Graphing Fleas

Daniel Grazian, Michael Mekonnen, Justin Venezuela

Massachusetts Institute of Technology

October 22^{nd}, 2012

Discrete Number Line With States and Flea

Discrete Number Line With States and Flea

$$\ldots \mid 1 \mid 0 \mid 2 \mid 1 \mid 0 \mid 0 \mid 1 \mid \ldots$$

Discrete Number Line With States and Flea

$$\ldots \mid 1 \mid 0 \mid 2 \mid \overrightarrow{1} \mid 0 \mid 0 \mid 1 \mid \ldots$$

Discrete Number Line With States and Flea

$$\ldots \mid 1 \mid 0 \mid 2 \mid 1 \mid \overrightarrow{0} \mid 0 \mid 1 \mid \ldots$$

Rule determines behavior

Rule determines behavior

State of point where flea is located determines:

Rule determines behavior

State of point where flea is located determines:

1. what state the point changes to when the flea steps off.
2. whether the flea changes direction before stepping off.

Rule determines behavior

State of point where flea is located determines:

1. what state the point changes to when the flea steps off.
2. whether the flea changes direction before stepping off.

- Many possible rules
- We focus on particularly interesting ones.
- Can a rule produce every sequence of states?

Rule determines behavior

- A rule consists of a series of statements of the form:
  'When the flea steps off a point in state X, the point changes to state Y, and the flea does/doesn't reverse direction before stepping off.'

Rule determines behavior

- A rule consists of a series of statements of the form:
  'When the flea steps off a point in state X, the point changes to state Y, and the flea does/doesn't reverse direction before stepping off.'
- One such statement for every possible state.

Rule determines behavior

- A rule consists of a series of statements of the form:
  'When the flea steps off a point in state X, the point changes to state Y, and the flea does/doesn't reverse direction before stepping off.'
- One such statement for every possible state.
- $(2k)^k$ possible rules, where k is the number of allowed states.

Rule determines behavior

$$\ldots \mid 1 \mid 0 \mid 1 \mid \overrightarrow{1} \mid 0 \mid 0 \mid 1 \mid \ldots$$

Rule determines behavior

$$\ldots \,\big|\, 1 \,\big|\, 0 \,\big|\, 1 \,\big|\, \overrightarrow{1} \,\big|\, 0 \,\big|\, 0 \,\big|\, 1 \,\big|\, \ldots$$

$$\ldots \,\big|\, 1 \,\big|\, 0 \,\big|\, 1 \,\big|\, 1 \,\big|\, \overrightarrow{0} \,\big|\, 0 \,\big|\, 1 \,\big|\, \ldots$$

'When the flea steps off a point in state 1, the point changes to state 1, and the flea doesn't reverse direction before stepping off.'

Rule determines behavior

$$\ldots \mid 1 \mid 0 \mid 1 \mid \overrightarrow{1} \mid 0 \mid 0 \mid 1 \mid \ldots$$

$$\ldots \mid 1 \mid 0 \mid 1 \mid 1 \mid \overrightarrow{0} \mid 0 \mid 1 \mid \ldots$$

'When the flea steps off a point in state 1, the point changes to state 1, and the flea doesn't reverse direction before stepping off.'

$$\ldots \mid 1 \mid 0 \mid 1 \mid \overleftarrow{1} \mid 1 \mid 0 \mid 1 \mid \ldots$$

'When the flea steps off a point in state 0, the point changes to state 1, and the flea reverses direction before stepping off.'

Rule determines behavior

Formally, a rule is a function from state to state and direction change

Rule determines behavior

Formally, a rule is a function from state to state and direction change

$r : \{0, \dots, k-1\} \to (\{0, \dots, k-1\}, \{-1, 1\})$

where k is the number of allowed states.

Rule determines behavior

Formally, a rule is a function from state to state and direction change

$r : \{0, \dots, k-1\} \to (\{0, \dots, k-1\}, \{-1, 1\})$

where k is the number of allowed states.

