GRAPHING FLEAS

DANIEL GRAZIAN, MICHAEL MEKONNEN, AGUSTIN O VENEZUELA III

ABSTRACT. We examine the evolution of a one-dimensinal lattice of labeled points. The points' labels change over time according to a rule. We prove that under our formulation of the problem every finite sequence of two distinct labels can be reached by a single rule. We then derive the amount of time required to reach every such sequence of a given length.

1. INTRODUCTION

In this paper, we examine patterns on a one-dimensional lattice, on which each point has a state associated with it. A flea travels along the lattice, and the state of a point may change when the flea visits it. Consider the following starting configuration:

$$W_0 : \quad \ldots \quad 4\ 0 \mid \overrightarrow{3}\ 1 \ldots$$

We number the points $\ldots, -2, -1, 0, 1, 2, \ldots$. A vertical bar $(|)$ separates points -1 and 0. The numbers at each point represent the state of that point. Here, points -1, 0, and 1 begin in states 0, 3, and 1 respectively. The flea, represented by an arrow, begins at the origin, pointed in the positive direction.

The state of the point on which the flea is located at time t determines both the state of that point at time $t+1$ and whether the flea moves forward or reverses direction to arrive at its new location at time $t+1$. The precise behavior will be determined by a rule associated with the setting. For example, the rule may specify that if the flea, at a given time step, is on a point in state 3, then that point will be in state 2 at the next time step, and the flea will reverse direction. In this case, the following configuration would succeed the one shown above:

$$W_1 : \quad \ldots \quad 4\ \overleftarrow{0} \mid 2\ 1 \ldots$$

We begin by precisely defining the terminology that we will use in the rest of the paper. We then present the main result of the paper, our finding that in a universe with three states, a single rule can generate

Date: September 24, 2012.

every possible finite sequence of two of those states. Finally, we derive the number of time steps required to reach every sequence of a given length.

2. Definitions and Notation

Before we prove our main result, we need to define several terms. Throughout this paper, all indices are zero-indexed (so if k is a sequence of elements, then $k[0]$ is the first element in the sequence. $\mathbb{N}$ refers to the set of non-negative integers, and S^* refers to the set of all (possibly empty) finite sequences whose elements are in the set S.

Definition 2.1. *A world W is a tuple $W = (t, \ell, d)$, $t : \mathbb{Z} \to \mathbb{N}$, $\ell \in \mathbb{Z}$, $d \in \{-1, 1\}$.*

Here, t maps each point on the lattice to a state. We use the nonnegative integers $\mathbb{N}$ to name the states. This is an arbitrary choice, but it is notationally simple. l specifies the location of the flea on the lattice, and d specifies the direction of the flea. The following is an example of a world:

$$W: \quad \ldots \quad 1\ 3\ 2 \mid \overleftarrow{2}\ 0\ 1 \ldots$$

As suggested in the introduction, the larger space is between points -1 and 0. Here, the flea is at point 0, so $\ell = 0$. The flea is pointed to the left, so $d = -1$. t maps each point to a state, so $t(-1) = 2$, $t(0) = 2$, and $t(1) = 0$, etc.

Definition 2.2. *For $s \in \mathbb{N}$, an s-rule r is a function $r : \{0, \ldots, s-1\} \to \{0, \ldots, s-1\} \times \{-1, 1\}$.*

r describes the change in worlds between time steps. The first element of the output of r describes the new state of the point ℓ where the flea existed. The second element of the output specifies whether or not the flea switches direction before stepping forward.

Consider the effect on the world $W = (t, \ell, d)$, described under Definition 2.1, of the 4-rule $r : r(0) = (2, 1), r(1) = (0, -1), r(2) = (1, -1), r(3) = (0, 1)$. The new world would look like this:

$$W' = (t', \ell', d') : \quad \ldots \quad 1\ 3\ 2 \mid\ 1\ \overrightarrow{0}\ 1 \ldots$$

In W, $\ell = 0$ (that is, the flea is on point 0), and point 0 is in state 2. Because $r(2)[0] = 1$, point 0 is in state 1 in W'. That is, $t'(0) = 1$. All other points' states are unchanged. Because $r(2)[1] = -1$, the flea reverses direction before stepping forward. So $d' = -d = 1$ and $\ell' = \ell + d \times -1 = 1$.

