\therefore$ $x_2 - x_1^2 = 0$ $\therefore$ $x_2 = x_1^2$ $\leftarrow$ from

$-400x_1(x_2 - x_1^2) - 2(1-x_1) = 0$

$1 - x_1 = 0$ $\therefore$ $x_1 = 1$ $\therefore$ $x_2 = 1$

$(1,1)^T$ is a critical point

$$\frac{d^2 f}{d x_1^2}(1,1) = 800 + 2 = 802; \quad 802 > 0$$

$$\nabla^2 f(1,1) = 400 > 0$$

$\therefore$ (1,1) is the only local mimizer

$$H_{(1,1)} = \begin{bmatrix} 802 & -400 \\ -400 & 200 \end{bmatrix}$$

If the eigen values for this matrix are positive, then we can say the matrix's positive definite.

Python work is attached below showing that the eigenvalues are positive $\therefore$ the matrix is positive definite.

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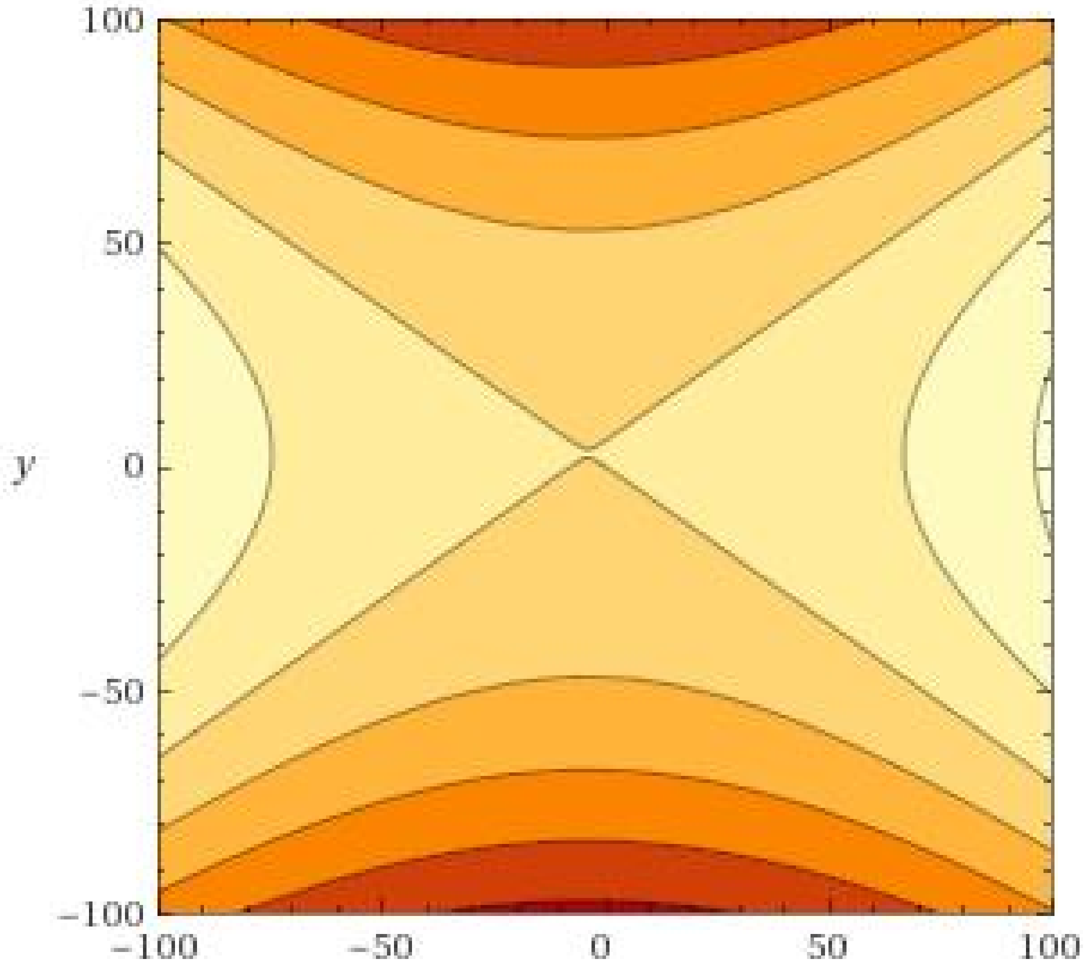
In [2]:
   ...: import numpy as np
   ...:
   ...: A= np.mat("802 -400; -400, 200")
   ...:
   ...: print("Eigenvalues : ", np.linalg.eigvals(A))
   ...:
Eigenvalues :  [  1.00160064e+03   3.99360767e-01]
```

contour plot	$8x + 12y + x^2 - 2y^2$	$x = -100$ to 100
		$y = -100$ to 100

Contour plot:



2.2

$f(x) = 8x_1 + 12x_2 + x_1^2 - 2x_2^2$

$\frac{df}{dx_1} = 8 + 0 + 2x_1 - 0 = 8 + 2x_1$

$\frac{df}{dx_2} = 0 + 12 + 0 - 4x_2 = 12 - 4x_2$

$2x + 8 = 0$
$2x = -8$
$x = -4$

$12 - 4x_2 = 0$
$-4x_2 = -12$
$4x_2 = 12$
$x_2 = 3$

Stationary Point $(-4, 3)$

Because there are unique solutions, this is the only stationary point.

$\frac{d^2f}{dx_1^2} = 2, \frac{d^2f}{dx_2 x_1} = 0, \frac{d^2f}{dx_2^2} = -4, \frac{d^2f}{dx_1 dx_2} = 0$

$\nabla^2 f(x) = \begin{bmatrix} 2 & 0 \\ 0 & -4 \end{bmatrix}$

if $f_{x_1x_1} f_{x_2x_2} - f_{x_1x_2}^2 < 0$ at $(-4,3)$; $(-4,3)$ is a saddle point

$(2)(-4) + 0 < 0$
$-8 < 0$ $\therefore$ $(-4,3)$ is a Saddle Point