Example

$$r : \begin{array}{ccc} 0 & \to & (1, -1) \\ 1 & \to & (1, 1) \end{array}$$

Definitions and Notation

Definitions and Notation

- A *world* describes the state at every point and gives the location and direction of the flea.

$$W_0: \quad \dots \quad 0 \quad 0 \quad 0 \quad 0 \quad | \quad \overrightarrow{0} \quad 0 \quad 0 \quad 0 \quad \dots$$

Definitions and Notation

- A *world* describes the state at every point and gives the location and direction of the flea.

$$W_0: \quad \ldots \quad 0 \quad 0 \quad 0 \quad 0 \quad | \quad \overrightarrow{0} \quad 0 \quad 0 \quad 0 \quad \ldots$$

- The *evolution* $E(r)$ of a rule r is the infinite sequence $(W_0, W_1, W_2, \ldots)$ of worlds produced by iteratively applying r to W_0:

Definitions and Notation

- A *world* describes the state at every point and gives the location and direction of the flea.

$$W_0: \quad \ldots \quad 0 \quad 0 \quad 0 \quad 0 \quad | \quad \overrightarrow{0} \quad 0 \quad 0 \quad 0 \quad \ldots$$

- The *evolution* $E(r)$ of a rule r is the infinite sequence $(W_0, W_1, W_2, \ldots)$ of worlds produced by iteratively applying r to W_0:
- A world *contains* a finite sequence $(s_0, s_1, \ldots, s_{n-1})$ of states if points $0, 1, \ldots, n-1$ are in states $s_0, s_1, \ldots, s_{n-1}$ respectively.

Definitions and Notation

- A *world* describes the state at every point and gives the location and direction of the flea.

$$W_0 : \quad \dots \quad 0 \quad 0 \quad 0 \quad 0 \quad | \quad \overrightarrow{0} \quad 0 \quad 0 \quad 0 \quad \dots$$

- The *evolution* $E(r)$ of a rule r is the infinite sequence $(W_0, W_1, W_2, \dots)$ of worlds produced by iteratively applying r to W_0:
- A world *contains* a finite sequence $(s_0, s_1, \dots, s_{n-1})$ of states if points $0, 1, \dots, n-1$ are in states $s_0, s_1, \dots, s_{n-1}$ respectively.
- An evolution E(r) *accepts* a sequence of states if any world in E(r) contains the sequence.

Example

Example

$$r:\begin{array}{ccc} 0 & \to & (1,-1) \\ 1 & \to & (1,1) \end{array}$$

Example

Example

$$r: \begin{array}{ccc} 0 & \to & (1,-1) \\ 1 & \to & (1,1) \end{array}$$

$$\begin{array}{cccccccccccc} W_0 & \dots & 0 & 0 & 0 & 0 & | & \overrightarrow{0} & 0 & 0 & 0 & \dots \\ W_1 & \dots & 0 & 0 & 0 & \overleftarrow{0} & | & 1 & 0 & 0 & 0 & \dots \end{array}$$

Example

Example

$$r: \begin{array}{ccc} 0 & \to & (1,-1) \\ 1 & \to & (1,1) \end{array}$$

$$\begin{array}{cccccc|ccccc}
W_0 & \dots & 0 & 0 & 0 & 0 & \overrightarrow{0} & 0 & 0 & 0 & \dots \\
W_1 & \dots & 0 & 0 & 0 & \overleftarrow{0} & 1 & 0 & 0 & 0 & \dots \\
W_2 & \dots & 0 & 0 & 0 & 1 & \overrightarrow{1} & 0 & 0 & 0 & \dots \\
W_3 & \dots & 0 & 0 & 0 & 1 & 1 & \overrightarrow{0} & 0 & 0 & \dots
\end{array}$$

Example

Example

$$r: \begin{array}{ccc} 0 & \rightarrow & (1,-1) \\ 1 & \rightarrow & (1,1) \end{array}$$

$$\begin{array}{cccccccccccc}
W_0 & \dots & 0 & 0 & 0 & 0 & | & \overrightarrow{0} & 0 & 0 & 0 & \dots \\
W_1 & \dots & 0 & 0 & 0 & \overleftarrow{0} & | & 1 & 0 & 0 & 0 & \dots \\
W_2 & \dots & 0 & 0 & 0 & 1 & | & \overrightarrow{1} & 0 & 0 & 0 & \dots \\
W_3 & \dots & 0 & 0 & 0 & 1 & | & 1 & \overrightarrow{0} & 0 & 0 & \dots
\end{array}$$

- The first four worlds of the *evolution* E(r) of r are shown above.
- W_2 *contains* $(1,0)$ and $(1,0,0)$
- Therefore E(r) *accepts* $(1,0)$ and $(1,0,0)$.