If r were applied again, this time to W', then we would get the world:

$$W'' = (t'', \ell'', d'') : \quad \ldots \quad 1\ 3\ 2\ |\ 1\ 2\ \overrightarrow{1} \ldots$$

Definition 2.3. *The* <u>*evolution*</u> *W^r of a world W under a rule r is the sequence $(W_i)_{i=0}^{\infty}$ of worlds obtained by iteratively applying r to W.:*

For example, the evolution of $W: \ \ldots \ 1\ 3\ 2\ \ \overleftarrow{2}\ 0\ 1\ \ldots$ under the rule, $r : r(0) = (2,1), r(1) = (0,-1), r(2) = (1,-1), r(3) = (0,1)$, begins as:

$$\begin{array}{ll} W_0 : & \ldots 1\ \ 3\ 2\ |\ \overleftarrow{2}\ 0\ 1 \ldots \\ W_1 : & \ldots 1\ \ 3\ 2\ |\ 1\ \overrightarrow{0}\ 1 \ldots \\ W_2 : & \ldots 1\ \ 3\ 2\ |\ 1\ 2\ \overrightarrow{1} \ldots \end{array}$$

This is just the sequence of worlds $W, W', W'', \ldots$ described below Definition 2.3. Each world in the sequence is obtained by applying r to the world before it.

Definition 2.4. *Let $b = (b_i)_{i=0}^{|b|-1} \in (\mathbb{N})^*$ be a finite sequence of positive integers. We say that world $W = (t, \ell, d)$* <u>*contains*</u> *b if $t(i) = b_i\ \forall 0 \leq i \leq |b| - 1$.*

For example, W_0, described below Definition 2.3, contains the sequences (2), $(2,0)$, and $(2,0,1)$.

Definition 2.5. *We say that an evolution W^r* <u>*accepts*</u> *$b \in (\mathbb{N})^*$ if $\exists W_i \in W^r$ such that W contains b.*

For example, the evolution W^r, described below Definition 2.3, accepts $(1,0,1)$, because $W_1 \in W^r$ contains $(1,0,1)$.

Definition 2.6. *For $k \in \mathbb{N}$, we say that a rule r is* <u>*k-complete*</u> *if the evolution $(t_0 : n \mapsto 0,0,1)^r$ accepts every element of $\overline{\{0,\ldots,k-1\}}^*$.*

In the following section, the main part of this paper, we show the existence of a 2-complete rule.

3. A 2-Complete Rule

Given our above notion of k-completeness, we now demonstrate by construction the existence of a 2-complete rule. Let $\mathring{r}_2$ be the following 3-rule:

$$\mathring{r}_2 : 0 \mapsto (1,-1),\ 1 \mapsto (2,1),\ 2 \mapsto (0,1)$$

We will also write $\mathring{W}_2$ to denote the evolution of the initial world under $\mathring{r}_2$. With these definitions, we can now present our main result.

$$\begin{array}{lcl} W_0: & \ldots & 0 \mid \overrightarrow{0} \ldots \\ W_1: & \ldots & \overleftarrow{0} \mid 1 \ldots \\ W_2: & \ldots & 1 \mid \overrightarrow{1} \ldots \end{array}$$

FIGURE 1. $\mathring{W}_2$ in the first 2 columns.

$$\begin{array}{lcl} W_i: & \ldots & 5\ 9\ 3\ \mathbf{0}\,\overrightarrow{\mathbf{0}}\ 7\ 0 \ldots \\ \downarrow & & \\ W_{i+2}: & \ldots & 5\ 9\ 3\ \mathbf{1}\,\overrightarrow{\mathbf{1}}\ 7\ 0 \ldots \end{array}$$

FIGURE 2. Example substitution of 00 with 11.

Theorem 3.1. *$\mathring{r}_2$ is 2-complete.*

Before we embark on a formal proof of the theorem, let's look at the worlds in $\mathring{W}_2$. Consider only the first few worlds (t, ℓ, d) in which $-1 \leq \ell \leq 0$, as shown in Figure **??**. The next world W_3 has $\ell > 0$, so we stop. The key observation is that, given any world W_i in the evolution containing $\overrightarrow{00}$, W_{i+2} will be the same world with $\overrightarrow{00}$ replaced by $\overrightarrow{11}$. Figure **??** shows an example.

Let's use this to list the first few worlds (t, ℓ, d) in which $-2 \leq \ell \leq 1$, as shown in Figure **??**. The next world W_{12} has $\ell > 1$ so we stop. Note how we were able to use the worlds W_0 to W_2 to determine worlds W_8 to W_{10}. We will perform one more iteration and list the first few worlds (t, ℓ, d) in which $-3 \leq \ell \leq 2$, as shown in Figure **??**.

