Hartshorne Solutions

mlwells

2025

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Contents

1.1 Affine Varieties

1.1

a) Let Y be the plane curve $y = x^2$ (i.e., Y is the zero set of the polynomial $f = y - x^2$). Show that $A(Y)$ is isomorphic to a polynomial ring in one variable over k.

b) Let Z be the plane curve $xy = 1$. Show that $A(Z)$ is not isomorphic to a polynomial ring in one variable over k.

c) Let f be any irreducible quadratic polynomial in $k[x, y]$, and let W be the conic defined by f. Show that $A(W)$ is isomorphic to $A(Y)$ or $A(Z)$. Which one is it when?

1.1 (a) edealba

Note that $A(Y)$ is the coordinate ring for the curve $y - x^2 = 0$, and since we're working over the plane, suppose we're working on $\mathbb{A}^2$ (affine plane) over the field k, such that the coordinate ring of the plane is just $k[x, y]$. We want to show that $A(Y) \cong k[x]$.
Note that the coordinate ring $A(Y)$ is defined as the following quotient ring:

$$A(Y) = k[x, y]/J,$$

where $J = (y - x^2)$, i.e. J is the ideal generated by $y - x^2$. Consider the following mapping:

$$\phi : k[x, y] \to k[x],$$

where ϕ is an evaluation mapping on y, where $y \mapsto x^2$. Evaluation maps like ϕ are ring homomorphisms. We will show that $\ker(\phi)$ is exactly equivalent to J since all polynomials generated by $y - x^2$ get mapped to $x^2 - x^2 = 0$ via ϕ, hence $J \subset \ker(\phi)$. To show equality, we need to show that $\ker(\phi) \subset J$. Let $a \in \ker(\phi)$, i.e. $\phi(a) = 0$. We want to show that $a \in J$, i.e. that we can write a as:

$$a = A(x, y) \cdot (y - x^2),$$

where $A(x, y) \in k[x, y]$. If a were *not* in J then we would have some non-zero remainder $r(x, y)$ with degree of y less than 1 (since the degree of y in $y - x^2$ is 1), so we can write the remainder as $r(x)$ such that:

$$a = A(x, y) \cdot (y - x^2) + r(x).$$

Let's apply ϕ to the RHS: $\phi(A(x, y)\cdot(y-x^2)+r(x)) = A(x, x^2)\cdot(x^2-x^2)+r(x) = A(x, x^2)\cdot 0+r(x) = r(x)$. Note that this implies that $\phi(a) = r(x) = 0$. This means that the remainder is zero, and hence $a = A(x, y) \cdot (y - x^2)$ which means that $a \in J$ like we wanted to show. Thus, $\ker(\phi) = J$.
Next, let's show that ϕ is surjective. Let $p(x) \in k[x]$ be an arbitary polynomial of degree m, so we can write:

$$p(x) = \sum_{i=0}^{m} c_i x^i$$

Note that $c_i \in k \subset k[x] \subset k[x, y]$ and also $x^i \in k[x] \subset k[x, y]$ for all i, then clearly $p(x) \in k[x, y]$ where $\phi(p(x)) = p(x)$. Since ϕ is surjective, then we know that $\text{im}(\phi) = k[x]$. By the first isomorphism theorem, we get:

$$k[x, y]/\ker(\phi) \cong \text{im}(\phi),$$

where $\ker(\phi) = J$ and $\text{im}(\phi) = k[x]$, thus

$$k[x, y]/J \cong k[x],$$

and finally since we know $A(Y) = k[x, y]/J$, then clearly we get that $A(Y) \cong k[x]$, so we've shown that the coordinate ring $A(Y)$ is, indeed, isomorphic to the polynomial rings in one variable over k. $\square$

1.1 (b) edealba

Note that $A(Z)$ is the following:

$$A(Z) = \left\{ \sum_{i,j\geq 0} c_{ij}\bar{x}^i\bar{y}^j \mid c_{ij} \in k, \text{ only a finite number of } c_{ij} \text{ are non-zero, and } \bar{x}\bar{y} = 1 \right\}$$

Suppose there were a ring isomorphism $\phi : A(Z) \to k[t]$, where we choose t to be the polynomial ring of one variable over k. Note that ring isomorphisms must map units to units. Since we're working over the plane curve $xy = 1$, this means that the image of x in $A(Z)$ is unit, i.e. $\bar{x} \in A(Z)$ is unit. Note that the units in $k[t]$ are all the non-zero constants, i.e. $c \in k^\times \subset k[t]$, where these are just the non-zero elements of the field k. Since ϕ maps units to units, then $\bar{x}$ must get mapped to some non-zero constant in $k[t]$. Note that if $\bar{x}$ gets mapped to a constant, then $\mathrm{im}(\phi) = k$. This is because if we have some arbitary $p \in A(Z)$ then we have:

$$p = \sum_{i,j\geq 0} c_{ij}\bar{x}^i\bar{y}^j,$$

but note that if $\bar{x} \mapsto c_x$ where c_x is some non-zero constant in k then this implies that $\bar{y} \mapsto c_y$ where c_y is also some non-zero constant such that $c_y = 1/c_x$. This means that $\phi(p)$ is:

$$\phi(p) = \sum_{i,j\geq 0} c_{ij}c_x^i c_y^j,$$

which is just some constant element of k for any $p \in A(Z)$. Note that $\mathrm{im}(\phi) = k$ is a contradiction since we assumed ϕ was an isomorphism, i.e. $\mathrm{im}(\phi) = k[t]$. This means there is no such ring isomorphism $\phi : A(Z) \to k[t]$. $\square$

1.3

Let Y be the algebraic set in $\mathbb{A}^3$ defined by the two polynomials $x^2 - yz = 0$ and $xz - x = 0$. Show that Y is a union of three irreducible components. Describe them and find their prime ideals.

