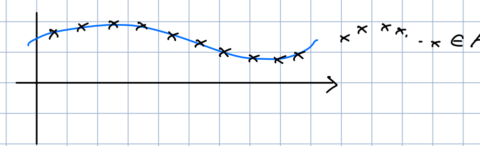
1. Inverse problems

A = input space (of signals to be reconstructed)

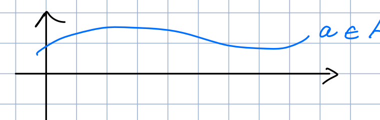
in general, A Hilbert space

e.g. $A = \mathbb{R}^n$ (discretised signals)



or A = function space (continuous signals)

$= L^2([0,1])$ or $L^2([0,1])^2$



B : output space (space of measurements, of data) e.g. Hilbert space or $\mathbb{R}^m$

$F : A \longrightarrow B$ (linear) forward map, models measurement process

Inverse problem: given

$$\underset{\text{data}}{b} = F \underset{\text{unknown to be reconstructed}}{a^+} + \underset{\text{noise}}{\varepsilon}$$

find a^+.

Key issue:

$F : A \longrightarrow B$ easy / stable

$F^{-1} : \operatorname{Im} F \longrightarrow A$ difficult / unstable

Hint: Naive approach (assume F injective $\Rightarrow$ F^{-1} exists):

$$a_{rec} = F^{-1} b = F^{-1}(F a^+ + \varepsilon)$$

$$= F^{-1} F a^+ + F^{-1} \varepsilon$$

$$= a^+ + F^{-1} \varepsilon$$

Hope: $\| F^{-1} \varepsilon \|$ small if $\| \varepsilon \|$ small

$$\Rightarrow \quad \| \underset{\text{reconstruction}}{a_{rec}} - \underset{\text{real one}}{a^+} \| = \| F^{-1} \varepsilon \|$$

$\rightarrow$ in general, not small

Example 1 (denoising)

$A = B = \mathbb{R}^n$ or $L^2([0,1])$ or $L^2([0,1]^2)$

$F : A \longrightarrow B, \quad F = Id$

$$b = Fa^+ + \varepsilon = a^+ + \varepsilon$$

IP: $\quad a^+ + \varepsilon \rightsquigarrow a^+$

Example 2 (deblurring)

$A = B = \mathbb{R}^n$ or $L^2([0,1])$ or $L^2([0,1]^2)$

$F a = g * a$

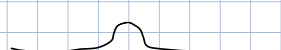
filter ↗ (g), convolution ↑ (*)

discrete: $Fa(j) = \sum_i a_i \, g_{j-i}$

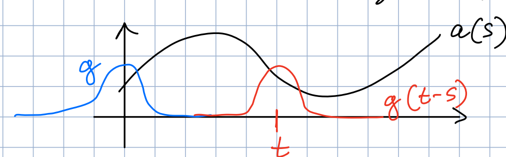
continuous: $Fa(t) = \int a(s)\, g(t-s)\, ds$

g:

Ⓓ $0 \; 0 \ldots 0 \; \frac{1}{4} \; \frac{1}{2} \; \frac{1}{4} \; 0 \ldots 0$

Ⓒ 

Fa: local averages of a

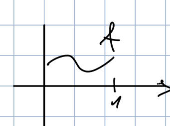


$a \longmapsto Fa$: blurring (easy) $\longleftrightarrow$ $+ \,/ \int$

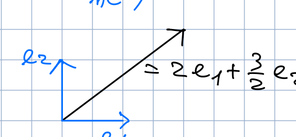
$Fa \longmapsto a$: deblurring (difficult) $\longleftrightarrow$ $- \,/ \frac{d}{ds}$