The next world has $\ell > 2$ so we stop. There is a very noticeable pattern: if we write 0^k and 1^k to mean a sequence of k zeroes and ones, respectively, we have that, after some number of steps, $0^k\overrightarrow{0}0^{k-1}$ turns into $1^k2^{k-1}\overrightarrow{1}$. In fact, we can strengthen this hypothesis and prove the resulting statement by induction:

Lemma 3.2. *For every $k \in \mathbb{Z}^+$, there exists $i_k \in \mathbb{Z}^+$ such that:*

(1) *$W_{i_k-1} = (t_{i_k}, k-1, 1)$, where*

$$t_{i_k-1}(n) = \begin{cases} 0 & , \; n < -k \text{ or } k \leq n \\ 1 & , \; -k \leq n < 0 \text{ or } n = k-1 \\ 2 & , \; 0 \leq n < k-2 \end{cases}$$

(2) *For each world $W_i = (t_i, \ell_i, d_i)$ in the first i_k worlds of $\mathring{W}_2$, $-k \leq \ell < k$.*

$$
\begin{array}{lll}
W_0: & \dots & 0\ \mathbf{0} \mid \overrightarrow{\mathbf{0}}\ 0 \dots \\
\downarrow & & \\
W_2: & \dots & 0\ \mathbf{1} \mid \overrightarrow{\mathbf{1}}\ 0 \dots \\
W_3: & \dots & 0\ 1 \mid 2\ \overrightarrow{0} \dots \\
W_4: & \dots & 0\ 1 \mid \overleftarrow{2}\ 1 \dots \\
W_5: & \dots & 0\ \overleftarrow{1} \mid 0\ 1 \dots \\
W_6: & \dots & \overleftarrow{0}\ 2 \mid 0\ 1 \dots \\
W_7: & \dots & 1\ \overrightarrow{2} \mid 0\ 1 \dots \\
W_8: & \dots & 1\ \mathbf{0} \mid \overrightarrow{\mathbf{0}}\ 1 \dots \\
\downarrow & & \\
W_{10} & \dots & 1\ \mathbf{1} \mid \overrightarrow{\mathbf{1}}\ 1 \dots \\
W_{11} & \dots & 1\ 1 \mid 2\ \overrightarrow{1} \dots
\end{array}
$$

FIGURE 3. $\mathring{W}_2$ in the first 4 columns.

$$
\begin{array}{lll}
W_0: & \dots & 0\ \mathbf{0}\ \mathbf{0} \mid \overrightarrow{\mathbf{0}}\ \mathbf{0}\ 0 \dots \\
\downarrow & & \\
W_{11}: & \dots & 0\ \mathbf{1}\ \mathbf{1} \mid \mathbf{2}\ \overrightarrow{\mathbf{1}}\ 0 \dots \\
W_{12}: & \dots & 0\ 1\ 1 \mid 2\ 2\ \overrightarrow{0} \dots \\
W_{13}: & \dots & 0\ 1\ 1 \mid 2\ \overleftarrow{2}\ 1 \dots \\
W_{14}: & \dots & 0\ 1\ 1 \mid \overleftarrow{2}\ 0\ 1 \dots \\
W_{15}: & \dots & 0\ 1\ \overleftarrow{1} \mid 0\ 0\ 1 \dots \\
W_{16}: & \dots & 0\ \overleftarrow{1}\ 2 \mid 0\ 0\ 1 \dots \\
W_{17}: & \dots & \overleftarrow{0}\ 2\ 2 \mid 0\ 0\ 1 \dots \\
W_{18}: & \dots & 1\ \overrightarrow{2}\ 2 \mid 0\ 0\ 1 \dots \\
W_{19}: & \dots & 1\ 0\ \overrightarrow{2} \mid 0\ 0\ 1 \dots \\
W_{20}: & \dots & 1\ \mathbf{0}\ \mathbf{0} \mid \overrightarrow{\mathbf{0}}\ \mathbf{0}\ 1 \dots \\
\downarrow & & \\
W_{31}: & \dots & 1\ \mathbf{1}\ \mathbf{1} \mid \mathbf{2}\ \overrightarrow{\mathbf{1}}\ 1 \dots \\
W_{32}: & \dots & 1\ 1\ 1 \mid 2\ 2\ \overrightarrow{1} \dots
\end{array}
$$

FIGURE 4. $\mathring{W}_2$ in the first 6 columns.

