Hartshorne Solutions

mlwells

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Contents

1.1 Affine Varieties

1.8

Let Y be an affine variety of dimension r in $\mathbb{A}^n$. Let H be a hypersurface in $\mathbb{A}^n$, and assume that $Y \not\subseteq H$. Then every irreducible component of $Y \cap H$ has dimension $r-1$.

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Suppose that $H = Z(f)$ with f irreducible in $k[x_1, \ldots, x_n]$. Then the projection of f in $A(Y) := k[x_1, \ldots, x_n]/I(Y)$ is not equal to $\overline{0}$ since by assumption $(f) \not\subset I(Y)$. Since $A(Y)$ is a domain (due to the irreducibility of Y), the element $\overline{f}$ is not a zero divisor. Assuming that $Y \cap H \neq \varnothing$, we have that $\overline{f}$ is not a unit in $A(Y)$. To see this, let $P \in Y \cap H$. Then $I(Y), (f) \subset \mathfrak{m}_P$, the maximal ideal of $k[x_1, \ldots, x_n]$ corresponding to P. This implies $(\overline{f}) \subset \mathfrak{m}_P/I(Y)$, the latter being a maximal ideal in $A(Y)$. Thus, $\overline{f}$ is not a unit.
We apply Theorem 1.11A to get that every minimal prime ideal $\mathfrak{p}$ in $A(Y)$ containing $\overline{f}$ has height 1. The irreducible components of $Y \cap H$ and the minimal prime ideals containing $(\overline{f})$ correspond, and since each of these prime ideals has height 1, the corresponding varieties have dimension $r-1$ by Theorem 1.8A.

1.2 Projective Varieties

2.12

The d-Uple Embedding. For given $n, d > 0$, let $M_0, M_1, \ldots, M_N$ be all the monomials of degree d in the $n+1$ variables $x_0, \ldots, x_n$, where $N = \binom{n+d}{n} - 1$. We define a mapping $\rho_d \colon \mathbb{P}^n \to \mathbb{P}^N$ by sending the point $P = (a_0, \ldots, a_n)$ to the point $\rho_d(P) = (M_0(a), \ldots, M_N(a))$ obtained by substituting the a_i in the monomials M_j. This is called the d-uple *embedding* of $\mathbb{P}^n$ in $\mathbb{P}^N$. For example, if $n = 1, d = 2$, then $N = 2$, and the image Y of the 2-uple embedding of $\mathbb{P}^1$ in $\mathbb{P}^2$ is a conic.

a) Let $\theta \colon k[y_0, \ldots, y_N] \to k[x_0, \ldots, x_n]$ be the homomorphism defined by sending y_i to M_i, and let $\mathfrak{a}$ be the kernel of θ. Then $\mathfrak{a}$ is a homogeneous prime ideal, and so $Z(\mathfrak{a})$ is a projective variety in $\mathbb{P}^N$.

b) Show that the image of ρ_d is exactly $Z(\mathfrak{a})$. (One inclusion is easy. The other will require some calculation.)

c) Now show that ρ_d is a homeomorphism of $\mathbb{P}^n$ onto the projective variety $Z(\mathfrak{a})$.

d) Show that the twisted cubic curve in $\mathbb{P}^3$ (Ex. 2.9) is equal to the 3-uple embedding of $\mathbb{P}^1$ in $\mathbb{P}^3$, for suitable choice of coordinates.

2.12 (b) mlwells

Let $Q = \rho_d(P) = (M_0(a), \ldots, M_N(a))$. Let $F \in \mathfrak{a} = \ker \theta$. Then $F(M_0, \ldots, M_N) = 0$ as a polynomial, which implies $F(M_0(a), \ldots, M_N(a)) = 0$. Since F was arbitrary, $Q \in Z(\mathfrak{a})$.
Now let $Q = (y_0, \ldots, y_N) \in Z(\mathfrak{a})$. Suppose without loss of generality that $y_0 \neq 0$. It follows that $Q = (y_0^d, y_0^{d-1}y_1, \ldots, y_0^{d-1}y_N)$. Suppose that $M_0 = x_0^d, M_1 = x_0^{d-1}x_1, \ldots, M_n = x_0^{d-1}x_n$.

Proposition 1.1. *For all $j = 0, \ldots, N$, we have that*

$$M_0^{d-1}M_j = M_0^{i_0}M_1^{i_1}\ldots M_n^{i_n} \tag{1}$$

for some $i_j \geq 0$ with $\sum_j i_j = d$.

Proof. Suppose $M_j = x_0^{i_0} \dots x_n^{i_n}$ where $\sum_j i_j = d$. We have

$$\begin{aligned} M_0^{d-1} M_j &= x_0^{(d-1)d+i_0} x_1^{i_1} \dots x_n^{i_n} && (2)\\ &= x_0^{(d-1)i_0+i_0+(d-1)\sum_{j\geq 1} i_j} x_1^{i_1} \dots x_n^{i_n} && (3)\\ &= x_0^{di_0}(x_0^{d-1}x_1)^{i_1} \dots (x_0^{d-1}x_n)^{i_n} && (4)\\ &= M_0^{i_0} M_1^{i_1} \dots M_n^{i_n} && (5) \end{aligned}$$

□

Let $P = (y_0, \dots, y_n)$. Then, by the proposition, we have that $Q = (M_0(P), M_1(P), \dots, M_N(P)) = \rho_d(P)$, as was to be shown.

