Hartshorne Solutions

mlwells

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1.1 Affine Varieties

1.8

Let Y be an affine variety of dimension r in $\mathbb{A}^n$. Let H be a hypersurface in $\mathbb{A}^n$, and assume that $Y \not\subseteq H$. Then every irreducible component of $Y \cap H$ has dimension $r-1$.

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Suppose that $H = Z(f)$ with f irreducible in $k[x_1, \ldots, x_n]$. Then the projection of f in $A(Y) := k[x_1, \ldots, x_n]/I(Y)$ is not equal to $\overline{0}$ since by assumption $(f) \not\subset I(Y)$. Since $A(Y)$ is a domain (due to the irreducibility of Y), the element $\overline{f}$ is not a zero divisor. Assuming that $Y \cap H \neq \varnothing$, we have that $\overline{f}$ is not a unit in $A(Y)$. To see this, let $P \in Y \cap H$. Then $I(Y), (f) \subset \mathfrak{m}_P$, the maximal ideal of $k[x_1, \ldots, x_n]$ corresponding to P. This implies $(\overline{f}) \subset \mathfrak{m}_P/I(Y)$, the latter being a maximal ideal in $A(Y)$. Thus, $\overline{f}$ is not a unit.
We apply Theorem 1.11A to get that every minimal prime ideal $\mathfrak{p}$ in $A(Y)$ containing $\overline{f}$ has height 1. The irreducible components of $Y \cap H$ and the minimal prime ideals containing $(\overline{f})$ correspond, and since each of these prime ideals has height 1, the corresponding varieties have dimension $r-1$ by Theorem 1.8A.