(3) *For every finite sequence $b \in \{0,1\}^k$ of length k, some world W_i in the first i_k worlds of $\mathring{W}_2$ contains b.*

Proof: We proceed by induction on k. The truth of this statement for the base case $k = 1$ was demonstrated in Figure **??** with $i_1 = 3$. Note that $W_2 = (t_2, 0, 1)$, where t_2 satisfies the properties in (1) in the statement of the lemma. Furthermore, W_0 contains 0, W_1 contains 1, and for all of W_0, W_1, W_2, we have $-1 \leq \ell < 1$. It follows that (2) and (3) are also satisfied.

Now suppose the statement holds for some $k = k' \in \mathbb{Z}^+$. We first show that property (1) holds for $k = k' + 1$. By our induction hypothesis, property (1) gives that $W_{i_{k'}}$ can be written as $01^{k'}2^{k'-1}\overrightarrow{1}0$. Using $\mathring{r}_2$, we find that the subsequent worlds in $\mathring{W}_2$ are:

$$01^{k'}2^{k'-1}\overrightarrow{1}0 \implies 01^{k'}2^{k'}\overrightarrow{0} \implies \dots \implies \overleftarrow{0}2^{k'}0^{k'}1 \implies \dots \\ \implies \mathbf{1}\mathbf{0}^{k'}\overrightarrow{\mathbf{0}}\mathbf{0}^{k'-1}1$$

Note that we have $\mathbf{0}^{k'}\overrightarrow{\mathbf{0}}\mathbf{0}^{k'-1}$ in the latter world. Using property (2), we have that we can use property (1) again, as exemplified in Figure **??**:

$$\mathbf{1}\mathbf{0}^{k'}\overrightarrow{\mathbf{0}}\mathbf{0}^{k'-1}1 \implies \dots \implies 1\mathbf{1}^{k'}\mathbf{2}^{k'-1}\overrightarrow{\mathbf{1}}1 = 1^{k'+1}2^{k'-1}\overrightarrow{1}1 \\ \implies 1^{k'+1}2^{k'}\overrightarrow{1}$$

This proves that (1) holds for $k = k' + 1$, taking $i_{k'+1}$ to be one more than the index of the world $1^{k'+1}2^{k'}\overrightarrow{1}$. Also, it is clear that $-(k'+1) \leq \ell < k' + 1$ in the above steps so that (2) also holds.

It suffices to show that (3) holds. That is, that every sequence in $\{0,1\}^{k'+1}$ is contained by some world in the first $i_{k'+1}$ worlds of $\mathring{W}_2$. By our induction hypothesis, every sequence in $\{0,1\}^{k'}$ is contained by some world in the first $i_{k'}$ worlds of $\mathring{W}_2$. Since $\ell_i < k'$ for each such world, we must have that $t_i(k') = 0$ for these worlds, so that every sequence in $\{0,1\}^{k'} \times \{0\}$ is also contained by these worlds. It remains to show that the sequences in $\{0,1\}^{k'} \times \{1\}$ are also so contained. But this follows from our proof of (1): we had that the world $10^{k'}\overrightarrow{0}0^{k'-1}1$ occurs in the first $i_{k'+1}$ worlds of $\mathring{W}_2$, $i_{k'}$ steps after which we have the world $11^{k'}2^{k'-1}\overrightarrow{1}1$. In between these worlds, however, we clearly have worlds that contain each sequence in $\{0,1\}^{k'} \times \{1\}$.

Thus our induction is complete and we conclude that the result holds for all $k \in \mathbb{Z}^+$. $\square$

This lemma immediately gives us the 2-completeness of $\mathring{r}_2$: given any sequence $b \in \{0,1\}^*$, it is clear that $\mathring{W}_2$ accepts it. In particular, there exists a world in the first $i_{|b|}$ worlds of $\mathring{W}_2$ that contains it. This completes the proof. $\blacksquare$

Our analysis of $\mathring{W}_2$ has thus far been confined to the non-negative axis. For completeness, it is worth considering how $\mathring{W}_2$ behaves on the negative axis. In fact, $\mathring{r}_2$ generates each sequence symmetrically on the negative axis. We will make a brief digression and make this claim more precise:

Definition 3.3. *We will say a sequence* <u>*negatively contains*</u> *a sequence if the string appears in reverse on the negative axis, starting at location* -1*. We will say an evolution* <u>*negatively accepts*</u> *a sequence if some world in it negatively contains it. For fear of obscuring what is a quite natural definition, we will omit formulating these definitions with our notation from Section* **??**.