1.3 Morphisms

3.20

Let Y be a variety of dimension ≥ 2, and let $P \in Y$ be a normal point. Let f be a regular function on $Y - P$.

a) Show that f extends to a regular function on Y.

b) Show this would be false for rm dim $Y = 1$.

See (III, Ex. 3.5) for generalization.

3.20 (a) mlwells

If we show that f restricted to $V - P$ extends to a regular function on $V \subseteq Y$ for any affine neighborhood V of P, then by gluing we will have shown that f extends to a regular function on Y. So, assume that Y is affine.
The function $f = c/d \in K(Y)$ for regular functions $c, d \in A(Y)$. If we can show that the ideal quotient $((c) : (d)) = \{h \in A(Y) \colon hc \in (d)\}$ is equal to $A(Y)$, then it follows that $c/d \in A(Y)$. By assumption, $f \in \mathcal{O}_Q$ for all $Q \neq P \in Y$. Let $\mathfrak{m}_Q$ denote the maximal ideal of $A(Y)$ corresponding to Q. Then the localized ideal $((c) : (d))_{\mathfrak{m}_Q}$ is equal to $A(Y)_{\mathfrak{m}_Q} = \mathcal{O}_Q$ for all $Q \neq P$ since $c/d \in \mathcal{O}_Q$. It remains to show that $((c) : (d))_{\mathfrak{m}_P} = A(Y)_{\mathfrak{m}_P} = \mathcal{O}_P$. If $\mathfrak{p}A(Y)_{\mathfrak{m}_P}$ is a prime ideal of $\mathcal{O}_P$ not equal to the maximal ideal $\mathfrak{m}_P A(Y)_{\mathfrak{m}_P}$, then $((c) : (d))_{\mathfrak{m}_P} \not\subseteq \mathfrak{p}A(Y)_{\mathfrak{m}_P}$ since c/d is in the local ring corresponding to the subvariety defined by $\mathfrak{p}$, and hence $((c) : (d))_{\mathfrak{p}} = A(Y)_{\mathfrak{p}}$. . Assume by way of contradiction that $((c) : (d))_{\mathfrak{m}_P} \subseteq \mathfrak{m}_P A(Y)_{\mathfrak{m}_P}$. Then, $\sqrt{((c) : (d))_{\mathfrak{m}_P}} = \mathfrak{m}_P$.
Let $a_1, \dots, a_s$ be a regular sequence in $\mathfrak{m}_P A(Y)_{\mathfrak{m}_P}$ with $s > 1$. By Theorem 8.22A of Chapter II of Hartshorne part (2), such a sequence exists. We have shown that $a_1^{r_1}, a_2^{r_2} \in ((c) : (d))_{\mathfrak{m}_P}$ for some $r_1, r_2 > 0$ with $a_1^{r_1-1}, a_2^{r_2-1} \notin ((c) : (d))_{\mathfrak{m}_P}$. Thus,

$$a_1^{r_1} c = e_1 d, \quad e_1 \notin (a_1) \tag{6}$$

and

$$a_2^{r_2} c = e_2 d, \quad e_2 \notin (a_2) \tag{7}$$

Thus,

$$a_1^{r_1} a_2^{r_2} c = e_1 a_2^{r_2} d = e_2 a_1^{r_1} d \tag{8}$$

which implies

$$e_1 a_2^{r_2} = e_2 a_1^{r_1} \tag{9}$$

Since $e_1 \notin (a_1)$ and $a_2 \notin (a_1)$, a_2 is a zero divisor in the ring $A(Y)_{\mathfrak{m}_P}/(a_1)$. This contradicts the fact that a_1, a_2 is a regular sequence in $\mathfrak{m}_P A(Y)_{\mathfrak{m}_P}$.
Thus, we conclude that $((c):(d))_{\mathfrak{m}_P} = A(Y)_{\mathfrak{m}_P}$, and hence $f \in \mathcal{O}_P$. This shows that $((c):(d)) = A(Y)$ which implies $f \in A(Y)$, and hence f extends to a regular function on Y.

2.1 Sheaves

1.2

a) For any morphism of sheaves $\phi : \mathscr{F} \to \mathscr{G}$, show that for each point P, $(\ker \phi)_P = \ker(\phi_P)$ and $(\operatorname{im} \phi)_P = \operatorname{im}(\phi_P)$.

b) Show that ϕ is injective (surjective) iff induced map on stalks ϕ_P is injective (surjective) for all P.

c) Show that the sequence of sheaves $\cdots \longrightarrow \mathscr{F}^{i-1} \longrightarrow \mathscr{F}^i \longrightarrow \mathscr{F}^{i+1} \longrightarrow \cdots$

1.2 yakimk

a) Since stalks are defined as a filtered colimit and kernels are a particular type of limit (pullback) and filtered colimits commute with finite limits isomorphism $(\ker \phi_P) = \ker(\phi_P)$ is automatic.

