

# Hartshorne Solutions

mlwells

2025

Feel free to add your Discord name to the author list. Let's add our usernames to our solutions, too, so that people get proper credit.

## Contents

<b>1.1 Affine Varieties</b>	<b>2</b>
1.1 . . . . .	2
1.4 . . . . .	2
1.5 . . . . .	2
1.8 . . . . .	2
<b>1.2 Projective Varieties</b>	<b>2</b>
2.12 . . . . .	2
<b>1.3 Morphisms</b>	<b>3</b>
3.20 . . . . .	3
<b>2.1 Sheaves</b>	<b>5</b>
1.2 . . . . .	5
1.8 . . . . .	5

## 1.1 Affine Varieties

### 1.1

- a) Let  $Y$  be the plane curve  $y = x^2$  (i.e.,  $Y$  is the zero set of the polynomial  $f = y - x^2$ ). Show that  $A(Y)$  is isomorphic to a polynomial ring in one variable over  $k$ .
- b) Let  $Z$  be the plane curve  $xy = 1$ . Show that  $A(Z)$  is not isomorphic to a polynomial ring in one variable over  $k$ .
- c) Let  $f$  be any irreducible quadratic polynomial in  $k[x, y]$ , and let  $W$  be the conic defined by  $f$ . Show that  $A(W)$  is isomorphic to  $A(Y)$  or  $A(Z)$ . Which one is it when? [Optional]

### 1.1 (a) edealba

Note that  $A(Y)$  is the coordinate ring for the curve  $y - x^2 = 0$ , and since we're working over the plane, suppose we're working on  $\mathbb{A}^2$  (affine plane) over the field  $k$ , such that the coordinate ring of the plane is just  $k[x, y]$ . We want to show that  $A(Y) \cong k[x]$ .

Note that the coordinate ring  $A(Y)$  is defined as the following quotient ring:

$$A(Y) = k[x, y]/J,$$

where  $J = (y - x^2)$ , i.e.  $J$  is the ideal generated by  $y - x^2$ . Consider the following mapping:

$$\phi : k[x, y] \rightarrow k[x],$$

where  $\phi$  is an evaluation mapping on  $y$ , where  $y \mapsto x^2$ . Evaluation maps like  $\phi$  are ring homomorphisms. We will show that  $\ker(\phi)$  is exactly equivalent to  $J$  since all polynomials generated by  $y - x^2$  get mapped to  $x^2 - x^2 = 0$  via  $\phi$ , hence  $J \subset \ker(\phi)$ . To show equality, we need to show that  $\ker(\phi) \subset J$ . Let  $a \in \ker(\phi)$ , i.e.  $\phi(a) = 0$ . We want to show that  $a \in J$ , i.e. that we can write  $a$  as:

$$a = A(x, y) \cdot (y - x^2),$$

where  $A(x, y) \in k[x, y]$ . If  $a$  were *not* in  $J$  then we would have some non-zero remainder  $r(x, y)$  with degree of  $y$  less than 1 (since the degree of  $y$  in  $y - x^2$  is 1), so we can write the remainder as  $r(x)$  such that:

$$a = A(x, y) \cdot (y - x^2) + r(x).$$

Let's apply  $\phi$  to the RHS:  $\phi(A(x, y) \cdot (y - x^2) + r(x)) = A(x, x^2) \cdot (x^2 - x^2) + r(x) = A(x, x^2) \cdot 0 + r(x) = r(x)$ . Note that this implies that  $\phi(a) = r(x) = 0$ . This means that the remainder is zero, and hence  $a = A(x, y) \cdot (y - x^2)$  which means that  $a \in J$  like we wanted to show. Thus,  $\ker(\phi) = J$ .

Next, let's show that  $\phi$  is surjective. Let  $p(x) \in k[x]$  be an arbitrary polynomial of degree  $m$ , so we can write:

$$p(x) = \sum_{i=0}^m c_i x^i$$

Note that  $c_i \in k \subset k[x] \subset k[x, y]$  and also  $x^i \in k[x] \subset k[x, y]$  for all  $i$ , then clearly  $p(x) \in k[x, y]$  where  $\phi(p(x)) = p(x)$ . Since  $\phi$  is surjective, then we know that  $\text{im}(\phi) = k[x]$ . By the first isomorphism theorem, we get:

$$k[x, y]/\ker(\phi) \cong \text{im}(\phi),$$

where  $\ker(\phi) = J$  and  $\text{im}(\phi) = k[x]$ , thus

$$k[x, y]/J \cong k[x],$$

and finally since we know  $A(Y) = k[x, y]/J$ , then clearly we get that  $A(Y) \cong k[x]$ , so we've shown that the coordinate ring  $A(Y)$  is, indeed, isomorphic to the polynomial rings in one variable over  $k$ . □