MINI-MINI-COURSE ON FOURIER ANALYSIS

$f : [0,1] \longrightarrow \mathbb{C}$



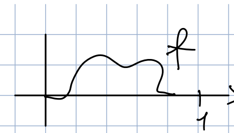
$$f(x) = \sum_{n \in \mathbb{Z}} \hat{f}(n) \, \underbrace{e^{\overbrace{2\pi i n x}}}_{e_n(x)}, \qquad e^{2\pi i n x} = \cos(2\pi n x) + i \sin(2\pi n x)$$



$$\hat{f}(n) = \int_0^1 f(x)\, e^{-2\pi i n x}\, dx$$

Convolution Theorem: $\quad \widehat{g * a}(n) = \hat{g}(n) \cdot \hat{a}(n)$

behaviour wrt d/dx :



$$\widehat{f'}(n) = \int_0^1 f'(x)\, e^{-2\pi i n x}\, dx$$

$$\overset{\text{IBP}}{=} (-2\pi i)\cdot n \int_0^1 f(x)\, e^{-2\pi i n x}\, dx$$

$$= (-2\pi i)\, n\, \hat{f}(n)$$

$f \longmapsto f'$ $\longleftrightarrow$ $\hat{f} \longmapsto (-2\pi i n)\hat{f}(n)$

in space (x) in frequency

$a \longmapsto Fa$: blurring (easy) $\Longleftrightarrow$ $+ / \int$ $\longleftrightarrow$ $\div(-2\pi i n)$ $^{(n\neq 0)}$ $\longrightarrow$ stable

$Fa \longmapsto a$: deblurring (difficult) $\Longleftrightarrow$ $- / \frac{d}{dx}$ $\longleftrightarrow$ $(-2\pi i n)\cdot$ $\rightarrow$ ill-posed unstable

$$\widehat{Fa} = \widehat{g * a} = \hat{g}\cdot\hat{a} = \sim\cdot\,\hat{a}$$ ✓ stable

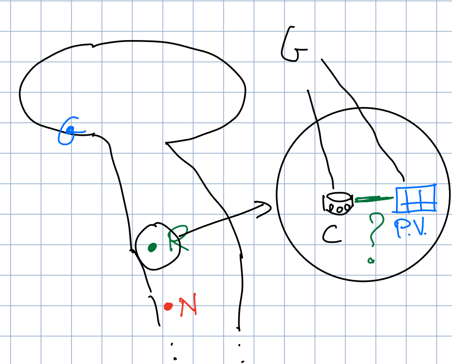
$Fa \longmapsto a$ in Fourier: $\dfrac{\hat{b}}{\hat{g}} = \hat{b}\cdot \smile$ ✓ unstable

$\uparrow \frac{1}{\hat{g}}$

$\hat{g}$:

<u>Example</u> (Italian geography)

By car:



G $\longrightarrow$ R : 5:15

R $\longrightarrow$ N : 2:35

G $\longrightarrow$ N : 7:08

+ 7:50

~ 10 %

G $\longrightarrow$ C : 5:50

G $\longrightarrow$ PV : 5:32

− 18 mins

$C \longrightarrow PV$: 3 mins $\sim 500\%$

This is True in general! Why?

$F : A \longrightarrow B$ compact, F has the SVD:

$$Fa = \sum_j \sigma_j \langle a, a_j \rangle b_j \qquad \sigma_{j+1} \le \sigma_j, \quad \sigma_j > 0$$

$\{a_j\}$ ONB of A, $\{b_j\}$ ONB of $\overline{\operatorname{Im} F}$

$$F^{-1} b = \sum_j \frac{1}{\sigma_j} \langle b, b_j \rangle a_j \qquad \boxed{\frac{1}{\sigma_j} \to +\infty} \qquad (\sigma_j > 0 \ \forall j)$$

$a \longmapsto Fa$: blurring (easy) $\longleftrightarrow$ $+ / \int$ $\longleftrightarrow$ $\div (-2\pi i n)$ $(n \neq 0)$ $\longrightarrow$ stable

$Fa \longmapsto a$: deblurring (difficult) $\longleftrightarrow$ $- / \frac{d}{dx}$ $\longleftrightarrow$ $(-2\pi i n) \cdot$ $\longrightarrow$ ill-posed unstable