1.3 azumaril.747

We have $Y = Z(x^2 - yz, xz - x) = Z(x^2 - yz) \cap Z(x(z-1)) = Z(x^2 - yz) \cap (Z(x) \cup Z(z-1)) = (Z(x^2 - yz) \cap Z(x)) \cup (Z(x^2 - yz) \cap Z(z-1))$. The first part $Z(x^2 - yz) \cap Z(x) = Z(x^2 - yz, x) = Z(yz, x) = Z(x, y) \cup Z(z, x)$. The second part $Z(x^2 - yz) \cap Z(z-1) = Z(x^2 - y, z-1)$. So the three irreducible components are $Z(x, y)$, $Z(x, z)$, and $Z(x^2 - y, z-1)$, with prime ideals (x, y), (x, z), and $(x^2 - y, z - 1)$, respectively. Geometrically, they are two affine lines and one parabolic curve.

1.4

If we identify $\mathbb{A}^2$ with $\mathbb{A}^1 \times \mathbb{A}^1$ in the natural way, show that the Zariski topology on $\mathbb{A}^2$ is not the product topology of the Zariski topologies on the two copies of $\mathbb{A}^1$.

1.4 edealba

Consider the diagonal line $y = x$ in $\mathbb{A}^2$ where we're working over the coordinate ring $k[x, y]$. Note that $V(y - x) \subset \mathbb{A}^2$ is closed in the Zariski topology of $\mathbb{A}^2$, and is the following:

$$V(y - x) = \left\{ p \in k^2 : f(p) = 0 \text{ for all } f \in (y - x) \right\}$$

However, when we look at the closed sets in $\mathbb{A}^1$, we only have the following: (1) sets with a finite number of points, (2) the entire affine line $\mathbb{A}^1$. In $\mathbb{A}^1 \times \mathbb{A}^1$ we have: (1) also sets with finite number of points, (2) the entire space $\mathbb{A}^1 \times \mathbb{A}^1$, and (3) we also finite unions of vertical and horizontal lines from $\mathbb{A}^1 \times \{p_i\}$ and $\{p_j\} \times \mathbb{A}^1$ for points p_i and p_j in $\mathbb{A}^1$. Note that it's *impossible* to construct the diagonal line $y - x = 0$ using any combination of closed sets in $\mathbb{A}^1 \times \mathbb{A}^1$, we would need an infinite number of points which is *not* allowed in the Zariski topology. Thus, although $\mathbb{A}^2$ and $\mathbb{A}^1 \times \mathbb{A}^1$ may be identical to each other, their topologies are not the same. $\square$

1.5

Show that a k-algebra B is isomorphic to the affine coordinate ring of some algebraic set in $\mathbb{A}^n$, for some n, if and only if B is a finitely generated k-algebra with no nilpotent elements.

1.5 edealba

Let B be a k-algebra, and let $W = V(J) \subset \mathbb{A}^n$ be some algebraic set of $\mathbb{A}^n$ for some ideal $J \subset k[x_1, \dots, x_n]$, where W is the set of all points that vanish for all polynomial functions in J. We then have the following coordinate ring (which is the following quotient ring):

$$A(W) = k[x_1, \dots, x_n]/I(W),$$

and since $W = V(J)$, by Hilbert's strong Nullstellensatz, we know that $I(V(J)) = \sqrt{J}$, hence

$$A(W) = k[x_1, \dots, x_n]/\sqrt{J}$$

We want to show that $B \cong A(W)$ iff B is a finitely generated k-algebra with no nilpotent elements.
($\Rightarrow$) First, suppose $B \cong A(W)$. Since we've defined $A(W)$ to be the quotient ring of $k[x_1, \dots, x_n]$ modded out by the radical of an ideal $\sqrt{J}$, then this is automatically a finitely generated k-algebra. We want to show that it has no nilpotent elements, i.e. there is *no* non-zero element $f \in B \cong A(W)$ such that $f^m = 0$ for some $m \geq 2$.
Suppose for contradiction that there exists some nilpotent element $f \in B \cong A(W)$ such that $f^m = 0$ for some $m \geq 2$. Since f is assumed to be non-zero, this means that $f \notin I(W) = \sqrt{J}$. Since $J \subset \sqrt{J}$, then $f \notin \sqrt{J}$ implies that $f \notin J$. Note that since $f \notin I(W)$ then this also means that there exists some point $p' \in W$ such that $f(p') \neq 0$. Next, since we have $f^m = 0$, then $f^m \in I(W) = \sqrt{J}$. This means that f^m vanishes for all points $p \in W$, and hence $f^m(p') = 0$. Since $f^m(p') = 0$, this implies that $f^m(p') = (f(p'))^m = 0$ which implies that $f(p') = 0$ which contradicts our previous $f(p') \neq 0$. With this, we've shown that there *can't* be a nilpotent element $f \in B \cong A(W)$.
($\Leftarrow$) Let B be a finitely generated k-algebra with generators $\{t_1, \dots, t_n\} \subset B$ such that any element in B can be written as a polynomial in these generators, and B also has no nilpotent elements. Consider the following mapping:

$$\phi : k[x_1, \dots, x_n] \to B,$$

where ϕ maps $x_i \mapsto t_i$ for all $1 \leq i \leq n$. This mapping is clearly surjective since any polynomial $f \in B$ can be written by swapping each t_i with its corresponding x_i. Note that if the generators $\{t_1, \dots, t_n\}$ have some algebraic relations, this is captured by $\ker(\phi)$. Let $J = \ker(\phi) \subset k[x_1, \dots, x_n]$ be the ideal generated by all the algebraic relations between the generators $\{t_1, \dots, t_n\}$. By the First isomorphism theorem, we get:

$$k[x_1, \dots, x_n]/J \cong B$$

Next, we want to show that J is a radical ideal. Note that there are *no* nilpotent elements in B. If there *were* some nilpotent element then suppose $f \in B$ were nilpotent, then this would mean that $f^m = 0$ for some $m \geq 1$.
(1) An ideal J is radical if for some $x^m \in J$ with $m \geq 1$ implies that $x \in J$, i.e.:

$$x^m \in J \implies x \in J$$

(2) Since $B \cong k[x_1, \ldots, x_n]/J$, then elements of B are of the coset form $\bar{f} = f + J$. If $\bar{f} \in B$ were nilpotent then this means that $\bar{f}^m = 0$ for some $m \geq 1$, which means that $f^m + J = 0$ which implies that $f^m \in J$ for $f \in k[x_1, \ldots, x_n]$.
Note that since there are *no* nilpotent elements in B, then this means for all non-zero $\bar{f} \in B$ we know that $\bar{f} = f + J \neq 0$, i.e. $f \notin J$ where $f^m \neq 0$ for all $m \geq 1$. Furthermore, note that the contrapositive of (1) means that if the following holds:

$$x \notin J \implies x^m \notin J \text{ for all } m \geq 1,$$

then J is a radical ideal. We know that for any non-zero $\bar{f}$ we have $f \notin J$, and since $\bar{f}$ is *not* nilpotent, then $f^m \notin J$ for all $m \geq 1$. With this, we've shown that B not having any nilpotent elements implies that J is a radical ideal, i.e. $J = \sqrt{J}$.
Finally, let's consider the algebraic set $W = V(J) \subset \mathbb{A}^n$. From Hilbert's strong Nullstellensatz, we know that the coordinate ring $A(W)$ is the following:

$$A(W) \cong k[x_1, \ldots, x_n]/I(V(J)) = k[x_1, \ldots, x_n]/\sqrt{J},$$

but note that since J is radical, then we get:

$$A(W) \cong k[x_1, \ldots, x_n]/J,$$

so we get that $B \cong k[x_1, \ldots, x_n]/J \cong A(W)$ like we wanted to show. □

1.8

Let Y be an affine variety of dimension r in $\mathbb{A}^n$. Let H be a hypersurface in $\mathbb{A}^n$, and assume that $Y \not\subseteq H$. Then every irreducible component of $Y \cap H$ has dimension $r - 1$.

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Suppose that $H = Z(f)$ with f irreducible in $k[x_1, \ldots, x_n]$. Then the projection of f in $A(Y) := k[x_1, \ldots, x_n]/I(Y)$ is not equal to $\overline{0}$ since by assumption $(f) \not\subset I(Y)$. Since $A(Y)$ is a domain (due to the irreducibility of Y), the element $\overline{f}$ is not a zero divisor. Assuming that $Y \cap H \neq \varnothing$, we have that $\overline{f}$ is not a unit in $A(Y)$. To see this, let $P \in Y \cap H$. Then $I(Y), (f) \subset \mathfrak{m}_P$, the maximal ideal of $k[x_1, \ldots, x_n]$ corresponding to P. This implies $(\overline{f}) \subset \mathfrak{m}_P/I(Y)$, the latter being a maximal ideal in $A(Y)$. Thus, $\overline{f}$ is not a unit.
We apply Theorem 1.11A to get that every minimal prime ideal $\mathfrak{p}$ in $A(Y)$ containing $\overline{f}$ has height 1. The irreducible components of $Y \cap H$ and the minimal prime ideals containing $(\overline{f})$ correspond, and since each of these prime ideals has height 1, the corresponding varieties have dimension $r - 1$ by Theorem 1.8A.

1.2 Projective Varieties

2.1

Prove the "homogeneous Nullstellensatz," which says if $J \subseteq S$ is a homogeneous ideal, and if $f \in S$ is a homogeneous polynomial with $\deg f > 0$, such that $f(P) = 0$ for all $P \in Z(J)$ in $\mathbb{P}^n$, then $f^q \in J$ for some $q > 0$. [*Hint:* Interpret the problem in terms of the affine $(n+1)$-space whose affine coordinate ring is S, and use the usual Nullstellensatz, (1.3A).]

2.1 edealba

Let $W = Z(J) \subset \mathbb{P}^n$ be a projective variety, and note that $f(P) = 0$ means that the homogeneous polynomial f vanishes at each point in $W \subset \mathbb{P}^n$ which represents a line going through the origin in $\mathbb{A}^{n+1}$. Consider the *affine cone* over $W = Z(J)$, which we denote $C(W) \subset \mathbb{A}^{n+1}$. Since $f(P) = 0$ for points P that represent lines going through the origin in $\mathbb{A}^{n+1}$, hence $f(p) = 0$ for all points $p \in C(W) \subset \mathbb{A}^{n+1}$ since these are all points on the lines projected in $W = Z(J) \subset \mathbb{P}^n$.
We want to show that $C(W) = C(Z(J)) = V(J)$, where $V(J) \subset \mathbb{A}^{n+1}$ is the variety of vanishing points of the ideal J.
First, let's show that $C(Z(J)) \subset V(J)$. Let $p = (a_0, \dots, a_n) \in C(Z(J))$ be a point. We have the following cases:
- If $p = (0, \dots, 0)$, i.e. this is the origin in $\mathbb{A}^{n+1}$, then for any homogeneous polynomial $g \in J$, we get $g(p) = 0$ since homogeneous polynomials don't have a constant term.
- Suppose $p = (a_0, \dots, a_n) \neq 0$. This is a point in the affine cone, so it's a representative for a point $P \in Z(J) \subset \mathbb{P}^n$ such that $P = [a_0 : \cdots : a_n]$. Note that any homogeneous polynomial $g \in J$ vanishes at P, by definition of $Z(J)$. Note that $g(P) = 0$ implies that $g(p) = 0$ since g vanishes at any point on the line corresponding to P, i.e. for any point $p_\lambda = (\lambda a_0, \dots, \lambda a_n)$ we get:

$$g(p_\lambda) = g(\lambda a_0, \dots, \lambda a_n) = \lambda^d \cdot g(a_0, \dots, a_n) = 0,$$

where g is a degree d homogeneous polynomial. What about non-homogeneous polynomials in J? Let h be any polynomial in J. Since J is a homogeneous ideal, then we can write:

$$h = h_1 + h_2 + \cdots + h_d,$$

where each h_i is a homogeneous polynomial in J. Since $h_i(p) = 0$ for each homogeneous $h_i \in J$, then:

$$h(p) = \sum h_i(p) = 0,$$

which means that any polynomial $h \in J$ vanishes at p.
This shows that $p \in C(Z(J))$ implies that $p \in V(J)$, thus