## 1.1 (b) edealba

Note that  $A(Z)$  is the following:

$$A(Z) = \left\{ \sum_{i,j \geq 0} c_{ij} \bar{x}^i \bar{y}^j \mid c_{ij} \in k, \text{ only a finite number of } c_{ij} \text{ are non-zero, and } \bar{x}\bar{y} = 1 \right\}$$

Suppose there were a ring isomorphism  $\phi : A(Z) \rightarrow k[t]$ , where we choose  $t$  to be the polynomial ring of one variable over  $k$ . Note that ring isomorphisms must map units to units. Since we're working over the plane curve  $xy = 1$ , this means that the image of  $x$  in  $A(Z)$  is unit, i.e.  $\bar{x} \in A(Z)$  is unit. Note that the units in  $k[t]$  are all the non-zero constants, i.e.  $c \in k^\times \subset k[t]$ , where these are just the non-zero elements of the field  $k$ . Since  $\phi$  maps units to units, then  $\bar{x}$  must get mapped to some non-zero constant in  $k[t]$ . Note that if  $\bar{x}$  gets mapped to a constant, then  $\text{im}(\phi) = k$ . This is because if we have some arbitrary  $p \in A(Z)$  then we have:

$$p = \sum_{i,j \geq 0} c_{ij} \bar{x}^i \bar{y}^j,$$

but note that if  $\bar{x} \mapsto c_x$  where  $c_x$  is some non-zero constant in  $k$  then this implies that  $\bar{y} \mapsto c_y$  where  $c_y$  is also some non-zero constant such that  $c_y = 1/c_x$ . This means that  $\phi(p)$  is:

$$\phi(p) = \sum_{i,j \geq 0} c_{ij} c_x^i c_y^j,$$

which is just some constant element of  $k$  for any  $p \in A(Z)$ . Note that  $\text{im}(\phi) = k$  is a contradiction since we assumed  $\phi$  was an isomorphism, i.e.  $\text{im}(\phi) = k[t]$ . This means there is no such ring isomorphism  $\phi : A(Z) \rightarrow k[t]$ .  $\square$

## 1.4

If we identify  $\mathbb{A}^2$  with  $\mathbb{A}^1 \times \mathbb{A}^1$  in the natural way, show that the Zariski topology on  $\mathbb{A}^2$  is not the product topology of the Zariski topologies on the two copies of  $\mathbb{A}^1$ .

### 1.4 edealba

Consider the diagonal line  $y = x$  in  $\mathbb{A}^2$  where we're working over the coordinate ring  $k[x, y]$ . Note that  $V(y - x) \subset \mathbb{A}^2$  is closed in the Zariski topology of  $\mathbb{A}^2$ , and is the following:

$$V(y - x) = \{p \in k^2 : f(p) = 0 \text{ for all } f \in (y - x)\}$$

However, when we look at the closed sets in  $\mathbb{A}^1$ , we only have the following: (1) sets with a finite number of points, (2) the entire affine line  $\mathbb{A}^1$ . In  $\mathbb{A}^1 \times \mathbb{A}^1$  we have: (1) also sets with finite number of points, (2) the entire space  $\mathbb{A}^1 \times \mathbb{A}^1$ , and (3) we also finite unions of vertical and horizontal lines from  $\mathbb{A}^1 \times \{p_i\}$  and  $\{p_j\} \times \mathbb{A}^1$  for points  $p_i$  and  $p_j$  in  $\mathbb{A}^1$ . Note that it's *impossible* to construct the diagonal line  $y - x = 0$  using any combination of closed sets in  $\mathbb{A}^1 \times \mathbb{A}^1$ , we would need an infinite number of points which is *not* allowed in the Zariski topology. Thus, although  $\mathbb{A}^2$  and  $\mathbb{A}^1 \times \mathbb{A}^1$  may be identical to each other, their topologies are not the same.  $\square$

## 1.5

Show that a  $k$ -algebra  $B$  is isomorphic to the affine coordinate ring of some algebraic set in  $\mathbb{A}^n$ , for some  $n$ , if and only if  $B$  is a finitely generated  $k$ -algebra with no nilpotent elements.