Our previous claim can now be written as follows:

Theorem 3.4. *If* $\mathring{W}_2$ *accepts* $b \in \{0,1,2\}^*$*, then* $\mathring{W}_2$ *negatively accepts* b*.*

As noted in Definition 3.3, we will avoid going into the same level of detail as we did in the proof of Theorem 3.1. A sketch of the proof is as follows.

Proof: For each world (t, ℓ, d), consider whether ℓ is non-negative or negative. Now consider splitting $\mathring{W}_2$ into contiguous subsequences where ℓ is either always non-negative or always negative. Finally pair the first two such subsequences, then the next two, and so on, ad infinitum.

For example, making reference to Figures **??** and **??**, we have that ℓ is non-negative in W_0, negative in W_1, non-negative from W_2 to W_4, negative from W_5 to W_7, etc., giving the following sequence:

$$((W_0),(W_1)),((W_2,W_3,W_4),(W_5,W_6,W_7)),((W_8),(W_9)),\ldots$$

Again, note that each element in this sequence is a pair of contiguous subsequences of $\mathring{W}_2$. It can be shown by induction that for each pair $((W_j),(W_k))$ of subsequences in this sequence, the following properties are satisfied:

(1) The length of (W_j) is equal to the length of (W_k).
(2) The r^{th} world in (W_k) can be derived as follows: the non-negative axis is the same as the non-negative axis of the last element of (W_j) and the negative axis is the same as the reverse of the non-negative axis of the r^{th} world in (W_j).

This immediately gives the desired result: if $\mathring{W}_2$ accepts some sequence b, then there exists a world with non-negative ℓ that contains b.

Thus the world must be an element of the second subsequence in some element of the above sequence. Suppose this world is the r^{th} world of (W_k) in the pair $((W_j),(W_k))$. By the second property above, the r^{th} world of (W_j) must negatively contain b. $\square$

4. "Efficiency" of Completeness

In the previous section, we showed that $\mathring{r}_2$ is 2-complete, i.e. the evolution $\mathring{W}_2$ accepts every sequence in $\{0,1\}^*$. We also showed that $\mathring{W}_2$ negatively accepts every sequence in $\{0,1\}^*$. Next, we look at the number of steps it takes $\mathring{r}_2$ to generate all sequences in $\{0,1\}^*$ of length at most n, both starting from 0 forwards and starting from -1 backwards. Since there are 2^n binary sequences of length n, we expect $\mathring{r}_2$ to take time at least on the order of 2^n to generate all these sequences. It turns out that $\mathring{r}_2$ takes time exactly on the order of 2^n, and we shall develop this result below.

Definition 4.1. *For a positive integer $n \in \mathbb{Z}^+$, let T_n denote the number of steps it takes $\mathring{r}_2$ to generate all binary sequences of length at most n. That is, let T_n be the minimum integer $i \in \mathbb{N}$ such that for every non-empty binary sequence $b \in \{0,1\}^*$ of length at most n, there exist integers j and k such that $0 \le j,k \le i$, the world $W_j \in \mathring{W}_2$ contains b, and the world $W_k \in \mathring{W}_2$ negatively contains b.*

The following Lemma describes the progression from W_0 to W_{T_n} in $\mathring{W}_2$. While developing the form of W_{T_n}, we shall come up with a recurrence describing T_n.

Lemma 4.2. *For any $n \in \mathbb{Z}^+$, the world $W_{T_n} \in \mathring{W}_2$ has the form*

$$W_{T_n}: \quad \dots \quad 0 \quad 1^n \Big| \overrightarrow{1} \quad 1^{n-1} \quad 0 \quad \dots$$

and the progression from W_0 to W_{T_n} does not step outside of the locations between $-n$ and $n-1$, inclusive.

Proof: We prove this by induction on n.

The result can be easily seen for the base case, $n = 1$, where we have $T_1 = 2$. There are exactly two non-empty binary sequences of length at most 1, namely 0 and 1. As shown in Figure **??**, W_0 contains and negatively contains 0, W_1 contains 1, and W_2 negatively contains 1. Note that the world W_2 has exactly the desired form, and the progression from W_0 to W_2 does not step outside of locations -1 and 0.