$$C(W) = C(Z(J)) \subset V(J).$$

Next, let's show that $V(J) \subset C(Z(J))$. Let $p = (a_0, \dots, a_n) \in V(J)$. This means that any polynomial $g \in J$ vanishes at p, i.e. $g(p) = 0$. This holds, in particular, for *homogeneous* polynomials in J. Note that this is precisely the condition for the projective point $P = [a_0 : a_1 : \cdots : a_n]$ to be in $Z(J)$. Since $P \in Z(J)$, then the point p lies on the line representing P. By definition of the affine cone, this means that $p \in C(Z(J))$. Thus,

$$V(J) \subset C(Z(J)).$$

With this, we've shown that $C(W) = C(Z(J)) = V(J)$. Since $V(J) \subset \mathbb{A}^{n+1}$, we can apply Hilbert's Nullstellensatz. Recall that f is a polynomial such that $f(P) = 0$ for all points $P \in Z(J)$. This implies that

f vanishes at all points in the affine cone $p \in C(Z(J)) = V(J) \subset \mathbb{A}^{n+1}$. By Hilbert's Nullstellensatz, since f vanishes at all points in $V(J)$, then $f \in \sqrt{J}$ which means that $f^q \in J$ for some $q > 0$ like we wanted to show. □

2.3

a) If $T_1 \subseteq T_2$ are subsets of S^h, then $Z(T_1) \supseteq Z(T_2)$.

b) If $Y_1 \subseteq Y_2$ are subsets of $\mathbb{P}^n$, then $I(Y_1) \supseteq I(Y_2)$.

c) For any two subsets Y_1, Y_2 of $\mathbb{P}^n$, $I(Y_1 \cup Y_2) = I(Y_1) \cap I(Y_2)$.

d) If $J \subset S$ is a homogenous ideal with $Z(J) \neq \varnothing$, then $I(Z(J)) = \sqrt{J}$.

e) For any subset $Y \subseteq \mathbb{P}^n$, $Z(I(Y)) = \overline{Y}$.

2.3 (a) (b) edealba

a) This follows since Z is a contravaiant functor (or operator), so when we apply Z to $T_1 \subseteq T_2$, we reverse the "arrows", in this case we get $\supseteq$, and thus:

$$Z(T_1) \supseteq Z(T_2).$$

To elaborate, let's consider $T_1 \subseteq T_2 \subseteq S^h$. Let $p \in Z(T_1)$, i.e. p is a point such that every homogeneous polynomial in T_1 vanishes at p. Does this imply that $p \in Z(T_2)$? Not necessarily. Suppose there exists some polynomial $g \in T_2 - T_1$, where $g \in T_2$ and $g \notin T_1$. Note that $p \in Z(T_1)$ does *not* guarantee that $g(p) = 0$ since $g \notin T_1$. On the other hand, suppose $p' \in Z(T_2)$. This means that every polynomial f in T_2 is such that $f(p') = 0$. Since $T_1 \subseteq T_2$, then this still holds for every polynomial in T_1, thus $p' \in Z(T_2)$ implies that $p' \in Z(T_1)$, i.e.

$$Z(T_2) \subseteq Z(T_1), \text{ or } Z(T_1) \supseteq Z(T_2),$$

like we wanted to show. □

b) If $Y_1 \subseteq Y_2$ are subsets of $\mathbb{P}^n$, then $I(Y_1) \supseteq I(Y_2)$. Similar to (a), I is also contravaiant functor (or operator), so when we apply I to $Y_1 \subseteq Y_2$, we get:

$$I(Y_1) \supseteq I(Y_2)$$

□

2.4

a) There is a 1-1 inclusion-reversing correspondence between algebraic sets in $\mathbb{P}^n$, and homogeneous radical ideals of S not equal to S_+, given by $Y \mapsto I(Y)$ and $J \mapsto Z(J)$. [*Note:* Since S_+ does not occur in this correspondence, it is sometimes called the *irrelevant* maximal ideal of S.]

b) An algebraic set $Y \subseteq \mathbb{P}^n$ is irreducible if and only if $I(Y)$ is a prime ideal.

c) Show that $\mathbb{P}^n$ itself is irreducible.

2.4 edealba

a) Let $J \subset S$ be a homogeneous radical ideal that's *not* S_+. We want to show that $I(Z(J)) = J$. Note that $Z(S_+) = \varnothing$. Suppose it were the case that $Z(J) = \varnothing$. This would imply that either $J = S$ or J contains S_+. Since J is a proper ideal, by assumption, then $J \neq S$. Furthermore, note that S_+ is a maximal *homogeneous* ideal, hence $S_+ \subseteq J$ implies $J = S_+$, but recall that we assumed that $J \neq S_+$. We've reached a contradiction, thus $Z(J) \neq \varnothing$. By **1.2.3d**, we know that if $\mathfrak{a} \subset S$ is a homogenous ideal with $Z(\mathfrak{a}) \neq \varnothing$, then $I(Z(\mathfrak{a})) = \sqrt{\mathfrak{a}}$. In our case, we know that J is radical by assumption, thus:

$$I(Z(J)) = \sqrt{J} = J.$$

Next, let $Y \subset \mathbb{P}^n$ be an algebraic set. From **1.2.3e**, we know that for any subset $W \subseteq \mathbb{P}^n$, we get $Z(I(W)) = \overline{W}$. Since Y is an algebraic set, this means that $\overline{Y} = Y$, hence:

$$Z(I(Y)) = \overline{Y} = Y,$$

like we wanted to show. □

b) ($\Leftarrow$) Suppose $I(Y)$ is a prime ideal. We want to show that $Y \subseteq \mathbb{P}^n$ is irreducible. Suppose, for contradiction, that Y is reducible, i.e. we can write $Y = C_1 \cup C_2$ where $C_1, C_2 \subsetneq Y$. Since $I(Y)$ is a prime ideal, this means that $f_1 f_2 \in I(Y)$ implies that $f_1 \in I(Y)$ or $f_2 \in I(Y)$. Note that since $Y = C_1 \cup C_2$, then we have:

$$I(Y) = I(C_1 \cup C_2) = I(C_1) \cap I(C_2).$$

Since we have $C_1 \subsetneq Y$, then we have:

$$I(C_1) \supsetneq I(Y),$$

i.e. $I(Y) \subsetneq I(C_1)$. Let $f_1 \in I(C_1) \setminus I(Y)$. Since C_1 is a strictly contained in Y, this means that $I(Y)$ is strcitly contained in $I(C_1)$, i.e. $I(C_1) \setminus I(Y) \neq \varnothing$. Similarly for $C_2 \subsetneq Y$, we get $I(C_2) \setminus I(Y) \neq \varnothing$, so let $f_2 \in I(C_2) \setminus I(Y)$. Note that since $I(C_1) \cap I(C_2) = I(Y)$, then clearly we have that $f_1, f_2 \notin I(Y)$. We want to show that $f_1 f_2 \in I(Y)$. Let $p \in Y$ be an arbitary point in $Y \subseteq \mathbb{P}^n$. Since $Y = C_1 \cup C_2$, then $p \in C_1$ or $p \in C_2$. If $p \in C_2$, then since $f_2 \in I(C_2) \setminus I(Y)$, then $f_1 f_2(p) = 0$, i.e. it vanishes at p. Similarly, if $p \in C_1$ then the f_1 part of $f_1 f_2$ vanishes at p, i.e. $f_1 f_2(p) = f_1(p) f_2(p) = 0 \cdot f_2(p) = 0$, and hence all of $f_1 f_2(p) = 0$. Either case, we get that $f_1 f_2 \in I(Y)$. Finally, since $I(Y)$ is a prime ideal, then this implies that either $f_1 \in I(Y)$ or $f_2 \in I(Y)$ which contradicts our earlier $f_1, f_2 \notin I(Y)$. Thus, Y must be irreducible.

($\Rightarrow$) Suppose $Y \subseteq \mathbb{P}^n$ is irreducible. Let $f_1 f_2 \in I(Y)$. We want to show that $f_1 \in I(Y)$ or $f_2 \in I(Y)$. Suppose for contradiction that $f_1, f_2 \notin I(Y)$, i.e. there exists some points $p_1, p_2 \in Y$ such that $f_1(p_1) \neq 0$ and $f_2(p_2) \neq 0$. Note that if $p_1 = p_2$, then we immediately reach a contradiction since $f_1 f_2(p_1) = f_1(p_1) f_2(p_1) \neq 0$ contradicts $f_1 f_2 \in I(Y)$. Therefore, we have $p_1 \neq p_2$. Note that since $f_1 f_2 \in I(Y)$ then $f_1 f_2$ vanishes at both p_1 and p_2, i.e. $f_1 f_2(p_1) = 0$ and $f_1 f_2(p_2) = 0$. Let's focus on $f_1 f_2(p_1) = 0$. Since $f_1 f_2(p_1) = f_1(p_1) f_2(p_1)$ and $f_1(p_1) \neq 0$, then $f_2(p_1) = 0$. Similarly, $f_1 f_2(p_2) = 0$ implies $f_1(p_2) = 0$. Consider the closed sets $C_1 = Y \cap V(f_1)$ and $C_2 = Y \cap V(f_2)$. Note that $p_2 \in C_1$ and $p_1 \in C_2$. Next, let's consider $C_1 \cup C_2$, where:

$$C_1 \cup C_2 = (Y \cap V(f_1)) \cup (Y \cap V(f_2)) = Y \cap (V(f_1) \cup V(f_2)).$$

Furthermore, note that $V(f_1 f_2) = V(f_1) \cup V(f_2)$, hence:

$$C_1 \cup C_2 = Y \cap V(f_1 f_2),$$

but since $f_1 f_2 \in I(Y)$, then this implies that $Y \subset V(f_1 f_2)$, which means that $Y \cap V(f_1 f_2) = Y$, thus

$$C_1 \cup C_2 = Y.$$

Note that since Y is irreducible, then this means that either $C_1 = Y$ or $C_2 = Y$. Without loss of generality, if $C_1 = Y$, then this means that $Y \subseteq V(f_1)$ which contradicts our assumption that $f_1 \notin I(Y)$. Therefore, $I(Y)$ *must* be a prime ideal.

c) First, note that $I(\mathbb{P}^n)$ is the ideal of homogeneous polynomials that vanish at every point in $\mathbb{P}^n$. Consider the *homogeneous constant* polynomial $f(x_0, \ldots, x_n) = 0$. This is the only homogeneous polynomial in $k[x_0, \ldots, x_n]$ that vanishes at all points in $\mathbb{P}^n$, hence $I(\mathbb{P}^n) = (0)$. To show that $I(\mathbb{P}^n)$ is a prime ideal, note that

$$k[x_0, \ldots, x_n]/(0) \cong k[x_0, \ldots, x_n],$$

and since $k[x_0, \ldots, x_n]$ itself is an integral domain, this implies that $I(\mathbb{P}^n) = (0)$ is a prime ideal. From **1.2.4b**, we know that $I(\mathbb{P}^n)$ being prime implies that $\mathbb{P}^n$ is irreducible. $\square$

2.10

The Cone Over a Projective Variety Let $Y \subseteq \mathbb{P}^n$ be a nonempty algebraic set, and let $\theta : \mathbb{A}^{n+1} \setminus \{(0, \ldots, 0)\} \to \mathbb{P}^n$ be the map which sends the point with affine coordinates $(a_0, \ldots, a_n)$ to the point with homogeneous coordinates $(a_0, \ldots, a_n)$. We define the *affine cone* over Y to be

$$C(Y) = \theta^{-1}(Y) \cup \{(0, \ldots, 0)\}.$$

a) Show that $C(Y)$ is an algebraic set in $\mathbb{A}^{n+1}$, whose ideal is equal to $I(Y)$, considered as an ordinary ideal in $k[x_0, \ldots, x_n]$.

b) $C(Y)$ is irreducible if and only if Y is.

c) $\dim C(Y) = \dim Y + 1$.

2.10 (a) (b) edealba

a) We know that $Y \subseteq \mathbb{P}^n$ is a nonempty algebraic set, so there exists a homogeneous ideal J such that $Z(J) = Y$. Note that since $Y = Z(J)$, then $C(Y) = C(Z(J))$. In **1.2.1** we showed that $C(Z(J)) = V(J) \subseteq \mathbb{A}^{n+1}$ for any ideal $J \subseteq k[x_0, \ldots, x_n]$, hence $C(Y) = V(J) \subseteq \mathbb{A}^{n+1}$. $C(Y) = V(J)$ is an algebraic set in $\mathbb{A}^{n+1}$ like we wanted to show.