## 1.5 edealba

Let  $B$  be a  $k$ -algebra, and let  $W = V(J) \subset \mathbb{A}^n$  be some algebraic set of  $\mathbb{A}^n$  for some ideal  $J \subset k[x_1, \dots, x_n]$ , where  $W$  is the set of all points that vanish for all polynomial functions in  $J$ . We then have the following coordinate ring (which is the following quotient ring):

$$A(W) = k[x_1, \dots, x_n]/I(W),$$

and since  $W = V(J)$ , by Hilbert's strong Nullstellensatz, we know that  $I(V(J)) = \sqrt{J}$ , hence

$$A(W) = k[x_1, \dots, x_n]/\sqrt{J}$$

We want to show that  $B \cong A(W)$  iff  $B$  is a finitely generated  $k$ -algebra with no nilpotent elements.

( $\Rightarrow$ ) First, suppose  $B \cong A(W)$ . Since we've defined  $A(W)$  to be the quotient ring of  $k[x_1, \dots, x_n]$  modded out by the radical of an ideal  $\sqrt{J}$ , then this is automatically a finitely generated  $k$ -algebra. We want to show that it has no nilpotent elements, i.e. there is *no* non-zero element  $f \in B \cong A(W)$  such that  $f^m = 0$  for some  $m \geq 2$ .

Suppose for contradiction that there exists some nilpotent element  $f \in B \cong A(W)$  such that  $f^m = 0$  for some  $m \geq 2$ . Since  $f$  is assumed to be non-zero, this means that  $f \notin I(W) = \sqrt{J}$ . Since  $J \subset \sqrt{J}$ , then  $f \notin \sqrt{J}$  implies that  $f \notin J$ . Note that since  $f \notin I(W)$  then this also means that there exists some point  $p' \in W$  such that  $f(p') \neq 0$ . Next, since we have  $f^m = 0$ , then  $f^m \in I(W) = \sqrt{J}$ . This means that  $f^m$  vanishes for all points  $p \in W$ , and hence  $f^m(p') = 0$ . Since  $f^m(p') = 0$ , this implies that  $f^m(p') = (f(p'))^m = 0$  which implies that  $f(p') = 0$  which contradicts our previous  $f(p') \neq 0$ . With this, we've shown that there *can't* be a nilpotent element  $f \in B \cong A(W)$ .

( $\Leftarrow$ ) Let  $B$  be a finitely generated  $k$ -algebra with generators  $\{t_1, \dots, t_n\} \subset B$  such that any element in  $B$  can be written as a polynomial in these generators, and  $B$  also has no nilpotent elements. Consider the following mapping:

$$\phi : k[x_1, \dots, x_n] \rightarrow B,$$

where  $\phi$  maps  $x_i \mapsto t_i$  for all  $1 \leq i \leq n$ . This mapping is clearly surjective since any polynomial  $f \in B$  can be written by swapping each  $t_i$  with its corresponding  $x_i$ . Note that if the generators  $\{t_1, \dots, t_n\}$  have some algebraic relations, this is captured by  $\ker(\phi)$ . Let  $J = \ker(\phi) \subset k[x_1, \dots, x_n]$  be the ideal generated by all the algebraic relations between the generators  $\{t_1, \dots, t_n\}$ . By the First isomorphism theorem, we get:

$$k[x_1, \dots, x_n]/J \cong B$$

Next, we want to show that  $J$  is a radical ideal. Note that there are *no* nilpotent elements in  $B$ . If there *were* some nilpotent element then suppose  $f \in B$  were nilpotent, then this would mean that  $f^m = 0$  for some  $m \geq 1$ .

(1) An ideal  $J$  is radical if for some  $x^m \in J$  with  $m \geq 1$  implies that  $x \in J$ , i.e.:

$$x^m \in J \implies x \in J$$

(2) Since  $B \cong k[x_1, \dots, x_n]/J$ , then elements of  $B$  are of the coset form  $\bar{f} = f + J$ . If  $\bar{f} \in B$  were nilpotent then this means that  $\bar{f}^m = 0$  for some  $m \geq 1$ , which means that  $f^m + J = 0$  which implies that  $f^m \in J$  for  $f \in k[x_1, \dots, x_n]$ .

