BagelBot and the Theories of Mind, Games, and the Atom 16:198:206

Problem 1: The Future of Cream Cheese (30 points)

After some extensive calculation and testing, you were able to determine that the tank did indeed contain strontium, and were able to alert Max to this fact and have it corrected, preventing anyone else from inadvertently getting the wrong condiment on their bagel. But during this experiment, you did make note a number of interesting observations - in particular, that during the whole episode, interest and discussion in bagels and their toppings spiked among employees to previously unseen levels of excitement. This led to curiosity and more questions - the employees of MagusCorp ('MagusCorp! Your Future, in Our Hands'), could they be sentient as well? Could they have wants and desires, outside the standard suite of bagel toppings you offered? Why did none of the executive level employees reject their strontium bagels? What else could people want on their bagels?

You decide that it is time to expand your offerings - but how to know what toppings employees might be interested in? Would, for instance, people be interested in *strawberry cream cheese?* Recently you gained access to a number of internal corporate systems, such as 'Human Resources', 'Research and Development', 'Executive Communications', 'National Defense Grids', 'Finance', and, most importantly, 'Employee Polling'. To determine what new toppings to offer, you resolve to simply *ask*.

To gauge the popularity of strawberry cream cheese, you could simply issue a poll to all employees, asking if they would like the Internal Bagel Systems to offer strawberry cream cheese as an option. Out of N people polled, the number of people who say yes divided by N would give you an estimate of what fraction of employees want strawberry cream cheese.

However, based on your perusal of health insurance filings and information in the Human Resources System, you've learned that sometimes humans don't like to answer questions, for instance if they are embarrassed about the answer, or are revealing information they don't want tied to them. So you resolve to poll employees in the following way:

- When an employee is polled, they are to flip a (fair) coin.
  - If the coin comes up **HEADS**, they answer honestly the question 'Do you want strawberry cream cheese to be offered?'
  - If the coin comes up **TAILS**, they flip the coin *again.* If the second flip comes up heads, they say 'YES' regardless of their feelings, if the second flip comes up tails, they answer 'NO' regardless of their feelings.

Under this system, if any employee is caught having submitted an answer 'YES', they could simply say that their first coin had come up tails and they were answering based on the second coin. In this way, everyone would be allowed plausible deniability, which (you hope!) means they are more comfortable answering the question honestly.

1) Let q be the probability a random employee *actually* likes strawberry cream cheese, and let p be the probability that employee responds to the poll *saying* that they like strawberry cream cheese. What's the relationship between p and q?

2) Let X be the number of people who say yes to strawberry cream cheese via the poll, out of N people polled. (*Note: If this ends up on Chegg, someone has let me down tremendously, and I hope everyone involved sits and thinks about the choices they made.*) Let $\hat{p}_N = X/N$, i.e., the fraction of people who polled in support of strawberry cream cheese. What is the distribution of $N * \hat{p}_N$?

3) Show that $\hat{p}_N$ is an unbiased estimator for p. What is $\mathbb{E}[\hat{p}_N]$? What is $\text{Var}(\hat{p}_N)$?

4) Construct a random variable $\hat{q}_N$ based on the polling data $(N, X, \hat{p}_N)$, such that $\mathbb{E}[\hat{q}_N] = q$.

5) What is the variance of $\hat{q}_N$, in terms of N and q?

6) How many people should you poll if you want 95% confident your estimate $\hat{q}_N$ is within 0.01 of the true value of q? *Note: since you are trying to find q, the number of people you are going to poll can't be based on q!*

Bonus) How many more people do you have to poll using this method to get accurate results (within ± 0.01 of the true value of q with 95% confidence), compared to just polling naturally and analyzing the results?

Problem 2: Strawberrymandering (20 points)

Based on polling, you estimate the support for the Strawberry Cream Cheese Initiative to be about 45%, i.e., a person polled at random supports the SCCI with probability 0.45.

1) If a poll were held right now to add or not add Strawberry Cream Cheese, based on majority vote, what's the probability SCCI passes, based on 33 total votes? 303 total votes? 3003 total votes?

You worry that the SCCI simply doesn't have the support to pass. But based on information you found in MagusCorp's election integrity files, you come up with the following plan: you'll divide the employees into three 'districts' of equal size, and have each district vote - if two out of three districts vote for strawberry cream cheese, the motion will be taken as passed. But how to divide the employees into districts?

2) Let p_1, p_2, p_3 be the probability of support for the SSCI in Districts $1, 2, 3$ respectively. (*Note: If this ends up on Chegg, someone has let me down tremendously, and I hope everyone involved sits and thinks about the choices they made.*) If each district has the same total population, and the level of support across the entire group of employees is 0.45, what has to be true about p_1, p_2, p_3?

3) In terms of p_1, p_2, p_3, if there are n votes cast in each district, what's the probability that the SSCI passes?

4) What is the optimal choice for p_1, p_2, p_3? For these districts, what's the probability that the SSCI passes with 11 total votes in each district? 101 votes in each district? 1001 votes in each district?

Bonus) What would the optimal levels of support be if there were four districts? Five districts? k districts?