A k-complete Rule

A k-complete Rule

A rule r is *k-complete* if the evolution of r accepts every finite sequence of the states $0, \ldots, k-1$.

A k-complete Rule

A rule r is *k-complete* if the evolution of r accepts every finite sequence of the states $0, \ldots, k-1$.

For example, a 2-complete rule must accept:

A k-complete Rule

A rule r is *k-complete* if the evolution of r accepts every finite sequence of the states $0, \ldots, k-1$.

For example, a 2-complete rule must accept:

(0)
(1)

A k-complete Rule

A rule r is *k-complete* if the evolution of r accepts every finite sequence of the states $0, \ldots, k-1$.

For example, a 2-complete rule must accept:

(0)
(1)
$(0,1)$
$(1,0)$
$(0,1)$
$(1,1)$

A k-complete Rule

A rule r is *k-complete* if the evolution of r accepts every finite sequence of the states $0, \ldots, k-1$.

For example, a 2-complete rule must accept:

(0)
(1)
$(0, 1)$
$(1, 0)$
$(0, 1)$
$(1, 1)$
$(0, 0, 0)$
$\ldots$
$(1, 1, 1)$
$\ldots$

Our Main Result

Our Main Result

Theorem

We can construct a k-complete rule over $k+1$ states for all $k \in \mathbb{Z}^+$.

Our Main Result

Theorem

We can construct a k-complete rule over $k+1$ states for all $k \in \mathbb{Z}^+$.

In this talk, we'll give a 2-complete rule over 3 states.

Our Main Result

Theorem

We can construct a k-complete rule over $k+1$ states for all $k \in \mathbb{Z}^+$.

In this talk, we'll give a 2-complete rule over 3 states.

We will then show how 'fast' this rule is in generating every possible sequence of 0's and 1's.

Example of a 2-complete rule

Example of a 2-complete rule

$$R : \begin{array}{ccc} 0 & \rightarrow & (1, -1) \\ 1 & \rightarrow & (2, 1) \\ 2 & \rightarrow & (0, 1) \end{array}$$

First few worlds of $E(R)$

$$R: \begin{array}{ccc} 0 & \rightarrow & (1,-1) \\ 1 & \rightarrow & (2,1) \\ 2 & \rightarrow & (0,1) \end{array}$$

First few worlds of $E(R)$

$$R: \begin{array}{ccc} 0 & \to & (1,-1) \\ 1 & \to & (2,1) \\ 2 & \to & (0,1) \end{array}$$

$$\ldots \quad 0 \quad 0 \quad 0 \quad | \quad \overrightarrow{0} \quad 0 \quad 0 \quad \ldots$$

First few worlds of $E(R)$

$$
R: \begin{array}{ccc} 0 & \rightarrow & (1,-1) \\ 1 & \rightarrow & (2,1) \\ 2 & \rightarrow & (0,1) \end{array}
$$

$$
\begin{array}{cccccccccc}
\dots & 0 & 0 & 0 & | & \overrightarrow{0} & 0 & 0 & \dots \\
\dots & 0 & 0 & \overleftarrow{0} & | & 1 & 0 & 0 & \dots
\end{array}
$$

First few worlds of $E(R)$

$$R: \begin{array}{rcl} 0 & \rightarrow & (1,-1) \\ 1 & \rightarrow & (2,1) \\ 2 & \rightarrow & (0,1) \end{array}$$

$$\begin{array}{cccccccc} \ldots & 0 & 0 & 0 & | & \overrightarrow{0} & 0 & 0 & \ldots \\ \ldots & 0 & 0 & \overleftarrow{0} & | & 1 & 0 & 0 & \ldots \\ \ldots & 0 & 0 & 1 & | & \overrightarrow{1} & 0 & 0 & \ldots \end{array}$$

First few worlds of $E(R)$

$$R: \begin{array}{ccc} 0 & \rightarrow & (1,-1) \\ 1 & \rightarrow & (2,1) \\ 2 & \rightarrow & (0,1) \end{array}$$

$$\begin{array}{cccccccc} \dots & 0 & 0 & 0 & | & \overrightarrow{0} & 0 & 0 & \dots \\ \dots & 0 & 0 & \overleftarrow{0} & | & 1 & 0 & 0 & \dots \\ \dots & 0 & 0 & 1 & | & \overrightarrow{1} & 0 & 0 & \dots \\ \dots & 0 & 0 & 1 & | & 2 & \overrightarrow{0} & 0 & \dots \end{array}$$