Note that since there are *no* nilpotent elements in  $B$ , then this means for all non-zero  $\bar{f} \in B$  we know that  $\bar{f} = f + J \neq 0$ , i.e.  $f \notin J$  where  $f^m \neq 0$  for all  $m \geq 1$ . Furthermore, note that the contrapositive of (1) means that if the following holds:

$$x \notin J \implies x^m \notin J \text{ for all } m \geq 1,$$

then  $J$  is a radical ideal. We know that for any non-zero  $\bar{f}$  we have  $f \notin J$ , and since  $\bar{f}$  is *not* nilpotent, then  $f^m \notin J$  for all  $m \geq 1$ . With this, we've shown that  $B$  not having any nilpotent elements implies that  $J$  is a radical ideal, i.e.  $J = \sqrt{J}$ .

Finally, let's consider the algebraic set  $W = V(J) \subset \mathbb{A}^n$ . From Hilbert's strong Nullstellensatz, we know that the coordinate ring  $A(W)$  is the following:

$$A(W) \cong k[x_1, \dots, x_n]/I(V(J)) = k[x_1, \dots, x_n]/\sqrt{J},$$

but note that since  $J$  is radical, then we get:

$$A(W) \cong k[x_1, \dots, x_n]/J,$$

so we get that  $B \cong k[x_1, \dots, x_n]/J \cong A(W)$  like we wanted to show.  $\square$

## 1.8

Let  $Y$  be an affine variety of dimension  $r$  in  $\mathbb{A}^n$ . Let  $H$  be a hypersurface in  $\mathbb{A}^n$ , and assume that  $Y \not\subseteq H$ . Then every irreducible component of  $Y \cap H$  has dimension  $r - 1$ .

### 1.8 mlwells

Suppose that  $H = Z(f)$  with  $f$  irreducible in  $k[x_1, \dots, x_n]$ . Then the projection of  $f$  in  $A(Y) := k[x_1, \dots, x_n]/I(Y)$  is not equal to  $\bar{0}$  since by assumption  $(f) \not\subset I(Y)$ . Since  $A(Y)$  is a domain (due to the irreducibility of  $Y$ ), the element  $\bar{f}$  is not a zero divisor. Assuming that  $Y \cap H \neq \emptyset$ , we have that  $\bar{f}$  is not a unit in  $A(Y)$ . To see this, let  $P \in Y \cap H$ . Then  $I(Y), (f) \subset \mathfrak{m}_P$ , the maximal ideal of  $k[x_1, \dots, x_n]$  corresponding to  $P$ . This implies  $(\bar{f}) \subset \mathfrak{m}_P/I(Y)$ , the latter being a maximal ideal in  $A(Y)$ . Thus,  $\bar{f}$  is not a unit.

We apply Theorem 1.11A to get that every minimal prime ideal  $\mathfrak{p}$  in  $A(Y)$  containing  $\bar{f}$  has height 1. The irreducible components of  $Y \cap H$  and the minimal prime ideals containing  $(\bar{f})$  correspond, and since each of these prime ideals has height 1, the corresponding varieties have dimension  $r - 1$  by Theorem 1.8A.

## 1.2 Projective Varieties

### 2.12

*The d-Uple Embedding.* For given  $n, d > 0$ , let  $M_0, M_1, \dots, M_N$  be all the monomials of degree  $d$  in the  $n + 1$  variables  $x_0, \dots, x_n$ , where  $N = \binom{n+d}{d} - 1$ . We define a mapping  $\rho_d: \mathbb{P}^n \rightarrow \mathbb{P}^N$  by sending the point  $P = (a_0, \dots, a_n)$  to the point  $\rho_d(P) = (M_0(a), \dots, M_N(a))$  obtained by substituting the  $a_i$  in the monomials  $M_j$ . This is called the *d-uple embedding* of  $\mathbb{P}^n$  in  $\mathbb{P}^N$ . For example, if  $n = 1, d = 2$ , then  $N = 2$ , and the image  $Y$  of the 2-uple embedding of  $\mathbb{P}^1$  in  $\mathbb{P}^2$  is a conic.

