Contents

1 7.1 Gauss-divergence theorem

Area (volume in 2D) is given by equation 7.40:

$$V_0 = \frac{1}{2}\left(\sum_{1}^{N_{f,O}} n_{x,f} x_f A_f + \sum_{1}^{N_{f,O}} n_{y,f} y_f A_f\right)$$

and in 3D:

$$V_0 = \frac{1}{3}\left(\sum_{1}^{N_{f,O}} n_{x,f} x_f A_f + \sum_{1}^{N_{f,O}} n_{y,f} y_f A_f + \sum_{1}^{N_{f,O}} n_{z,f} z_f A_f\right)$$

The face normals in 2D, $\mathrm{n_{x,f}}$, are computed by first calculating the unit tangent along the faces:

$$t_{x,f} = \frac{x_2 - x_1}{A_f}$$

$$t_{y,f} = \frac{x_2 - x_1}{A_f}$$

Next the normal vector now be computed in 2D:

$$n_{y,f} = -t_{x,f}$$

$$n_{x,f} = t_{y,f}$$

For 3D, using the tangent vectors t_1, t_2 (just vector substraction of vertices, not the **unit** tangent vector):

$$n = t_1 \times t_2$$

or

$$n = (t_{1y}t_{2z} - t_{1z}t_{2y})\hat{i} + (t_{1z}t_{2x} - t_{1x}t_{2z})\hat{j} + (t_{1x}t_{2y} - t_{1y}t_{2x})\hat{j}$$

The face areas, A_f in 2D are just the line length given by the Pythagorean theorem. While in 3D, they are given as:

$$A_f = \frac{1}{1}|t_1 \times t_2| = \frac{1}{2}|n|$$

Lastly, the face centroid components, x_f, y_f are given as:

$$x_f = \frac{1}{A_f}\int_{S_f} xdA$$

$$y_f = \frac{1}{A_f}\int_{S_f} ydA$$

$$z_f = \frac{1}{A_f}\int_{S_f} zdA$$

1.1 Computations:

1.1.1 a. Triangle given by points: (1,2), (3,5), (-1,1)

Faces numbering:

1. (1,2) to (3,5)
2. (3,5) to (-1,1)
3. (-1,1) to (1,2)

$$A_{f,1} = \sqrt{(5-2)^2 + (3-1)^2} = \sqrt{13}$$

$$A_{f,2} = \sqrt{(1-5)^2 + (-1-3)^2} = \sqrt{32}$$

$$A_{f,3} = \sqrt{(2-1)^2 + (1+1)^2} = \sqrt{5}$$

$$x_{f,1} = \frac{1+3}{2} = 2,\ y_{f,1} = \frac{2+5}{2} = 3.5$$

$$x_{f,2} = \frac{3-1}{2} = 1,\ y_{f,2} = \frac{5+1}{2} = 3$$

$$x_{f,3} = \frac{-1+1}{2} = 0,\ y_{f,3} = \frac{1+2}{2} = 1.5$$

$$t_{x,f,1} = \frac{3-1}{A_{f,1}} = 0.554 = -n_{y,f,1},\ t_{y,f,1} = \frac{5-2}{A_{f,1}} = 0.832 = n_{x,f,1}$$

$$t_{x,f,2} = \frac{-1-3}{A_{f,2}} = -0.707 = -n_{y,f,2},\ t_{y,f,2} = \frac{1-5}{A_{f,2}} = -0.707 = n_{x,f,2}$$

$$t_{x,f,3} = \frac{1+1}{A_{f,3}} = 0.894 = -n_{y,f,3},\ t_{y,f,3} = \frac{2-1}{A_{f,3}} = 0.447 = n_{x,f,3}$$

Finally the area of the triangle can be computed:

$$V_0 = \frac{1}{2}\left(6-4+0\right) + \left(-7+12-3\right)\right) = 2$$

1.1.2 b.Tetrahedron given by points: (1,1,1), (1,4,3), (-2,1,4), (4,-2,1)

Points numbering:

1. (1,1,1)
2. (1,4,3)
3. (-2,1,4)
4. (4,-2,1)

faces numbering:

1. includes points: 1,2,3
2. includes points: 2,3,4
3. includes points: 3,4,1
4. includes points: 4,1,2

Tangent vectors can be calculated as the vector difference :

- Face 1:

$$t_{1,2} = [0, 3, 2]$$
$$t_{1,3} = [-3, 0, 3]$$
$$n_1 = t_{1,2} \times t_{1,3} = [9, -6, 9]$$

- Face 2:

$$t_{3,4} = [6, -3, -3]$$
$$n_2 = t_{2,3} \times t_{3,4} = [12, -3, 27]$$

- Face 3:

$$t_{4,1} = [-3, 3, 0]$$

$$n_3 = t_{3,4} \times t_{4,1} = [9, 9, 9]$$

- Face 4:

$$n_4 = t_{4,1} \times t_{1,2} = [6, 6, -9]$$

Using the fact that $A_f = \frac{1}{2}|n|$, and face centroid values:

$$V_0 = \frac{1}{3}\left(\sum_{1}^{N_{f,O}} n_{x,f} x_f A_f + \sum_{1}^{N_{f,O}} n_{y,f} y_f A_f + \sum_{1}^{N_{f,O}} n_{z,f} z_f A_f\right) = 7.5$$

2 7.2 Calculating average $\nabla\phi$: