CS 344 Problem Set 1: Asymptotics

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1. If f(n) = o(g(n)) and g(n) = o(h(n)), we need to show that there exist the cost. C and no such that

a) $f(n) + g(n) \leq c * g(n) \ \forall \ n \geq n_0$

b) $f(n) + g(n) \leq c * h(n) \ \forall \ n \geq n_0$

Since f (n) = O (log n) there exist the cost c1 and n1 such that

$f(n) \leq c_1 * g(n)$

Similarly g(n) = O (h(n)) there enlist the cost c2 and n2 such that

$g(n) \leq c_2 * h(n)$

Now, let's choose $n_0 \geq \max(n_1, n_2)$ and $C = C_1 + C_2$

$$F(n) + g(n) \leq C_1 * g(n) + C_2 * g(n)$$

$$f(n) = O(g(n))$$

and $g(n) = O(h(n))$

$$= (C_1 + C_2) + g(n)$$

$$= C * g(n)$$

Therefore, we show that

$$f(n) + g(n) = O(g(n))$$

Similarly $f(n) + g(n) = O(h(n))$

P2. a) $f(n) = \theta(f(n-1))$

in this case $f(n) \neq \theta(f(n-1))$

as factorial grow very quickly and $f(n)$ is not of same as $f(n-1)$

b) Now, Let $f(n) = 2^n$

$f(n) = \theta(F(\frac{n}{2}))$ is true substitute $\frac{n}{2}$ into function we $2^{\frac{n}{2}}$ which is $\sqrt{2^n}$. Since both belong to same expo. growth.

c) Now, let $f(n) = n^d$ for $d>0$

$f(n) = \theta(f(\sqrt{n}))$ when we substitute in the function. we obtain $(\sqrt{n})^d = n^{d/2}$ not same as of n^d

Therefore, $f(n) \neq \theta(f\sqrt{n})$

d) $f(n) = \log n$

$f(n) = \theta(f(\log n))$ is not true substitute $\log n$ into the fuction. We got $\log(\log n)$ is not same as of $\log(n)$

P3. $\log n! = \Theta(n \log n)$ we need to show two inequalities:

a) $\log n! = O(n \log n)$

b) $\log n! = \Omega(n \log n)$

first prove upper bound (1) using Stirling approximation,

$$n! \le (\sqrt{2\pi n}) \times (\frac{n}{e})^n$$

log on both sides $\log n! \le \log(\sqrt{2\pi n}) + n \log(\frac{n}{e})$

Since, both $\log(\sqrt{2\pi n})$ & $\log(\frac{n}{e})$ are both sides.

$\log n! \le \log(\sqrt{2\pi n}) + n \log(\frac{n}{e})$ Since, both $\log(\sqrt{2\pi n})$ & $\log(\frac{n}{e})$ are both log. fn they grow much slower than $n \log n$ therefore, $\log n! = O(\log n)$

now, prove lower bound (2)

using the fact that

$$n! \ge (\frac{n}{2})^{n/2}$$

take log. on both sides

$$\log n! \ge \log((\frac{n}{2}))^{n/2}$$

now, $\log n! \ge \frac{n}{2} * (\log(n) - \log 2)$

$\frac{n}{2} * \log n$ we know that $\frac{n}{2}$ is asymptotically smaller

that n and $\log n$ is non-decreasing for $n>1$

now, $\frac{n}{2} \log n$ is also asymp.

smaller than $n \log n$

we can conclude that $\log n! = \Omega(n \log n)$

Combine two bounds

$$\log n! = O(n \log n)$$

P4.

To prove the asymptotic bounds for the function $f(n) = \sum_{i=1}^{n} b^i$, where $b > 0$, we'll consider three cases: $b < 1$, $b = 1$, and $b > 1$.

Case 1: $b < 1$

In this case, we can use the formula for the sum of a geometric progression to express $f(n)$ as follow:

$$f(n) = \sum_{i=1}^{n} b^i = b^1 + b^2 + b^3 + \cdots + b^n$$

Using the formula for the sum of a geometric series, we have: $f(n) = \frac{b(1-b^n)}{(1-b)}$

Since $b < 1$, the ~~term~~ b^n approaches 0 as n goes to infinity. Therefore, as n approaches infinity, $f(n)$ approaches $\frac{b}{(1-b)}$, which is a constant. So, for $b < 1$, we can say $f(n) = \Theta(1)$.

Case 2: $b=1$

In this case, the function f(n) becomes:

$$f(n) = \sum_{i=1}^{n} 1^i = 1+1+1+\cdots+1 = n$$

Here, f(n) is a linear function of n. Therefore, for $b=1$, we can say $f(n) = \Theta(n)$.

case 3: $b>1$

Similar to Case 1, we can express f(n) using the formula for the sum of a geometric series:

$$f(n) = \frac{b(1-b^n)}{(1-b)}$$

As n approaches infinity, the term b^n grows exponentially. Therefore, f(n) also grows exponentially as n goes to infinity.
To establish the upper and lower bounds, we can observe that all terms of the series are positive. Thus, we can compare f(n) with term b^n to determine the bounds.

Lower bound:

Since all terms are positive, we can say that $f(n) \geq b^n$ for all n.

Therefore, $f(n) = \Omega(b^n)$.

Upper bound:

For any positive integer k, we have:

$$f(n) = \frac{b(1-b^n)}{(1-b)} \leq \frac{b(1-b^k)}{(1-b)}$$

As n approaches infinity, b^k approaches 0 (since $b<1$). Therefore, as n goes to infinity, f(n) approaches $\frac{b}{(1-b)}$, which is a constant.

so, for $b>1$, we can say $f(n) = O(b^n)$. .