- Let  $\theta: k[y_0, \dots, y_N] \rightarrow k[x_0, \dots, x_n]$  be the homomorphism defined by sending  $y_i$  to  $M_i$ , and let  $\mathfrak{a}$  be the kernel of  $\theta$ . Then  $\mathfrak{a}$  is a homogeneous prime ideal, and so  $Z(\mathfrak{a})$  is a projective variety in  $\mathbb{P}^N$ .
- Show that the image of  $\rho_d$  is exactly  $Z(\mathfrak{a})$ . (One inclusion is easy. The other will require some calculation.)
- Now show that  $\rho_d$  is a homeomorphism of  $\mathbb{P}^n$  onto the projective variety  $Z(\mathfrak{a})$ .
- Show that the twisted cubic curve in  $\mathbb{P}^3$  (Ex. 2.9) is equal to the 3-uple embedding of  $\mathbb{P}^1$  in  $\mathbb{P}^3$ , for suitable choice of coordinates.

## 2.12 (b) mlwells

Let  $Q = \rho_d(P) = (M_0(a), \dots, M_N(a))$ . Let  $F \in \mathfrak{a} = \ker \theta$ . Then  $F(M_0, \dots, M_N) = 0$  as a polynomial, which implies  $F(M_0(a), \dots, M_N(a)) = 0$ . Since  $F$  was arbitrary,  $Q \in Z(\mathfrak{a})$ .

Now let  $Q = (y_0, \dots, y_N) \in Z(\mathfrak{a})$ . Suppose without loss of generality that  $y_0 \neq 0$ . It follows that  $Q = (y_0^d, y_0^{d-1}y_1, \dots, y_0^{d-1}y_N)$ . Suppose that  $M_0 = x_0^d, M_1 = x_0^{d-1}x_1, \dots, M_n = x_0^{d-1}x_n$ .

**Proposition 1.1.** *For all  $j = 0, \dots, N$ , we have that*

$$M_0^{d-1}M_j = M_0^{i_0}M_1^{i_1} \dots M_n^{i_n} \quad (1)$$

for some  $i_j \geq 0$  with  $\sum_j i_j = d$ .

*Proof.* Suppose  $M_j = x_0^{i_0} \dots x_n^{i_n}$  where  $\sum_j i_j = d$ . We have

$$M_0^{d-1}M_j = x_0^{(d-1)d+i_0}x_1^{i_1} \dots x_n^{i_n} \quad (2)$$

$$= x_0^{(d-1)i_0+i_0+(d-1)\sum_{j \geq 1} i_j}x_1^{i_1} \dots x_n^{i_n} \quad (3)$$

$$= x_0^{di_0}(x_0^{d-1}x_1)^{i_1} \dots (x_0^{d-1}x_n)^{i_n} \quad (4)$$

$$= M_0^{i_0}M_1^{i_1} \dots M_n^{i_n} \quad (5)$$

□

Let  $P = (y_0, \dots, y_n)$ . Then, by the proposition, we have that  $Q = (M_0(P), M_1(P), \dots, M_N(P)) = \rho_d(P)$ , as was to be shown.

## 1.3 Morphisms

### 3.20

Let  $Y$  be a variety of dimension  $\geq 2$ , and let  $P \in Y$  be a normal point. Let  $f$  be a regular function on  $Y - P$ .

a) Show that  $f$  extends to a regular function on  $Y$ .

b) Show this would be false for  $\dim Y = 1$ .

See (III, Ex. 3.5) for generalization.

### 3.20 (a) mlwells

If we show that  $f$  restricted to  $V - P$  extends to a regular function on  $V \subseteq Y$  for any affine neighborhood  $V$  of  $P$ , then by gluing we will have shown that  $f$  extends to a regular function on  $Y$ . So, assume that  $Y$  is affine.

The function  $f = c/d \in K(Y)$  for regular functions  $c, d \in A(Y)$ . If we can show that the ideal quotient  $((c) : (d)) = \{h \in A(Y) : hc \in (d)\}$  is equal to  $A(Y)$ , then it follows that  $c/d \in A(Y)$ . By assumption,  $f \in \mathcal{O}_Q$  for all  $Q \neq P \in Y$ . Let  $\mathfrak{m}_Q$  denote the maximal ideal of  $A(Y)$  corresponding to  $Q$ . Then the localized ideal  $((c) : (d))_{\mathfrak{m}_Q}$  is equal to  $A(Y)_{\mathfrak{m}_Q} = \mathcal{O}_Q$  for all  $Q \neq P$  since  $c/d \in \mathcal{O}_Q$ . It remains to show that  $((c) : (d))_{\mathfrak{m}_P} = A(Y)_{\mathfrak{m}_P} = \mathcal{O}_P$ . If  $\mathfrak{p}A(Y)_{\mathfrak{m}_P}$  is a prime ideal of  $\mathcal{O}_P$  not equal to the maximal ideal  $\mathfrak{m}_PA(Y)_{\mathfrak{m}_P}$ , then  $((c) : (d))_{\mathfrak{m}_P} \not\subseteq \mathfrak{p}A(Y)_{\mathfrak{m}_P}$  since  $c/d$  is in the local ring corresponding to the subvariety defined by  $\mathfrak{p}$ , and hence  $((c) : (d))_{\mathfrak{p}} = A(Y)_{\mathfrak{p}}$ . Assume by way of contradiction that  $((c) : (d))_{\mathfrak{m}_P} \subseteq \mathfrak{m}_PA(Y)_{\mathfrak{m}_P}$ . Then,  $\sqrt{((c) : (d))_{\mathfrak{m}_P}} = \mathfrak{m}_P$ .

Let  $a_1, \dots, a_s$  be a regular sequence in  $\mathfrak{m}_P A(Y)_{\mathfrak{m}_P}$  with  $s > 1$ . By Theorem 8.22A of Chapter II of Hartshorne part (2), such a sequence exists. We have shown that  $a_1^{r_1}, a_2^{r_2} \in ((c) : (d))_{\mathfrak{m}_P}$  for some  $r_1, r_2 > 0$  with  $a_1^{r_1-1}, a_2^{r_2-1} \notin ((c) : (d))_{\mathfrak{m}_P}$ . Thus,

$$a_1^{r_1} c = e_1 d, \quad e_1 \notin (a_1) \tag{6}$$

and

$$a_2^{r_2} c = e_2 d, \quad e_2 \notin (a_2) \tag{7}$$

Thus,

$$a_1^{r_1} a_2^{r_2} c = e_1 a_2^{r_2} d = e_2 a_1^{r_1} d \tag{8}$$

which implies

$$e_1 a_2^{r_2} = e_2 a_1^{r_1} \tag{9}$$

Since  $e_1 \notin (a_1)$  and  $a_2 \notin (a_1)$ ,  $a_2$  is a zero divisor in the ring  $A(Y)_{\mathfrak{m}_P}/(a_1)$ . This contradicts the fact that  $a_1, a_2$  is a regular sequence in  $\mathfrak{m}_P A(Y)_{\mathfrak{m}_P}$ .

Thus, we conclude that  $((c) : (d))_{\mathfrak{m}_P} = A(Y)_{\mathfrak{m}_P}$ , and hence  $f \in \mathcal{O}_P$ . This shows that  $((c) : (d)) = A(Y)$  which implies  $f \in A(Y)$ , and hence  $f$  extends to a regular function on  $Y$ .

## 2.1 Sheaves

### 1.2

- a) For any morphism of sheaves  $\phi : \mathcal{F} \rightarrow \mathcal{G}$ , show that for each point  $P$ ,  $(\ker \phi)_P = \ker(\phi_P)$  and  $(\operatorname{im} \phi)_P = \operatorname{im}(\phi_P)$ .
- b) Show that  $\phi$  is injective (surjective) iff induced map on stalks  $\phi_P$  is injective (surjective) for all  $P$ .
- c) Show that the sequence of sheaves  $\cdots \rightarrow \mathcal{F}^{i-1} \rightarrow \mathcal{F}^i \rightarrow \mathcal{F}^{i+1} \rightarrow \cdots$

### 1.2 yakimk

- a) Since stalks are defined as a filtered colimit and kernels are a particular type of limit (pullback) and [filtered colimits commute with finite limits](#) isomorphism  $(\ker \phi)_P = \ker(\phi_P)$  is automatic.

To show a similar thing for the image we first note that  $(\operatorname{im} \phi)_P \simeq ((\operatorname{im}_{pre} \phi)^+)_P \simeq (\operatorname{im}_{pre} \phi)_P$ , where the first isomorphism is by definition and the second one using the fact that sheafification has the same stalks as original presheaf. Now we conclude noticing that  $\operatorname{im}_{pre} \phi \simeq \ker(\operatorname{coker} \phi)$ , i.e. it is a finite limit of finite colimits and since again stalks are filtered colimits they commute with finite limits and colimits (in for instance abelian category) we conclude that  $(\operatorname{im} \phi)_P \simeq \operatorname{im}(\phi_P)$ .