First few worlds of $E(R)$

$$R: \begin{array}{ccc} 0 & \to & (1,-1) \\ 1 & \to & (2,1) \\ 2 & \to & (0,1) \end{array}$$

$$\begin{array}{cccc|cccc} \dots & 0 & 0 & 0 & \overrightarrow{0} & 0 & 0 & \dots \\ \dots & 0 & 0 & \overleftarrow{0} & 1 & 0 & 0 & \dots \\ \dots & 0 & 0 & 1 & \overrightarrow{1} & 0 & 0 & \dots \\ \dots & 0 & 0 & 1 & 2 & \overrightarrow{0} & 0 & \dots \\ \dots & 0 & 0 & 1 & \overleftarrow{2} & 1 & 0 & \dots \end{array}$$

First few worlds of $E(R)$

$$R: \begin{array}{ccc} 0 & \to & (1,-1) \\ 1 & \to & (2,1) \\ 2 & \to & (0,1) \end{array}$$

$$\begin{array}{cccccccccc}
\dots & 0 & 0 & 0 & | & \overrightarrow{0} & 0 & 0 & \dots \\
\dots & 0 & 0 & \overleftarrow{0} & | & 1 & 0 & 0 & \dots \\
\dots & 0 & 0 & 1 & | & \overrightarrow{1} & 0 & 0 & \dots \\
\dots & 0 & 0 & 1 & | & 2 & \overrightarrow{0} & 0 & \dots \\
\dots & 0 & 0 & 1 & | & \overleftarrow{2} & 1 & 0 & \dots \\
\dots & 0 & 0 & \overleftarrow{1} & | & 0 & 1 & 0 & \dots
\end{array}$$

Demo!

Claim

Claim

Theorem

R is 2-complete.

Claim

Theorem

R is 2-complete.

Key observation:

$E(R)$ accepts all sequences of length n before the flea leaves $[-n, n-1]$.

$E(R)$ in $[-1, 0]$

$E(R)$ in $[-1, 0]$

$$\ldots \quad 0 \quad | \quad \overrightarrow{0} \quad \ldots$$

$E(R)$ in $[-1, 0]$

$$\begin{array}{ccc|ccc} \dots & & 0 & \overrightarrow{0} & & \dots \\ \dots & \overleftarrow{0} & & & 1 & \dots \end{array}$$

$E(R)$ in $[-1, 0]$

$$
\begin{array}{cccccc}
\ldots & 0 & | & \overrightarrow{0} & \ldots \\
\ldots & \overleftarrow{0} & | & 1 & \ldots \\
\ldots & 1 & | & \overrightarrow{1} & \ldots
\end{array}
$$

$E(R)$ in $[-2, 1]$

$E(R)$ in $[-2, 1]$

$$\ldots \quad 0 \quad \mathbf{0} \quad | \quad \overrightarrow{\mathbf{0}} \quad 0 \quad \ldots$$

$E(R)$ in $[-2, 1]$

$$\begin{array}{cccccc} \ldots & 0 & \mathbf{0} & \mid \overrightarrow{\mathbf{0}} & 0 & \ldots \\ \downarrow & & & & & \\ \ldots & 0 & \mathbf{1} & \mid \overrightarrow{\mathbf{1}} & 0 & \ldots \end{array}$$

$E(R)$ in $[-2, 1]$

$$
\begin{array}{ccccccc}
\dots & 0 & \mathbf{0} & | & \overrightarrow{\mathbf{0}} & 0 & \dots \\
\downarrow & & & & & & \\
\dots & 0 & \mathbf{1} & | & \overrightarrow{\mathbf{1}} & 0 & \dots \\
\dots & 0 & 1 & | & 2 & \overrightarrow{0} & \dots
\end{array}
$$