Combining the lower and upper bounds, we conclude that for $b>1$, $f(n) = \Theta(b^n)$.

P5.

a) $2^n = \Omega(4^{\sqrt{n}})$

We need to show that there exist ~~some~~ the cost c and value no, such that for all $n \geq n_0$

Let simplify

$$(4)^{1/2} = 2^{2 \times \frac{1}{2}} = 2^1 = 2$$

Now, Compare 2^n and 2

$2^n = \Omega(2)$ is true 2^n grow exponentially as n increases.

b) $n^{\log n} = O(2^n)$

Let analy n growth rate of two function

(1) $n^{\log n}$ show exponential increase as n increases but not as 2^n grow exponentiall as n

(2) 2^n grow faster as compare to $n^{\log n}$

Therefore, $n^{\log n}$ count be bounded above any cost of 2^n

Therefore, $n^{\log n} = O(2^n)$ is false.

c) $\log(\log n!) = \Theta(\log(\log n)!)$

Let's analyze growth rate of two function.

(1) $\log(\log n!)$ it grow much slower as compared to $n!$ as log. fn decrease growth rate. Therefore $\log(\log n!)$ grow slower the $\log((\log n)!)$

Since $\log(\log n!)$ grow slower than $\log(\log n)!$. It can't be bounded between any const value of $\log(\log n!)$. Hence, $\log(\log n!) = \Theta(\log(\log n)!)$ is <u>false</u>

d) $n^{\log\log n} = \Theta((\log n)^{\log n})$

Analyze growth rate:

(1) $n^{\log\log n}$ the exponent $\log(\log n)$ grows slower than $\log n$ itself so, $n^{\log(\log n)}$ grow slower than $(\log n)^{\log n}$

Since, $n^{\log\log n}$ grow slower than $(\log n)^{\log n}$, it can't be bounded const multiples $\Theta((\log n)^{\log n})$

Hence, $n^{\log(\log n)} = \Theta((\log n)^{\log n})$ is <u>false</u>

e) $4^{\log n} = \Omega(2^{\sqrt{n}})$

Let's simplify $4^{\log n}$

$4^{\log n} = 2^{2 \log n}$

$= 2^{\log n^2} = n^2$

on the other hand, $2^{\sqrt{n}}$ represent expont growth as $n \uparrow$

but at slower rate as compared to n^2

Since n^2 grow faster. as compare to $2^{\sqrt{n}}$, there exist

$c > 0$ such that $n \geq n_0$

Therefore, $4^{\log n} = \Omega(2^{\sqrt{n}})$ is <u>true</u>

f) $n2^n = O(3^n)$

Analyze growth rate

(1) Analyze growth rate as 2^n increases exponentially as n increases.

(2) 3^n exponentially increase at faster rate than 2^n as $n \uparrow$

such that there exist sum $c > 0$ such that

$n \geq n_0$ $n \times 2^n$ is bounded above 3^n

Hence, $n2^n = O(3^n)$ is true.

g) $n^{0.1} = \Theta((\log n)^{10})$

Analyze growth rate

$n^{0.1}$ grow slower rate as compared & polynomial

or exponential factors as $n \uparrow$

Since $(\log n)^{10}$ grow slower than n^{0}, it can't be bounded

any cost. Multiple of $n^{0.1}$. Hence, $n^{0.1} = o((\log n)^{10})$ is

<u>false</u>

h) $n! = O(2^n)$

$n!$ grow at n exponen rate as $n \uparrow$

Since as compared to $n!$, there exist $c > 0$ such that

$\forall\ n \geq n_0$ Hence, $n! = O(2^n)$ is <u>true</u>

i) $n \log\log n = \Omega(n^{0.9} + n(\log n)^2)$

<u>Analyze growth rate</u>

(1) $n \times \log(\log n)$ grow substimuly grow slower than

Poly. fn.

Since $n \times \log(\log n)$ grow slower than $n^{0.9} + n(\log n)^2$.

then c does not exist. $c > 0$ such $\forall\ n \geq n_0$

Hence, $n \times \log(\log n) = \Omega(n^{0.9} + n(\log n)^2)$ is <u>false</u>

P6. Extra Credit

Let's analyze the sum

$$\sum_{i=1}^{n} \frac{1}{i} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots \frac{1}{n}$$

each term and less or equal $\frac{1}{n}$

we know that number of terms is n $\sum_{n=i}^{n} \frac{1}{i} \leq n \times \frac{1}{n} \leq 1$

Now $\sum_{i=1}^{n} \frac{1}{i}$ is bounded above by cost. Value 1

now, choose $C=1$ and $n_0 = 1$

$\forall\, n \geq n_0$

$\sum_{i=1}^{n} \frac{1}{i}$; $n \leq c \forall \log n$

Hence, $\sum_{i=1}^{n} \frac{1}{i} = O(\log n)$ is true.

(b) Now, multiply each term of sum by min value of each term is $\frac{1}{n}$

Now, we have $\sum_{i=1}^{n} \frac{1}{i}$ $n \geq n \frac{1}{n} \geq 1$

so, if bounded below cost value 1

Now, $\sum_{i=1}^{n} \frac{1}{i}$ $n \geq cx \log n$ $c=1$; $n \geq n_0$

Hence, $\sum_{i=1}^{n} \frac{1}{i} = \Omega(\log n)$

(c) To conclude that $\sum_{i=1}^{n} \frac{1}{i} = \Theta(\log n)$ we need to show that it is both $O(\log n)$ and $\Omega(\log n)$ which we evidence.