# Hartshorne Solutions

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## 1.1 Affine Varieties

### 1.8

Let $Y$ be an affine variety of dimension $r$ in $\mathbb{A}^n$. Let $H$ be a hypersurface in $\mathbb{A}^n$, and assume that $Y \not\subseteq H$. Then every irreducible component of $Y \cap H$ has dimension $r-1$.

### 1.8 mlwells

Suppose that $H = Z(f)$ with $f$ irreducible in $k[x_1, \ldots, x_n]$. Then the projection of $f$ in $A(Y) := k[x_1, \ldots, x_n]/I(Y)$ is not equal to $\overline{0}$ since by assumption $(f) \not\subset I(Y)$. Since $A(Y)$ is a domain (due to the irreducibility of $Y$), the element $\overline{f}$ is not a zero divisor. Assuming that $Y \cap H \neq \varnothing$, we have that $\overline{f}$ is not a unit in $A(Y)$. To see this, let $P \in Y \cap H$. Then $I(Y), (f) \subset \mathfrak{m}_P$, the maximal ideal of $k[x_1, \ldots, x_n]$ corresponding to $P$. This implies $(\overline{f}) \subset \mathfrak{m}_P/I(Y)$, the latter being a maximal ideal in $A(Y)$. Thus, $\overline{f}$ is not a unit.
We apply Theorem 1.11A to get that every minimal prime ideal $\mathfrak{p}$ in $A(Y)$ containing $\overline{f}$ has height 1. The irreducible components of $Y \cap H$ and the minimal prime ideals containing $(\overline{f})$ correspond, and since each of these prime ideals has height 1, the corresponding varieties have dimension $r-1$ by Theorem 1.8A.

## 2.1 Sheaves

### 1.2

a) For any morphism of sheaves $\phi : \mathscr{F} \to \mathscr{G}$, show that for each point $P$, $(\ker \phi)_P = \ker(\phi_P)$ and $(\operatorname{im} \phi)_P = \operatorname{im}(\phi_P)$.

b) Show that $\phi$ is injective (surjective) iff induced map on stalks $\phi_P$ is injective (surjective) for all $P$.

c) Show that the sequence of sheaves $\cdots \longrightarrow \mathscr{F}^{i-1} \longrightarrow \mathscr{F}^i \longrightarrow \mathscr{F}^{i+1} \longrightarrow \cdots$ is exact iff coresponding sequences of stalks are exact for all $P$.

### 1.2 yakimk

a) Since stalks are defined as a filtered colimit and kernels are a particular type of limit (pullback) and filtered colimits commute with finite limits isomorphism $(\ker \phi_P) = \ker(\phi_P)$ is automatic.

To show a similar thing for the image we first note that $(\operatorname{im} \phi)_P \simeq ((\operatorname{im}_{pre} \phi)^+)_P \simeq (\operatorname{im}_{pre} \phi)_P$, where the first isomorphism is by definition and the second one using the fact that sheafification has the same stalks as original presheaf. Now we conclude noticing that $\operatorname{im}_{pre} \phi \simeq \ker(\operatorname{coker} \phi)$, i.e. it is a finite limit of finite colimits and since again stalks are filtered colimits they commute with finite limits and colimits (in for instance abelian category) we conclude that $(\operatorname{im} \phi)_P \simeq \operatorname{im}(\phi_P)$.

b) We show that a map of sheaves $\phi : \mathscr{F} \to \mathscr{G}$ is injective iff it induced maps on stalks are injective (proof for surjectivity is essentially identical).

Consider the following commutative diagram

$$\begin{tikzcd}
\mathscr{F}(U) \arrow[r, "\phi_U"] \arrow[d, "\iota_1"'] & \mathscr{G}(U) \arrow[d, "\iota_2"] \\
\prod_{P \in U} \mathscr{F}_p \arrow[r, "\prod \tilde{\phi}_P"] & \prod_{P \in U} \mathscr{G}_p
\end{tikzcd}$$

Note that since section of a sheaf on an open set is determined by its stalks, $\iota_1$ and $\iota_2$ are both injective.

Suppose $\tilde{\phi}_P$ is injective for all $P$. Take two sections $f, g \in \mathscr{F}(U)$, since $\prod \tilde{\phi}_P$ and $\iota_1$ are injective (former by the hypothesis that $\tilde{\phi}_P$ are injective), their composition is injective. If $\phi_U$ were not injective, we could find two different sections that go to the same class in $\prod \mathscr{G}_P$, which would contradict injectivity of $\prod \tilde{\phi}_P \circ \iota_1$.

Similarly if $\phi_U$ is injective then if some of $\tilde{\phi}_P$ were not injective, its composition with $\iota_1$ would not be injective contradicting commutativity of the diagram above.

c) We have to show that $\cdots \longrightarrow \mathscr{F}^{i-1} \longrightarrow \mathscr{F}^i \longrightarrow \mathscr{F}^{i+1} \longrightarrow \cdots$ is exact iff induced sequence of stalks are exact for all $P$.

Assuming that sequence of sheaves is exact we show induced sequences are exact at all stalks. By the previuos section $\operatorname{im} \phi^i_P \simeq (\operatorname{im} \phi^i)_P \simeq (\ker \phi^{i+1})_P \simeq \ker \phi^{i+1}_P$.

Now assume that all sequences of stalks are exact. By the same reasoning as before we see that $\operatorname{im} \phi^i \simeq \ker \phi^{i+1}$ (since they are isomorphic on each stalk and hence are equal).