Now for $n > 1$, suppose that the world $W_{T_{n-1}}$ has the form

$$W_{T_{n-1}}: \quad \dots \quad 0 \quad 1^{n-1} \Big| \quad \overrightarrow{1} \quad 1^{n-2} \quad 0 \quad \dots$$

$$
\begin{array}{lccccc|ccccc}
W_0: & \ldots & 0 & 0 & 0^{n-1} & \overrightarrow{0} & 0^{n-2} & 0 & 0 & \ldots \\
\downarrow & & & & & & & & & \\
W_{T_{n-1}}: & \ldots & 0 & 0 & 1^{n-1} & \overrightarrow{1} & 1^{n-2} & 0 & 0 & \ldots \\
\downarrow & & & & & & & & & \\
W_{T_{n-1}+n-1}: & \ldots & 0 & 0 & 1^{n-1} & 2 & 2^{n-2} & \overrightarrow{0} & 0 & \ldots \\
\downarrow & & & & & & & & & \\
W_{T_{n-1}+3n-2}: & \ldots & 0 & \overleftarrow{0} & 2^{n-1} & 0 & 0^{n-2} & 1 & 0 & \ldots \\
\downarrow & & & & & & & & & \\
W_{T_{n-1}+4n-2}: & \ldots & 0 & 1 & 0^{n-1} & \overrightarrow{0} & 0^{n-2} & 1 & 0 & \ldots \\
\downarrow & & & & & & & & & \\
W_{2T_{n-1}+4n-2}: & \ldots & 0 & 1 & 1^{n-1} & \overrightarrow{1} & 1^{n-2} & 1 & 0 & \ldots
\end{array}
$$

FIGURE 5. Progression from W_0 to $W_{T_n} = W_{2T_{n-1}+4n-2}$.

and that the progression from W_0 to $W_{T_{n-1}}$ does not step outside of the locations between $-(n-1)$ and $n-2$, inclusive. Let us use this inductive assumption to step through the evolution $\mathring{W}_2$ until we reach the world W_{T_n}. The progression from W_0 to W_{T_n} is shown in Figure **??**. Based on our assumption, between worlds W_0 and $W_{T_{n-1}}$, we generate all binary sequences of length n *ending in a 0*. Next, between worlds $W_{T_{n-1}+4n-2}$ and $W_{2T_{n-1}+4n-2}$, we generate all the remaining sequences of length n, namely the ones *ending in a 1*. Therefore, the last world in the sequence, $W_{2T_{n-1}+4n-2}$, is indeed W_{T_n}. It has the desired form, and the progression to it from W_0 does not step outside of the locations between $-n$ and $n-1$, inclusive. This completes the induction.

□

The following Corollary gives us a recurrence for T_n, which we shall solve to get a closed form expression.

Corollary 4.3. *For any $n \in \mathbb{Z}^+$, we have*

$$T_n = 2 \cdot T_{n-1} + 4n - 2.$$

Proof: This recurrence follows directly from the conclusion in Lemma **??** that

$$W_{T_n} = W_{2T_{n-1}+4n-2}.$$

□

Finally, we state T_n in closed form by solving the recurrence.

Theorem 4.4. $T_n = 6 \cdot 2^n - 4n - 6.$

Proof: We can prove this equality by induction. First we check the base case, $n = 1$:

$$T_1 = 6 \cdot 2^1 - 4 \cdot 1 - 6 = 2.$$

Next, for $n > 1$, let us assume that $T_{n-1} = 6 \cdot 2^{n-1} - 4(n-1) - 6$. With this inductive assumption and the recurrence from Corollary **??**, we can solve for T_n:

$$\begin{aligned} T_n &= 2T_{n-1} + 4n - 2 \\ &= 2(6 \cdot 2^{n-1} - 4(n-1) - 6) + 4n - 2 \\ &= 6 \cdot 2^n - 4n - 6. \end{aligned}$$

This completes the induction.

□

Corollary 4.5. *T_n grows on the order of 2^n.*

Proof: This follows directly from the expression for T_n. □

5. Further Work

In this paper, we have discussed in detail a 3-rule $\mathring{r}_2$ that is 2-complete. It is easy to show that there does not exist a 2-rule that is 2-complete by looking at all 16 2-rules. An interesting question to ask is whether there exists an s-rule that is s-complete for some integer $s > 2$.

Another unrelated question we considered is identifying the conditions on an s-rule r that would result in the flea being bounded between two locations in $\mathbb{Z}$. There are some very simple conditions we can develop such as if $r(0) = (\sigma, 1)$ for any $\sigma \in \{0, \ldots, s-1\}$, then, starting from the world $W_0 = (t(n) = 0 \forall n, 0, 1)$, the flea will always step to the right, which means that it would be unbounded. It would be interesting to develop an algorithm that can check the boundedness of a flea under any given rule.