HW5 Numerical Methods Fall 2017

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Euler Explicit

Problem

Given the 1D heat conduction equation with no source term:

$$\frac{\partial \phi}{\partial t} = \frac{\partial^2 \phi}{\partial x^2}$$

given the initial condition:

$$\phi(t, 0) = 1$$

and the boundary conditions:

$$\frac{\partial \phi}{\partial x}(t, 0) = 0$$

$$\frac{\partial \phi}{\partial x}(t, 1) = -\phi(t, 1)$$

The analytical solution to the problem is given by:

$$\phi(t, x) = \sum_{n=1}^{\infty} C_n exp(-\lambda_n^2 t) \cos(\lambda_n x)$$

where:

$$C_n = \frac{4 \sin \lambda_n}{2\lambda_n + 2\sin(2\lambda_n)}$$

We will use the explicit (forward Euler) method to solve the problem numerically. For our purposes, it is sufficient to use just the first four terms of C_n for an accurate enough answer.

Forward Euler Scheme

The governing equation is discretized using a second order central-differencing for the spatial derivative for the interior nodes, and a first order forward differencing scheme for the temporal derivative. The second order central-differencing for the spatial derivative:

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{\phi_{i+1} - 2\phi_i + \phi_{i-1}}{(\Delta x)^2}$$

First order forward difference for temporal derivative:

$$\frac{\partial \phi}{\partial t} = \frac{\phi_{i,n+1} - \phi_{i,n}}{\Delta t}$$

Rearranging for $\phi_{i,n+1}$ leads to the scheme:

$$\phi_{i,n+1} = \phi_{i,n} + \Delta t \left[\frac{\phi_{i+1,n} - 2\phi_{i,n} + \phi_{i-1,n}}{(\Delta x)^2} \right]$$

Regarding the boundary conditions, we will perform a first order forward/backward differencing to the boundary equations:

$$\frac{\partial \phi}{\partial x}(t, 0) = 0$$

hence:

$$\frac{\phi_{2,n+1} - \phi_{1,n}}{\Delta x} = 0 \implies \phi_{1,n+1} = \phi_{2,n}$$

$$\frac{\partial \phi}{\partial x}(t, 1) = -\phi(t, 1)$$

hence

$$\frac{\phi_{m,n+1} - \phi_{m-1,n}}{\Delta x} = -\phi_{m,n} \implies \phi_{m,n+1} = -\Delta x \phi_{m-1,n} + \phi_{m,n}$$

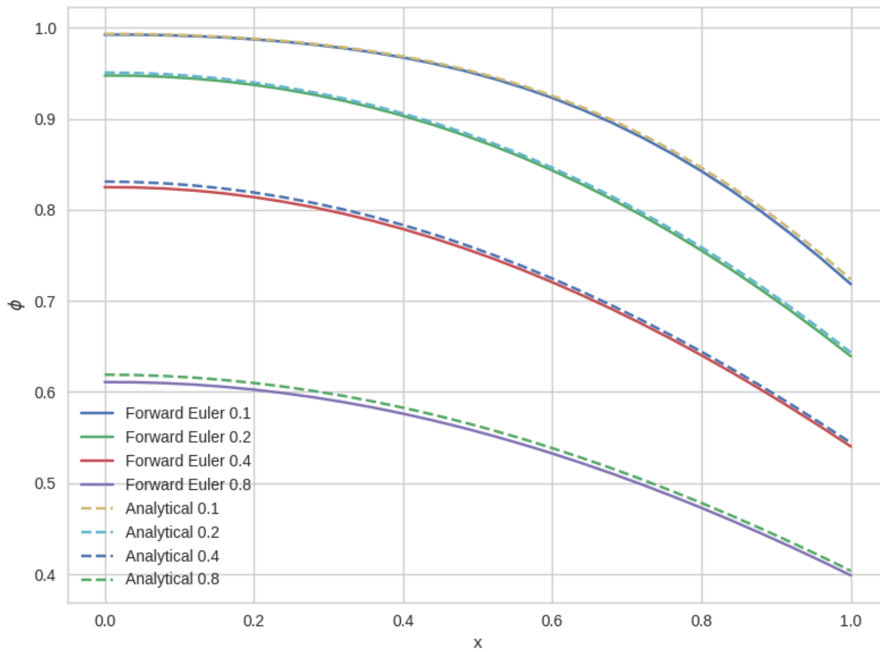
The stability criterion for the forward Euler method is given by:

$$\Delta t \leq \frac{1}{2} \frac{(\Delta x)^2}{\alpha}$$

Choosing 26 nodes in the x-direction leads to a maximum stable time step of: $\Delta t_{max} \leq 0.0008$ We will choose to use a time step of $\frac{1}{2}\Delta t_{max} = 0.0004$ sec.

Results

The equations are solved with a time-marching approach, and results were recorded at times $t = 0.1, 0.2, 0.4, 0.8$ and plotted on a single graph.



Discussion

It is clear that a good agreement is seen between the analytical solution and the numerical one. But we can observe that the numerical solution drifts from the analytical as more time passes. Choosing a higher order time-stepping method such as the Crank-Nicholson method should help reduce this error. Alternatively, the time-step can be further reduced.

Appendix: Python Code

```
import numpy as np
import matplotlib.pyplot as plt
import seaborn as sns
sns.set_style('whitegrid')

if __name__ == "__main__":
    # parameters
    nx = 26
    dx = 1.0 / (nx - 1)
    dt = 0.25*(dx**2)
    finaltime = 1.0

    x = np.linspace(0,1,nx)
    nt = int(finaltime/dt)
    dx2 = dx**2
    phi = np.ones(nx, dtype=float)

    for n in range(1,nt):
        phi_n = phi.copy()
        phi[1:-1] = phi_n[1:-1] + (dt/dx2)*(phi_n[2:] -2*phi_n[1:-1] + phi_n[0:-2])
        phi[0] = phi_n[1]
        phi[-1] = -dx*phi_n[-1] + phi_n[-2]

        if n*dt == 0.1:
            phi_01 = phi.copy()
        elif n*dt == 0.2:
            phi_02 = phi.copy()
        elif n*dt == 0.4:
            phi_04 = phi.copy()
        elif n*dt == 0.8:
            phi_08 = phi.copy()

    lamb = np.array([0.8603, 3.4256, 6.4373, 9.5293])
    Cn = (4*np.sin(lamb))/(2*lamb + np.sin(2*lamb))
    phi_anal = np.zeros((nt,nx))
    for n in range(nt):
        phi_anal[n] = Cn[0]*np.exp((-(lamb[0]**2))*n*dt)*np.cos(lamb[0]*x) +\
                      Cn[1]*np.exp((-(lamb[1]**2))*n*dt)*np.cos(lamb[1]*x) +\
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                  Cn[2]*np.exp((-(lamb[2]**2))*n*dt)*np.cos(lamb[2]*x) +\
                  Cn[3]*np.exp((-(lamb[3]**2))*n*dt)*np.cos(lamb[3]*x)

plt.plot(x, phi_01, label='Forward Euler 0.1')
plt.plot(x, phi_02, label='Forward Euler 0.2')
plt.plot(x, phi_04, label='Forward Euler 0.4')
plt.plot(x, phi_08, label='Forward Euler 0.8')
plt.plot(x, phi_anal[int(0.1/dt)], '--', label='Analytical 0.1')
plt.plot(x, phi_anal[int(0.2/dt)], '--', label='Analytical 0.2')
plt.plot(x, phi_anal[int(0.4/dt)], '--', label='Analytical 0.4')
plt.plot(x, phi_anal[int(0.8/dt)], '--', label='Analytical 0.8')
plt.xlabel('x')
plt.ylabel(r'$\phi$')
plt.legend()
plt.show()
```