$E(R)$ in $[-2, 1]$

$$
\begin{array}{ccccccc}
\ldots & 0 & \mathbf{0} & | & \overrightarrow{\mathbf{0}} & 0 & \ldots \\
\downarrow & & & & & & \\
\ldots & 0 & \mathbf{1} & | & \overrightarrow{\mathbf{1}} & 0 & \ldots \\
\ldots & 0 & 1 & | & 2 & \overrightarrow{0} & \ldots \\
\ldots & 0 & 1 & | & \overleftarrow{2} & 1 & \ldots
\end{array}
$$

$E(R)$ in $[-2, 1]$

$$
\begin{array}{cccccc}
\ldots & 0 & \mathbf{0} & \big| & \overrightarrow{\mathbf{0}} & 0 & \ldots \\
\downarrow & & & & & & \\
\ldots & 0 & \mathbf{1} & \big| & \overrightarrow{\mathbf{1}} & 0 & \ldots \\
\ldots & 0 & 1 & \big| & 2 & \overrightarrow{0} & \ldots \\
\ldots & 0 & 1 & \big| & \overleftarrow{2} & 1 & \ldots \\
\ldots & 0 & \overleftarrow{1} & \big| & 0 & 1 & \ldots
\end{array}
$$

$E(R)$ in $[-2, 1]$

$$
\begin{array}{ccc|ccc}
\dots & 0 & \mathbf{0} & \overrightarrow{\mathbf{0}} & 0 & \dots \\
\downarrow & & & & & \\
\dots & 0 & \mathbf{1} & \overrightarrow{\mathbf{1}} & 0 & \dots \\
\dots & 0 & 1 & 2 & \overrightarrow{0} & \dots \\
\dots & 0 & 1 & \overleftarrow{2} & 1 & \dots \\
\dots & 0 & \overleftarrow{1} & 0 & 1 & \dots \\
\dots & \overleftarrow{0} & 2 & 0 & 1 & \dots
\end{array}
$$

$E(R)$ in $[-2, 1]$

$$
\begin{array}{cccc|ccc}
\dots & 0 & \mathbf{0} & & \overrightarrow{\mathbf{0}} & 0 & \dots \\
\downarrow & & & & & & \\
\dots & 0 & \mathbf{1} & & \overrightarrow{\mathbf{1}} & 0 & \dots \\
\dots & 0 & 1 & & 2 & \overrightarrow{0} & \dots \\
\dots & 0 & 1 & & \overleftarrow{2} & 1 & \dots \\
\dots & 0 & \overleftarrow{1} & & 0 & 1 & \dots \\
\dots & \overleftarrow{0} & 2 & & 0 & 1 & \dots \\
\dots & 1 & \overrightarrow{2} & & 0 & 1 & \dots
\end{array}
$$

$E(R)$ in $[-2, 1]$

$$
\begin{array}{cccccc}
\ldots & 0 & \mathbf{0} & | & \overrightarrow{\mathbf{0}} & 0 & \ldots \\
\downarrow & & & & & & \\
\ldots & 0 & \mathbf{1} & | & \overrightarrow{\mathbf{1}} & 0 & \ldots \\
\ldots & 0 & 1 & | & 2 & \overrightarrow{0} & \ldots \\
\ldots & 0 & 1 & | & \overleftarrow{2} & 1 & \ldots \\
\ldots & 0 & \overleftarrow{1} & | & 0 & 1 & \ldots \\
\ldots & \overleftarrow{0} & 2 & | & 0 & 1 & \ldots \\
\ldots & 1 & \overrightarrow{2} & | & 0 & 1 & \ldots \\
\ldots & 1 & \mathbf{0} & | & \overrightarrow{\mathbf{0}} & 1 & \ldots
\end{array}
$$

$E(R)$ in $[-2, 1]$

$$
\begin{array}{cccc|ccc}
\ldots & 0 & \mathbf{0} & & \overrightarrow{\mathbf{0}} & 0 & \ldots \\
\downarrow & & & & & & \\
\ldots & 0 & \mathbf{1} & & \overrightarrow{\mathbf{1}} & 0 & \ldots \\
\ldots & 0 & 1 & & 2 & \overrightarrow{0} & \ldots \\
\ldots & 0 & 1 & & \overleftarrow{2} & 1 & \ldots \\
\ldots & 0 & \overleftarrow{1} & & 0 & 1 & \ldots \\
\ldots & \overleftarrow{0} & 2 & & 0 & 1 & \ldots \\
\ldots & 1 & \overrightarrow{2} & & 0 & 1 & \ldots \\
\ldots & 1 & \mathbf{0} & & \overrightarrow{\mathbf{0}} & 1 & \ldots \\
\downarrow & & & & & & \\
\ldots & 1 & \mathbf{1} & & \overrightarrow{\mathbf{1}} & 1 & \ldots
\end{array}
$$