To show a similar thing for the image we first note that $(\operatorname{im} \phi)_P \simeq ((\operatorname{im}_{pre} \phi)^+)_P \simeq (\operatorname{im}_{pre} \phi)_P$, where the first isomorphism is by definition and the second one using the fact that sheafification has the same stalks as original presheaf. Now we conclude noticing that $\operatorname{im}_{pre} \phi \simeq \ker(\operatorname{coker} \phi)$, i.e. it is a finite limit of finite colimits and since again stalks are filtered colimits they commute with finite limits and colimits (in for instance abelian category) we conclude that $(\operatorname{im} \phi)_P \simeq \operatorname{im}(\phi_P)$.

b) We show that a map of sheaves $\phi : \mathscr{F} \to \mathscr{G}$ is injective iff it induced maps on stalks are injective (proof for surjectivity is essentially identical).

Consider the following commutative diagram

$$\begin{tikzcd} \mathscr{F}(U) \arrow[r, "\phi_U"] \arrow[d, "\iota_1"'] & \mathscr{G}(U) \arrow[d, "\iota_2"] \\ \prod_{P\in U} \mathscr{F}_p \arrow[r, "\prod \tilde{\phi}_P"] & \prod_{P\in U} \mathscr{G}_p \end{tikzcd}$$

Note that since section of a sheaf on an open set is determined by its stalks, ι_1 and ι_2 are both injective.

Suppose $\tilde{\phi}_P$ is injective for all P. Take two sections $f, g \in \mathscr{F}(U)$, since $\prod \tilde{\phi}_P$ and ι_1 are injective (former by the hypothesis that $\tilde{\phi}_P$ are injective), their composition is injective. If ϕ_U were not injective, we could find two different sections that go to the same class in $\prod \mathscr{G}_P$, which would contradict injectivity of $\prod \tilde{\phi}_P \circ \iota_1$.

Similarly if ϕ_U is injective then if some of $\tilde{\phi}_P$ were not injective, its composition with ι_1 would not be injective contradicting commutativity of the diagram above.

c) We have to show that $\cdots \longrightarrow \mathscr{F}^{i-1} \longrightarrow \mathscr{F}^i \longrightarrow \mathscr{F}^{i+1} \longrightarrow \cdots$ is exact iff induced sequence of stalks are exact for all P.

Assuming that sequence of sheaves is exact we show induced sequences are exact at all stalks. By the previuos section $\operatorname{im} \phi_P^i \simeq (\operatorname{im} \phi^i)_P \simeq (\ker \phi^{i+1})_P \simeq \ker \phi_P^{i+1}$.

Now assume that all sequences of stalks are exact. By the same reasoning as before we see that $\operatorname{im} \phi^i \simeq \ker \phi^{i+1}$ (since they are isomorphic on each stalk and hence are equal).

1.8

For any open set $U \subseteq X$ the functor $\Gamma(U, \cdot)$ from sheaves on X to abelian groups is left-exact.

1.8 yakimk

Given exact sequence of sheaves

$$0 \longrightarrow \mathscr{F}' \xrightarrow{\phi} \mathscr{F} \xrightarrow{\gamma} \mathscr{F}'' \longrightarrow 0$$

We have to show that

$$0 \longrightarrow \Gamma(U, \mathscr{F}') \xrightarrow{\phi_U} \Gamma(U, \mathscr{F}) \xrightarrow{\gamma_U} \Gamma(U, \mathscr{F}'')$$

exact.
Injectivity of ϕ_U follows from definition.
We show that $\ker \gamma_U = \operatorname{im} \phi_U$.
($\ker \gamma_U \supseteq \operatorname{im} \phi_U$) Is immidiate by functoriality of $\Gamma(U, \cdot)$ and hypothesis.
($\ker \gamma_U \subseteq \operatorname{im} \phi_U$) Let $s \in \mathscr{F}(U)$ and $\gamma_U(s) = 0$. By hypothesis and "1.2" we know that the induced sequences of stalks are all exact. In particular for all $P \in U$ induced sequences are all exact. Which means that each for stalk $s_P = \phi_P(k_P)$, i.e. there is some section $k \in \mathscr{F}'(U)$ and an open set V_P such that $s|_{V_P} = \phi(k)|_{V_P}$. Hence we can form an open cover $U = \bigcup_{P \in U} V_P$. Notice that by definition they agree on overlaps and hence we get a full section of $\mathscr{F}'$ on U (call it $q \in \mathscr{F}'(U)$) and since all the stalks of $\phi_P(q_P) = s_P$ for all $P \in U$ and hence are equal on U.