Attached is the sketch of the Contour Lines

2.3 $f_1(x) = a^T x$

$f_2(x) = x^T A x$

$a^T x = a_1 x_1 + \dots + a_n x_n$

$\nabla' f_1(x) = \frac{d f_1(x)}{d x_1} = a_1 = a$

if $\nabla' = a$ then there will be no x in ∇^2 $\therefore$ $\nabla f_1(x) = 0$

$\therefore \nabla^2 f_1(x) = 0$

$f_2 = x^T A x$

$\nabla' f_2(x) = \frac{d f_2(x)}{d x_1}, \frac{d f_2(x)}{d x_2}, \vdots \frac{d f_2(x)}{d x_n} = 2Ax \therefore \nabla^2 f_2(x) = 2A$

2.4 Part 1

$$f(x) = f(a) + (x-a)f'(a) + \left(\frac{x-a}{2!}\right)^2 f(a) + \frac{(x-a)^3}{6} f'''(a) + \dots$$

$$f(x) = \cos\frac{1}{x} \qquad f(a) = \cos\left(\frac{1}{a}\right)$$

$$f'(x) = -\sin\left(\frac{1}{x}\right)\frac{-1}{x^2} = \frac{1}{x^2}\sin\left(\frac{1}{x}\right)$$

$$f'(a) = \frac{1}{a^2}\sin\left(\frac{1}{a}\right)$$

$$f''(x) = \frac{1}{x^2}\left(\cos\left(\frac{1}{x}\right)\right)\left(\frac{-1}{x^2}\right) + \sin\left(\frac{1}{x}\right)\left(\frac{-2}{x^3}\right)$$

$$= \frac{-1}{x^4}\cos\left(\frac{1}{x}\right) - \frac{2}{x^3}\sin\left(\frac{1}{x}\right)$$

$$f''(a) = \frac{-1}{a^4}\cos\left(\frac{1}{a}\right) - \frac{2}{a^3}\sin\left(\frac{1}{a}\right)$$

$$\cos\frac{1}{x} = \cos\left(\frac{1}{a}\right) + \frac{(x-a)}{a^2}\sin\left(\frac{1}{a}\right) - \frac{(x-a)^2}{2!}\left[\frac{1}{a^4}\cos\left(\frac{1}{a}\right) + \frac{2}{a^3}\sin\frac{1}{a}\right]$$

2.4 Part 2

$f(x) = \cos x \quad f'(x) = -\sin x \quad f''(x) = -\cos x$

$f^3(x) = \sin x$

$$\cos(x) = \cos(a) - (x-a)\sin a - \frac{(x-a)^2}{2}\cos(a) + \frac{(x-a)^3}{6}\sin(a)$$

~~[illegible]~~ $a = 1$

$$\cos(x) = \cos(1) - (x-1)\sin(1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{6}\sin(1)$$