$E(R)$ in $[-2, 1]$

$$
\begin{array}{cccccc}
\ldots & 0 & \mathbf{0} & | & \overrightarrow{\mathbf{0}} & 0 & \ldots \\
\downarrow & & & & & & \\
\ldots & 0 & \mathbf{1} & | & \overrightarrow{\mathbf{1}} & 0 & \ldots \\
\ldots & 0 & 1 & | & 2 & \overrightarrow{0} & \ldots \\
\ldots & 0 & 1 & | & \overleftarrow{2} & 1 & \ldots \\
\ldots & 0 & \overleftarrow{1} & | & 0 & 1 & \ldots \\
\ldots & \overleftarrow{0} & 2 & | & 0 & 1 & \ldots \\
\ldots & 1 & \overrightarrow{2} & | & 0 & 1 & \ldots \\
\ldots & 1 & \mathbf{0} & | & \overrightarrow{\mathbf{0}} & 1 & \ldots \\
\downarrow & & & & & & \\
\ldots & 1 & \mathbf{1} & | & \overrightarrow{\mathbf{1}} & 1 & \ldots \\
\ldots & 1 & 1 & | & 2 & \overrightarrow{1} & \ldots
\end{array}
$$

$E(R)$ in $[-3, 2]$

$E(R)$ in $[-3, 2]$

$$\ldots \quad 0 \quad \mathbf{0} \quad \mathbf{0} \quad | \quad \overrightarrow{\mathbf{0}} \quad \mathbf{0} \quad 0 \quad \ldots$$

$E(R)$ in $[-3, 2]$

$$
\begin{array}{ccccccccc}
\dots & 0 & \mathbf{0} & \mathbf{0} & | & \overrightarrow{\mathbf{0}} & \mathbf{0} & 0 & \dots \\
\downarrow & & & & & & & & \\
\dots & 0 & \mathbf{1} & \mathbf{1} & | & \mathbf{2} & \overrightarrow{\mathbf{1}} & 0 & \dots
\end{array}
$$

$E(R)$ in $[-3, 2]$

$$
\begin{array}{cccc|cccc}
\dots & 0 & \mathbf{0} & \mathbf{0} & \overrightarrow{\mathbf{0}} & \mathbf{0} & 0 & \dots \\
\downarrow & & & & & & & \\
\dots & 0 & \mathbf{1} & \mathbf{1} & \mathbf{2} & \overrightarrow{\mathbf{1}} & 0 & \dots \\
\dots & 0 & 1 & 1 & 2 & 2 & \overrightarrow{0} & \dots
\end{array}
$$

$E(R)$ in $[-3, 2]$

$$
\begin{array}{cccccccc}
\ldots & 0 & \mathbf{0} & \mathbf{0} & | & \overrightarrow{\mathbf{0}} & \mathbf{0} & 0 & \ldots \\
\downarrow & & & & & & & & \\
\ldots & 0 & \mathbf{1} & \mathbf{1} & | & \mathbf{2} & \overrightarrow{\mathbf{1}} & 0 & \ldots \\
\ldots & 0 & 1 & 1 & | & 2 & 2 & \overrightarrow{0} & \ldots \\
\downarrow & & & & & & & & \\
\ldots & \overleftarrow{0} & 2 & 2 & | & 0 & 0 & 1 & \ldots
\end{array}
$$

$E(R)$ in $[-3, 2]$

$$
\begin{array}{cccccccc}
\ldots & 0 & \mathbf{0} & \mathbf{0} & | & \overrightarrow{\mathbf{0}} & \mathbf{0} & 0 & \ldots \\
\downarrow & & & & & & & & \\
\ldots & 0 & \mathbf{1} & \mathbf{1} & | & \mathbf{2} & \overrightarrow{\mathbf{1}} & 0 & \ldots \\
\ldots & 0 & 1 & 1 & | & 2 & 2 & \overrightarrow{0} & \ldots \\
\downarrow & & & & & & & & \\
\ldots & \overleftarrow{0} & 2 & 2 & | & 0 & 0 & 1 & \ldots \\
\downarrow & & & & & & & & \\
\ldots & 1 & \mathbf{0} & \mathbf{0} & | & \overrightarrow{\mathbf{0}} & \mathbf{0} & 1 & \ldots
\end{array}
$$

