

# Hartshorne Solutions

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2025

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## Contents

<b>1.1 Affine Varieties</b>	<b>2</b>
1.8 . . . . .	2
<b>2.1 Sheaves</b>	<b>3</b>
1.2 . . . . .	3
1.8 . . . . .	3

## 1.1 Affine Varieties

### 1.8

Let  $Y$  be an affine variety of dimension  $r$  in  $\mathbb{A}^n$ . Let  $H$  be a hypersurface in  $\mathbb{A}^n$ , and assume that  $Y \not\subseteq H$ . Then every irreducible component of  $Y \cap H$  has dimension  $r - 1$ .

### 1.8 mlwells

Suppose that  $H = Z(f)$  with  $f$  irreducible in  $k[x_1, \dots, x_n]$ . Then the projection of  $f$  in  $A(Y) := k[x_1, \dots, x_n]/I(Y)$  is not equal to  $\bar{0}$  since by assumption  $(f) \not\subseteq I(Y)$ . Since  $A(Y)$  is a domain (due to the irreducibility of  $Y$ ), the element  $\bar{f}$  is not a zero divisor. Assuming that  $Y \cap H \neq \emptyset$ , we have that  $\bar{f}$  is not a unit in  $A(Y)$ . To see this, let  $P \in Y \cap H$ . Then  $I(Y), (f) \subset \mathfrak{m}_P$ , the maximal ideal of  $k[x_1, \dots, x_n]$  corresponding to  $P$ . This implies  $(\bar{f}) \subset \mathfrak{m}_P/I(Y)$ , the latter being a maximal ideal in  $A(Y)$ . Thus,  $\bar{f}$  is not a unit.

We apply Theorem 1.11A to get that every minimal prime ideal  $\mathfrak{p}$  in  $A(Y)$  containing  $\bar{f}$  has height 1. The irreducible components of  $Y \cap H$  and the minimal prime ideals containing  $(\bar{f})$  correspond, and since each of these prime ideals has height 1, the corresponding varieties have dimension  $r - 1$  by Theorem 1.8A.

## 2.1 Sheaves

### 1.2

- For any morphism of sheaves  $\phi : \mathcal{F} \rightarrow \mathcal{G}$ , show that for each point  $P$ ,  $(\ker \phi)_P = \ker(\phi_P)$  and  $(\operatorname{im} \phi)_P = \operatorname{im}(\phi_P)$ .
- Show that  $\phi$  is injective (surjective) iff induced map on stalks  $\phi_P$  is injective (surjective) for all  $P$ .
- Show that the sequence of sheaves  $\cdots \rightarrow \mathcal{F}^{i-1} \rightarrow \mathcal{F}^i \rightarrow \mathcal{F}^{i+1} \rightarrow \cdots$

### 1.2 yakimk

- Since stalks are defined as a filtered colimit and kernels are a particular type of limit (pullback) and [filtered colimits commute with finite limits](#) isomorphism  $(\ker \phi)_P = \ker(\phi_P)$  is automatic.

To show a similar thing for the image we first note that  $(\operatorname{im} \phi)_P \simeq ((\operatorname{im}_{pre} \phi)^+)_P \simeq (\operatorname{im}_{pre} \phi)_P$ , where the first isomorphism is by definition and the second one using the fact that sheafification has the same stalks as original presheaf. Now we conclude noticing that  $\operatorname{im}_{pre} \phi \simeq \ker(\operatorname{coker} \phi)$ , i.e. it is a finite limit of finite colimits and since again stalks are filtered colimits they commute with finite limits and colimits (in for instance abelian category) we conclude that  $(\operatorname{im} \phi)_P \simeq \operatorname{im}(\phi_P)$ .

- We show that a map of sheaves  $\phi : \mathcal{F} \rightarrow \mathcal{G}$  is injective iff it induced maps on stalks are injective (proof for surjectivity is essentially identical).

Consider the following commutative diagram

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\phi_U} & \mathcal{G}(U) \\ \downarrow \iota_1 & & \downarrow \iota_2 \\ \prod_{P \in U} \mathcal{F}_P & \xrightarrow{\prod \tilde{\phi}_P} & \prod_{P \in U} \mathcal{G}_P \end{array}$$

Note that since section of a sheaf on an open set is determined by its stalks,  $\iota_1$  and  $\iota_2$  are both injective.

Suppose  $\tilde{\phi}_P$  is injective for all  $P$ . Take two sections  $f, g \in \mathcal{F}(U)$ , since  $\prod \tilde{\phi}_P$  and  $\iota_1$  are injective (former by the hypothesis that  $\tilde{\phi}_P$  are injective), their composition is injective. If  $\phi_U$  were not injective, we could find two different sections that go to the same class in  $\prod \mathcal{G}_P$ , which would contradict injectivity of  $\prod \tilde{\phi}_P \circ \iota_1$ .

Similarly if  $\phi_U$  is injective then if some of  $\tilde{\phi}_P$  were not injective, its composition with  $\iota_1$  would not be injective contradicting commutativity of the diagram above.

- We have to show that  $\cdots \rightarrow \mathcal{F}^{i-1} \rightarrow \mathcal{F}^i \rightarrow \mathcal{F}^{i+1} \rightarrow \cdots$  is exact iff induced sequence of stalks are exact for all  $P$ .

Assuming that sequence of sheaves is exact we show induced sequences are exact at all stalks. By the previous section  $\operatorname{im} \phi_P^i \simeq (\operatorname{im} \phi^i)_P \simeq (\ker \phi^{i+1})_P \simeq \ker \phi_P^{i+1}$ .

Now assume that all sequences of stalks are exact. By the same reasoning as before we see that  $\operatorname{im} \phi^i \simeq \ker \phi^{i+1}$  (since they are isomorphic on each stalk and hence are equal).

### 1.8

For any open set  $U \subseteq X$  the functor  $\Gamma(U, \cdot)$  from sheaves on  $X$  to abelian groups is left-exact.

## 1.8 yakimk

Given exact sequence of sheaves

$$0 \rightarrow \mathcal{F}' \xrightarrow{\phi} \mathcal{F} \xrightarrow{\gamma} \mathcal{F}'' \rightarrow 0$$

We have to show that

$$0 \rightarrow \Gamma(U, \mathcal{F}') \xrightarrow{\phi_U} \Gamma(U, \mathcal{F}) \xrightarrow{\gamma_U} \Gamma(U, \mathcal{F}'')$$

exact.

Injectivity of  $\phi_U$  follows from definition.

We show that  $\ker \gamma_U = \operatorname{im} \phi_U$ .

( $\ker \gamma_U \supseteq \operatorname{im} \phi_U$ ) Is immediate by functoriality of  $\Gamma(U, \cdot)$  and hypothesis.

( $\ker \gamma_U \subseteq \operatorname{im} \phi_U$ ) Let  $s \in \mathcal{F}(U)$  and  $\gamma_U(s) = 0$ . By hypothesis and “1.2” we know that the induced sequences of stalks are all exact. In particular for all  $P \in U$  induced sequences are all exact. Which means that each for stalk  $s_P = \phi_P(k_P)$ , i.e. there is some section  $k \in \mathcal{F}'(U)$  and an open set  $V_P$  such that  $s|_{V_P} = \phi(k)|_{V_P}$ . Hence we can form an open cover  $U = \bigcup_{P \in U} V_P$ . Notice that by definition they agree on overlaps and hence we get a full section of  $\mathcal{F}'$  on  $U$  (call it  $q \in \mathcal{F}'(U)$ ) and since all the stalks of  $\phi_P(q_P) = s_P$  for all  $P \in U$  and hence are equal on  $U$ .