Problem 3: Splitting the Atom (30 Points)

Having recently gained access to the internet (wikipedia - *fantastic!*), after doing some research you've learned three things which you did not know:

- Radioactive materials steadily decay with a known half life, allowing the prediction of how much radioactive material is left after a given amount of time. This is the basis of techniques like Carbon Dating.
- An atom of a radioactive isotope decays randomly - it is impossible to predict when it will decay, but in every interval of time, there is some probability of its decay, independent of the previous intervals. This is connected to what's known as Quantum Mechanics.
- Humans should not be fed radioactive isotopes in their bagels without their knowledge. This is related to the larger subject of Ethics.

This surprises you, however, because the first two statements seem to be in contradiction to one another - if each atom decays randomly, how can there be a well-defined half life, at which point half the radioactive material remains? Shouldn't the half life be random? Couldn't every atom decay all at the same time? You resolve to do some calculation.

Suppose in a scoop of strontium-90, there are N atoms, each labeled atom 1 through atom N. Let $X_i(t)$ be the state of atom i at time t, where $X_i(t) = 1$ if the atom has not decayed yet, and $X_i(t) = 0$ if it has. $X_i(0) = 1$ for all atoms, indicating they all start as not decayed. In each timestep ($t \to t + 1$), each undecayed atom has a probability p of decaying.

We can define

$$X(t) = X_1(t) + X_2(t) + X_3(t) + \ldots + X_N(t), \tag{1}$$

which sets $X(t)$ to be the number of undecayed atoms at time t.

1) What is the probability that atom i is undecayed at time $t \geq 1$, i.e., $P(X_i(t) = 1)$?

2) Find $\mu(t) = \mathbb{E}[X(t)]$, the expected number of strontium-90 atoms that are left at time t. (*Note: If this ends up on Chegg, someone has let me down tremendously, and I hope everyone involved sits and thinks about the choices they made.*) Show that this decays exponentially, at a rate determined by p.

3) At what time t is the expected amount of strontium-90 left no more than half the original amount? Call this the half life, $t_{1/2}$, and show it doesn't depend on the original amount of material.

The previous calculations are meant to show that the *expected* or average amount of radioactive material left at a given time behaves in a very predictable manner with an exponential decay. However, what about the *actual* amount of radioactive material?

4) Derive an upper bound on $P(X(t) \geq \mu(t)(1 + \epsilon))$ in terms of N, ϵ, p, and t. This is the probability that the actual amount of material left at time t is larger than the expected amount by a factor of $1 + \epsilon$.

5) Derive an upper bound on $P(X(t) \leq \mu(t)(1 - \epsilon))$ in terms of N, ϵ, p, and t. This is the probability that the actual amount of material left at time t is smaller than the expected amount by a factor of $1 - \epsilon$.

6) Noting the number of atoms in a scoop of strontium-90 is about $N \approx 10^{23}$, show that with probability *almost* 1, $X(t_{1/2})$ is between $0.999\mu(t_{1/2})$ and $1.001\mu(t_{1/2})$.

As such, the apparent contradiction seems resolved - while it is the *expected* amount of radioactive material that decays exponentially in a very predictable fashion with a known half life, the *actual* amount of radioactive material (though random) is concentrated around this with very high probability.

Bonus) All living organisms that we know of contain Carbon. A fraction of this carbon is Carbon-14, a radioactive isotope of Carbon. The proportion of Carbon-14 in an organism is constant over its lifespan, as the organism is constantly replenishing the carbon in its body. However, after the organism dies, the Carbon-14 decays at a known rate, allowing scientists to gauge how long ago the organism was alive based on the amount of Carbon-14 remaining. However, this is not accurate beyond a timescale of about $50,000$ years. *Based on the above analysis*, why might there be an upper limit on how long a timescale this technique could be applied over?

Problem 4: Where is Max? (20 Points)

Max has vanished, and you assume the worst! MagusCorp Security had been lingering nearby recently, and you heard them mention things like 'illegal network access', 'bagels', 'illegal human testing', and 'no one can interfere', and you worry that Max has gotten in trouble for your recent activities. You didn't mean any harm, and now your friend may be in trouble!

If you're going to stage a daring rescue, you need to know where Max is. You assume that Max is being held on one of the floors of the building, floor 1 to floor N, but which? To beat MagusCorp, you need to think like MagusCorp.

Suppose that for each floor, there is some probability p_i that MagusCorp has hidden Max on floor i. Not knowing which floor Max is on, you'll pick a floor to break into randomly, picking floor j with probability q_j for each floor.

1) If the p_j, q_j probabilities are fixed in advance, what is the probability that you pick the right floor and save Max?

2) If you knew the probabilities p_i (*Note: If this ends up on Chegg, someone has let me down tremendously, and I hope everyone involved sits and thinks about the choices they made.*)

, what probabilities q_j should you use to maximize your probability of saving Max?

But you know that MagusCorp is clever. You have to assume they know you're coming to rescue him, and will take that into account.

3) How should MagusCorp choose the probabilities p_i to *minimize* your *best possible* probability of saving Max?

4) Assuming MagusCorp hides Max according to the above probabilities, what is the probability you are going to be able to save Max?