Next, let's show that the ideal of $C(Y)$, i.e. $I(C(Y))$, is equal to $I(Y)$.

First, let's show that $I(Y) \subseteq I(C(Y))$. Let $f \in I(Y)$ such that f is a *homogeneous* polynomial of degree d, then this means that f is a polynomial that vanishes at every point $p \in Y$. We want to show that $f \in I(C(Y))$. Let $q \in C(Y)$ be an arbitary point in the affine cone over Y. First case, suppose $q = (0, \ldots, 0)$. Since $I(Y)$ is the set of polynomials that vanish at every point in $Y \subseteq \mathbb{P}^n$, then these must be polynomials with *zero* constant, hence $f(0, \ldots, 0) = 0$ (this is also clear since f is homogeneous). Suppose $q = (a_0, \ldots, a_n)$ is an arbitary non-zero point in $C(Y)$ with some non-zero a_i. Note that, by definition of $I(Y)$, since f vanishes at every *projective* point in Y, then f must vanish at *all* of its representative points that exist in affine space. q has a corresponding projective point $\overline{q} \in \mathbb{P}^n$ such that $f(\overline{q}) = 0$ which implies that $f(q) = 0$. If $\overline{q} = [q_0 : \cdots : q_n] \in \mathbb{P}^n$, then q is of the form:

$$q = (\lambda q_0, \ldots, \lambda q_n), \text{ for } \lambda \in k,$$

and since f is a homogeneous polynomial of degree d, then we can write:

$$f(q) = f(\lambda q_0, \ldots, \lambda q_n) = \lambda^d f(q_0, \ldots, q_n) = \lambda^d \cdot 0 = 0,$$

so we see that any homogeneous polynomial $f \in I(Y)$ is in $I(C(Y))$. Furthermore, if we have some other *non-homogeneous* polynomial $g \in I(Y)$, since $I(Y)$ is a homogeneous ideal, this means that it's generated by a set of homogeneous polynomials. Suppose $g = \sum_i h_i$, where h_i is a homogeneous polynomial. Since each h_i component vanishes for any $q \in C(Y)$ (since $h_i \in I(C(Y))$ for each homogenous h_i), then this means that g overall also vanishes at q, hence $g \in I(C(Y))$. This shows that $f \in I(Y)$ then $f \in I(C(Y))$, thus $I(Y) \subseteq I(C(Y))$.

Second, le't show that $I(C(Y)) \subset I(Y)$. Let $f \in I(C(Y))$. Since f vanishes at all points in $C(Y)$, and $(0, \ldots, 0) \in C(Y)$, then we know $f(0, \ldots, 0) = 0$, so like before, we know that f is a polynomial with *zero* constant term. f is a graded ring over homogeneous polynomials, so we can write:

$$f = \sum_i h_i,$$

where h_i is a homogeneous polynomial of degree i. Note that if a point $q \in C(Y)$ then the entire line is in $C(Y)$, i.e. $\{\lambda q : \lambda \in k\} \subseteq C(Y)$. Since $f \in I(C(Y))$, then we know that $f(\lambda q) = 0$ for all $\lambda \in k$. This means that:

$$f(\lambda q) = \sum_i^n h_i(\lambda q) = \sum_i^n \lambda^i h_i(q) = 0,$$

and we can view this as a polynomial $g(\lambda) = f(\lambda q)$ where the $h_i(q)$ terms are seen as coefficients to λ. Note that:

$$g(\lambda) = \sum_i^n \lambda^i h_i(q) = 0,$$

for all $\lambda \in k$ implies that g has infinitely many roots. However, note that by the Fundamental Theorem of Algebra, g is a non-zero polynomial of degree n and must have at most n many roots, which is a contradiction. $g(\lambda) = 0$ for all $\lambda \in k$ if and only if g is the *zero polynomial*. This means that the coefficients of g are all zero, i.e. $h_i(q) = 0$ for all i. With this, we've shown that all the homogeneous components of f must vanish at any point $q \in C(Y)$. Since each h_i is a homogeneous polynomial that vanishes at any point $q \in C(Y)$, then this means that for any projective point $[q] \in Y$, h_i vanishes on any of its corresponding coordinates q, which implies that $h_i \in I(Y)$. Since $f = \sum_i h_i$, then f also vanishes for any projective point $[q] \in Y$, i.e. $f \in I(Y)$. Finally, we see that $I(C(Y)) \subset I(Y)$ like we wanted to show.

Putting everything together, we see that, indeed, $I(C(Y)) = I(Y)$. □

b) ($\Rightarrow$) Suppose $C(Y)$ is irreducible. From (a), we know that the ideal of $C(Y)$ is $I(Y)$, and since $C(Y)$ is irreducible, then $I(Y)$ must be a prime ideal. From **1.2.4b**, we know that $I(Y)$ being prime implies that Y itself must be irreducible.

($\Leftarrow$) Suppose Y is irreducible. Then this means that $I(Y)$ is a prime ideal. Furthermore, since $I(Y)$ is the ideal of $C(Y)$, i.e. $I(C(Y)) = I(Y)$, then this also implies that $C(Y)$ itself is irreducible. □

2.12

The d-Uple Embedding. For given $n, d > 0$, let $M_0, M_1, \ldots, M_N$ be all the monomials of degree d in the $n+1$ variables $x_0, \ldots, x_n$, where $N = \binom{n+d}{n} - 1$. We define a mapping $\rho_d \colon \mathbb{P}^n \to \mathbb{P}^N$ by sending the point

$P = (a_0, \ldots, a_n)$ to the point $\rho_d(P) = (M_0(a), \ldots, M_N(a))$ obtained by substituting the a_i in the monomials M_j. This is called the d-uple *embedding* of $\mathbb{P}^n$ in $\mathbb{P}^N$. For example, if $n = 1, d = 2$, then $N = 2$, and the image Y of the 2-uple embedding of $\mathbb{P}^1$ in $\mathbb{P}^2$ is a conic.

a) Let $\theta\colon k[y_0, \ldots, y_N] \to k[x_0, \ldots, x_n]$ be the homomorphism defined by sending y_i to M_i, and let $\mathfrak{a}$ be the kernel of θ. Then $\mathfrak{a}$ is a homogeneous prime ideal, and so $Z(\mathfrak{a})$ is a projective variety in $\mathbb{P}^N$.

b) Show that the image of ρ_d is exactly $Z(\mathfrak{a})$. (One inclusion is easy. The other will require some calculation.)

c) Now show that ρ_d is a homeomorphism of $\mathbb{P}^n$ onto the projective variety $Z(\mathfrak{a})$.

d) Show that the twisted cubic curve in $\mathbb{P}^3$ (Ex. 2.9) is equal to the 3-uple embedding of $\mathbb{P}^1$ in $\mathbb{P}^3$, for suitable choice of coordinates.