- b) We show that a map of sheaves  $\phi : \mathcal{F} \rightarrow \mathcal{G}$  is injective iff it induced maps on stalks are injective (proof for surjectivity is essentially identical).

Consider the following commutative diagram

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\phi_U} & \mathcal{G}(U) \\ \iota_1 \downarrow & & \downarrow \iota_2 \\ \prod_{P \in U} \mathcal{F}_P & \xrightarrow{\prod \tilde{\phi}_P} & \prod_{P \in U} \mathcal{G}_P \end{array}$$

Note that since section of a sheaf on an open set is determined by its stalks,  $\iota_1$  and  $\iota_2$  are both injective.

Suppose  $\tilde{\phi}_P$  is injective for all  $P$ . Take two sections  $f, g \in \mathcal{F}(U)$ , since  $\prod \tilde{\phi}_P$  and  $\iota_1$  are injective (former by the hypothesis that  $\tilde{\phi}_P$  are injective), their composition is injective. If  $\phi_U$  were not injective, we could find two different sections that go to the same class in  $\prod \mathcal{G}_P$ , which would contradict injectivity of  $\prod \tilde{\phi}_P \circ \iota_1$ .

Similarly if  $\phi_U$  is injective then if some of  $\tilde{\phi}_P$  were not injective, its composition with  $\iota_1$  would not be injective contradicting commutativity of the diagram above.

- c) We have to show that  $\cdots \rightarrow \mathcal{F}^{i-1} \rightarrow \mathcal{F}^i \rightarrow \mathcal{F}^{i+1} \rightarrow \cdots$  is exact iff induced sequence of stalks are exact for all  $P$ .

Assuming that sequence of sheaves is exact we show induced sequences are exact at all stalks. By the previous section  $\operatorname{im} \phi_P^i \simeq (\operatorname{im} \phi^i)_P \simeq (\ker \phi^{i+1})_P \simeq \ker \phi_P^{i+1}$ .

Now assume that all sequences of stalks are exact. By the same reasoning as before we see that  $\operatorname{im} \phi^i \simeq \ker \phi^{i+1}$  (since they are isomorphic on each stalk and hence are equal).

### 1.8

For any open set  $U \subseteq X$  the functor  $\Gamma(U, \cdot)$  from sheaves on  $X$  to abelian groups is left-exact.



## 1.8 yakimk

Given exact sequence of sheaves

$$0 \rightarrow \mathcal{F}' \xrightarrow{\phi} \mathcal{F} \xrightarrow{\gamma} \mathcal{F}'' \rightarrow 0$$

We have to show that

$$0 \rightarrow \Gamma(U, \mathcal{F}') \xrightarrow{\phi_U} \Gamma(U, \mathcal{F}) \xrightarrow{\gamma_U} \Gamma(U, \mathcal{F}'')$$

exact.

Injectivity of  $\phi_U$  follows from definition.

We show that  $\ker \gamma_U = \operatorname{im} \phi_U$ .

( $\ker \gamma_U \supseteq \operatorname{im} \phi_U$ ) Is immediate by functoriality of  $\Gamma(U, \cdot)$  and hypothesis.

( $\ker \gamma_U \subseteq \operatorname{im} \phi_U$ ) Let  $s \in \mathcal{F}(U)$  and  $\gamma_U(s) = 0$ . By hypothesis and “1.2” we know that the induced sequences of stalks are all exact. In particular for all  $P \in U$  induced sequences are all exact. Which means that each stalk  $s_P = \phi_P(k_P)$ , i.e. there is some section  $k \in \mathcal{F}'(U)$  and an open set  $V_P$  such that  $s|_{V_P} = \phi(k)|_{V_P}$ . Hence we can form an open cover  $U = \bigcup_{P \in U} V_P$ . Notice that by definition they agree on overlaps and hence we get a full section of  $\mathcal{F}'$  on  $U$  (call it  $q \in \mathcal{F}'(U)$ ) and since all the stalks of  $\phi_P(q_P) = s_P$  for all  $P \in U$  and hence are equal on  $U$ .