$E(R)$ in $[-3, 2]$

$$
\begin{array}{cccc|cccc}
\ldots & 0 & \mathbf{0} & \mathbf{0} & \overrightarrow{\mathbf{0}} & \mathbf{0} & 0 & \ldots \\
\downarrow & & & & & & & \\
\ldots & 0 & \mathbf{1} & \mathbf{1} & \mathbf{2} & \overrightarrow{\mathbf{1}} & 0 & \ldots \\
\ldots & 0 & 1 & 1 & 2 & 2 & \overrightarrow{0} & \ldots \\
\downarrow & & & & & & & \\
\ldots & \overleftarrow{0} & 2 & 2 & 0 & 0 & 1 & \ldots \\
\downarrow & & & & & & & \\
\ldots & 1 & \mathbf{0} & \mathbf{0} & \overrightarrow{\mathbf{0}} & \mathbf{0} & 1 & \ldots \\
\downarrow & & & & & & & \\
\ldots & 1 & \mathbf{1} & \mathbf{1} & \mathbf{2} & \overrightarrow{\mathbf{1}} & 1 & \ldots
\end{array}
$$

$E(R)$ in $[-3,2]$

$$
\begin{array}{cccc|cccc}
\dots & 0 & \mathbf{0} & \mathbf{0} & \overrightarrow{\mathbf{0}} & \mathbf{0} & 0 & \dots \\
\downarrow & & & & & & & \\
\dots & 0 & \mathbf{1} & \mathbf{1} & \mathbf{2} & \overrightarrow{\mathbf{1}} & 0 & \dots \\
\dots & 0 & 1 & 1 & 2 & 2 & \overrightarrow{0} & \dots \\
\downarrow & & & & & & & \\
\dots & \overleftarrow{0} & 2 & 2 & 0 & 0 & 1 & \dots \\
\downarrow & & & & & & & \\
\dots & 1 & \mathbf{0} & \mathbf{0} & \overrightarrow{\mathbf{0}} & \mathbf{0} & 1 & \dots \\
\downarrow & & & & & & & \\
\dots & 1 & \mathbf{1} & \mathbf{1} & \mathbf{2} & \overrightarrow{\mathbf{1}} & 1 & \dots \\
\dots & 1 & 1 & 1 & 2 & 2 & \overrightarrow{1} & \dots
\end{array}
$$

Proof of 2-completeness

Proof of 2-completeness

By induction.

Proof of 2-completeness

By induction. Induction hypothesis:

- $0^n \overrightarrow{0}\, 0^{n-1} \to 1^n 2^{n-1} \overrightarrow{1}$

Proof of 2-completeness

By induction. Induction hypothesis:

- $0^n \overrightarrow{0} 0^{n-1} \to 1^n 2^{n-1} \overrightarrow{1}$
- ... while staying within these columns

Proof of 2-completeness

By induction. Induction hypothesis:

- $0^n \overrightarrow{0} 0^{n-1} \rightarrow 1^n 2^{n-1} \overrightarrow{1}$
- ... while staying within these columns
- ... while accepting all sequences of length n

Proof of 2-completeness

By induction. Induction hypothesis:

- $0^n\overrightarrow{0}\,0^{n-1} \to 1^n 2^{n-1}\overrightarrow{1}$
- ... while staying within these columns
- ... while accepting all sequences of length n $\square$

Proof of 2-completeness

By induction. Induction hypothesis:

- $0^n \overrightarrow{0}\, 0^{n-1} \rightarrow 1^n 2^{n-1} \overrightarrow{1}$
- ... while staying within these columns
- ... while accepting all sequences of length n $\square$

Corollary

We can similarly define the notions of <u>*negative containment*</u> *and* <u>*negative acceptance*</u>*.*

Proof of 2-completeness

By induction. Induction hypothesis:

- $0^n \overrightarrow{0}\, 0^{n-1} \to 1^n 2^{n-1} \overrightarrow{1}$
- ... while staying within these columns
- ... while accepting all sequences of length n $\square$

Corollary

We can similarly define the notions of <u>*negative containment*</u> *and* <u>*negative acceptance*</u>*.*

Every sequence accepted by $E(R)$ is also negatively contained.