2.12 (b) mlwells

Let $Q = \rho_d(P) = (M_0(a), \ldots, M_N(a))$. Let $F \in \mathfrak{a} = \ker\theta$. Then $F(M_0, \ldots, M_N) = 0$ as a polynomial, which implies $F(M_0(a), \ldots, M_N(a)) = 0$. Since F was arbitrary, $Q \in Z(\mathfrak{a})$.
Now let $Q = (y_0, \ldots, y_N) \in Z(\mathfrak{a})$. Suppose without loss of generality that $y_0 \neq 0$. It follows that $Q = (y_0^d, y_0^{d-1}y_1, \ldots, y_0^{d-1}y_N)$. Suppose that $M_0 = x_0^d, M_1 = x_0^{d-1}x_1, \ldots, M_n = x_0^{d-1}x_n$.

Proposition 1.1. *For all $j = 0, \ldots, N$, we have that*

$$M_0^{d-1}M_j = M_0^{i_0}M_1^{i_1}\ldots M_n^{i_n} \tag{1}$$

for some $i_j \geq 0$ with $\sum_j i_j = d$.

Proof. Suppose $M_j = x_0^{i_0}\ldots x_n^{i_n}$ where $\sum_j i_j = d$. We have

$$\begin{aligned} M_0^{d-1}M_j &= x_0^{(d-1)d+i_0}x_1^{i_1}\ldots x_n^{i_n} &(2)\\ &= x_0^{(d-1)i_0+i_0+(d-1)\sum_{j\geq 1}i_j}x_1^{i_1}\ldots x_n^{i_n} &(3)\\ &= x_0^{di_0}(x_0^{d-1}x_1)^{i_1}\ldots(x_0^{d-1}x_n)^{i_n} &(4)\\ &= M_0^{i_0}M_1^{i_1}\ldots M_n^{i_n} &(5)\end{aligned}$$

□

Let $P = (y_0, \ldots, y_n)$. Then, by the proposition, we have that $Q = (M_0(P), M_1(P), \ldots, M_N(P)) = \rho_d(P)$, as was to be shown.

1.3 Morphisms

3.20

Let Y be a variety of dimension ≥ 2, and let $P \in Y$ be a normal point. Let f be a regular function on $Y - P$.

a) Show that f extends to a regular function on Y.

b) Show this would be false for rm dim $Y = 1$.

See (III, Ex. 3.5) for generalization.

3.20 (a) mlwells

If we show that f restricted to $V - P$ extends to a regular function on $V \subseteq Y$ for any affine neighborhood V of P, then by gluing we will have shown that f extends to a regular function on Y. So, assume that Y is affine.

The function $f = c/d \in K(Y)$ for regular functions $c, d \in A(Y)$. If we can show that the ideal quotient $((c) : (d)) = \{h \in A(Y) \colon hc \in (d)\}$ is equal to $A(Y)$, then it follows that $c/d \in A(Y)$. By assumption, $f \in \mathcal{O}_Q$ for all $Q \neq P \in Y$. Let $\mathfrak{m}_Q$ denote the maximal ideal of $A(Y)$ corresponding to Q. Then the localized ideal $((c) : (d))_{\mathfrak{m}_Q}$ is equal to $A(Y)_{\mathfrak{m}_Q} = \mathcal{O}_Q$ for all $Q \neq P$ since $c/d \in \mathcal{O}_Q$. It remains to show that $((c) : (d))_{\mathfrak{m}_P} = A(Y)_{\mathfrak{m}_P} = \mathcal{O}_P$. If $\mathfrak{p}A(Y)_{\mathfrak{m}_P}$ is a prime ideal of $\mathcal{O}_P$ not equal to the maximal ideal $\mathfrak{m}_P A(Y)_{\mathfrak{m}_P}$, then $((c) : (d))_{\mathfrak{m}_P} \not\subseteq \mathfrak{p}A(Y)_{\mathfrak{m}_P}$ since c/d is in the local ring corresponding to the subvariety defined by $\mathfrak{p}$, and hence $((c) : (d))_{\mathfrak{p}} = A(Y)_{\mathfrak{p}}$. . Assume by way of contradiction that $((c) : (d))_{\mathfrak{m}_P} \subseteq \mathfrak{m}_P A(Y)_{\mathfrak{m}_P}$. Then, $\sqrt{((c) : (d))_{\mathfrak{m}_P}} = \mathfrak{m}_P$.

Let $a_1, \ldots, a_s$ be a regular sequence in $\mathfrak{m}_P A(Y)_{\mathfrak{m}_P}$ with $s > 1$. By Theorem 8.22A of Chapter II of Hartshorne part (2), such a sequence exists. We have shown that $a_1^{r_1}, a_2^{r_2} \in ((c) : (d))_{\mathfrak{m}_P}$ for some $r_1, r_2 > 0$ with $a_1^{r_1 - 1}, a_2^{r_2 - 1} \notin ((c) : (d))_{\mathfrak{m}_P}$. Thus,

$$a_1^{r_1} c = e_1 d, \quad e_1 \notin (a_1) \tag{6}$$

and

$$a_2^{r_2} c = e_2 d, \quad e_2 \notin (a_2) \tag{7}$$

Thus,

$$a_1^{r_1} a_2^{r_2} c = e_1 a_2^{r_2} d = e_2 a_1^{r_1} d \tag{8}$$

which implies

$$e_1 a_2^{r_2} = e_2 a_1^{r_1} \tag{9}$$

Since $e_1 \notin (a_1)$ and $a_2 \notin (a_1)$, a_2 is a zero divisor in the ring $A(Y)_{\mathfrak{m}_P}/(a_1)$. This contradicts the fact that a_1, a_2 is a regular sequence in $\mathfrak{m}_P A(Y)_{\mathfrak{m}_P}$.

Thus, we conclude that $((c) : (d))_{\mathfrak{m}_P} = A(Y)_{\mathfrak{m}_P}$, and hence $f \in \mathcal{O}_P$. This shows that $((c) : (d)) = A(Y)$ which implies $f \in A(Y)$, and hence f extends to a regular function on Y.

2.1 Sheaves

1.2

a) For any morphism of sheaves $\phi : \mathscr{F} \to \mathscr{G}$, show that for each point P, $(\ker\phi)_P = \ker(\phi_P)$ and $(\operatorname{im}\phi)_P = \operatorname{im}(\phi_P)$.

b) Show that ϕ is injective (surjective) iff induced map on stalks ϕ_P is injective (surjective) for all P.

c) Show that the sequence of sheaves $\cdots \longrightarrow \mathscr{F}^{i-1} \longrightarrow \mathscr{F}^{i} \longrightarrow \mathscr{F}^{i+1} \longrightarrow \cdots$

1.2 yakimk

a) Since stalks are defined as a filtered colimit and kernels are a particular type of limit (pullback) and filtered colimits commute with finite limits isomorphism $(\ker\phi_P) = \ker(\phi_P)$ is automatic.

To show a similar thing for the image we first note that $(\operatorname{im}\phi)_P \simeq ((\operatorname{im}_{pre}\phi)^+)_P \simeq (\operatorname{im}_{pre}\phi)_P$, where the first isomorphism is by definition and the second one using the fact that sheafification has the same stalks as original presheaf. Now we conclude noticing that $\operatorname{im}_{pre}\phi \simeq \ker(\operatorname{coker}\phi)$, i.e. it is a finite limit of finite colimits and since again stalks are filtered colimits they commute with finite limits and colimits (in for instance abelian category) we conclude that $(\operatorname{im}\phi)_P \simeq \operatorname{im}(\phi_P)$.

b) We show that a map of sheaves $\phi : \mathscr{F} \to \mathscr{G}$ is injective iff it induced maps on stalks are injective (proof for surjectivity is essentially identical).

Consider the following commutative diagram

$$\begin{tikzcd}
\mathscr{F}(U) \arrow[r, "\phi_U"] \arrow[d, "\iota_1"'] & \mathscr{G}(U) \arrow[d, "\iota_2"] \\
\prod_{P\in U}\mathscr{F}_p \arrow[r, "\prod\tilde{\phi}_P"] & \prod_{P\in U}\mathscr{G}_p
\end{tikzcd}$$

Note that since section of a sheaf on an open set is determined by its stalks, ι_1 and ι_2 are both injective.

Suppose $\tilde{\phi}_P$ is injective for all P. Take two sections $f, g \in \mathscr{F}(U)$, since $\prod\tilde{\phi}_P$ and ι_1 are injective (former by the hypothesis that $\tilde{\phi}_P$ are injective), their composition is injective. If ϕ_U were not injective, we could find two different sections that go to the same class in $\prod\mathscr{G}_P$, which would contradict injectivity of $\prod\tilde{\phi}_P \circ \iota_1$.

Similarly if ϕ_U is injective then if some of $\tilde{\phi}_P$ were not injective, its composition with ι_1 would not be injective contradicting commutativity of the diagram above.

c) We have to show that $\cdots \longrightarrow \mathscr{F}^{i-1} \longrightarrow \mathscr{F}^{i} \longrightarrow \mathscr{F}^{i+1} \longrightarrow \cdots$ is exact iff induced sequence of stalks are exact for all P.

Assuming that sequence of sheaves is exact we show induced sequences are exact at all stalks. By the previuos section $\operatorname{im}\phi^i_P \simeq (\operatorname{im}\phi^i)_P \simeq (\ker\phi^{i+1})_P \simeq \ker\phi^{i+1}_P$.

Now assume that all sequences of stalks are exact. By the same reasoning as before we see that $\operatorname{im}\phi^i \simeq \ker\phi^{i+1}$ (since they are isomorphic on each stalk and hence are equal).

1.8

For any open set $U \subseteq X$ the functor $\Gamma(U, \cdot)$ from sheaves on X to abelian groups is left-exact.

1.8 yakimk

Given exact sequence of sheaves

$$0 \longrightarrow \mathscr{F}' \xrightarrow{\phi} \mathscr{F} \xrightarrow{\gamma} \mathscr{F}'' \longrightarrow 0$$

We have to show that

$$0 \longrightarrow \Gamma(U, \mathscr{F}') \xrightarrow{\phi_U} \Gamma(U, \mathscr{F}) \xrightarrow{\gamma_U} \Gamma(U, \mathscr{F}'')$$

exact.
Injectivity of ϕ_U follows from definition.
We show that $\ker \gamma_U = \operatorname{im} \phi_U$.
($\ker \gamma_U \supseteq \operatorname{im} \phi_U$) Is immidiate by functoriality of $\Gamma(U, \cdot)$ and hypothesis.
($\ker \gamma_U \subseteq \operatorname{im} \phi_U$) Let $s \in \mathscr{F}(U)$ and $\gamma_U(s) = 0$. By hypothesis and "1.2" we know that the induced sequences of stalks are all exact. In particular for all $P \in U$ induced sequences are all exact. Which means that each for stalk $s_P = \phi_P(k_P)$, i.e. there is some section $k \in \mathscr{F}'(U)$ and an open set V_P such that $s|_{V_P} = \phi(k)|_{V_P}$. Hence we can form an open cover $U = \bigcup_{P \in U} V_P$. Notice that by definition they agree on overlaps and hence we get a full section of $\mathscr{F}'$ on U (call it $q \in \mathscr{F}'(U)$) and since all the stalks of $\phi_P(q_P) = s_P$ for all $P \in U$ and hence are equal on U.