R is 2-complete

$E(R)$ accepts and negatively accepts every finite binary sequence.

R is 2-complete

Example

$b = 01$

R is 2-complete

Example

$b = 01$

$$\ldots \quad 0 \quad 0 \quad 0 \quad | \quad \overrightarrow{0} \quad 0 \quad 0 \quad \ldots$$

R is 2-complete

Example

$b = 01$

$$\begin{array}{cccccccccc} \ldots & 0 & 0 & 0 & | & \overrightarrow{0} & 0 & 0 & \ldots \\ \ldots & 0 & 0 & \overleftarrow{0} & | & 1 & 0 & 0 & \ldots \end{array}$$

R is 2-complete

Example

$b = 01$

$$\begin{array}{cccccccccc} \dots & 0 & 0 & 0 & | & \overrightarrow{0} & 0 & 0 & \dots \\ \dots & 0 & 0 & \overleftarrow{0} & | & 1 & 0 & 0 & \dots \\ \dots & 0 & 0 & 1 & | & \overrightarrow{1} & 0 & 0 & \dots \end{array}$$

R is 2-complete

Example

$b = 01$

$$\begin{array}{cccccccccc}
\ldots & 0 & 0 & 0 & | & \overrightarrow{0} & 0 & 0 & \ldots \\
\ldots & 0 & 0 & \overleftarrow{0} & | & 1 & 0 & 0 & \ldots \\
\ldots & 0 & 0 & 1 & | & \overrightarrow{1} & 0 & 0 & \ldots \\
\ldots & 0 & 0 & 1 & | & 2 & \overrightarrow{0} & 0 & \ldots
\end{array}$$

R is 2-complete

Example

$b = 01$

$$\begin{array}{ccccccccc}
\ldots & 0 & 0 & 0 & | & \overrightarrow{0} & 0 & 0 & \ldots \\
\ldots & 0 & 0 & \overleftarrow{0} & | & 1 & 0 & 0 & \ldots \\
\ldots & 0 & 0 & 1 & | & \overrightarrow{1} & 0 & 0 & \ldots \\
\ldots & 0 & 0 & 1 & | & 2 & \overrightarrow{0} & 0 & \ldots \\
\ldots & 0 & 0 & 1 & | & \overleftarrow{2} & 1 & 0 & \ldots
\end{array}$$

R is 2-complete

Example

$b = 01$

$$
\begin{array}{ccccccccc}
\dots & 0 & 0 & 0 & | & \overrightarrow{0} & 0 & 0 & \dots \\
\dots & 0 & 0 & \overleftarrow{0} & | & 1 & 0 & 0 & \dots \\
\dots & 0 & 0 & 1 & | & \overrightarrow{1} & 0 & 0 & \dots \\
\dots & 0 & 0 & 1 & | & 2 & \overrightarrow{0} & 0 & \dots \\
\dots & 0 & 0 & 1 & | & \overleftarrow{2} & 1 & 0 & \dots \\
\dots & 0 & 0 & \overleftarrow{1} & | & \mathbf{0} & \mathbf{1} & 0 & \dots
\end{array}
$$

R is 2-complete

Example

$b = 01$

$$\begin{array}{ccccccccc}
\dots & 0 & 0 & 0 & | & \overrightarrow{0} & 0 & 0 & \dots \\
\dots & 0 & 0 & \overleftarrow{0} & | & 1 & 0 & 0 & \dots \\
\dots & 0 & 0 & 1 & | & \overrightarrow{1} & 0 & 0 & \dots \\
\dots & 0 & 0 & 1 & | & 2 & \overrightarrow{0} & 0 & \dots \\
\dots & 0 & 0 & 1 & | & \overleftarrow{2} & 1 & 0 & \dots \\
\dots & 0 & 0 & \overleftarrow{1} & | & \mathbf{0} & \mathbf{1} & 0 & \dots \\
\dots & 0 & \overleftarrow{0} & 2 & | & 0 & 1 & 0 & \dots
\end{array}$$

R is 2-complete

Example

$b = 01$

$$
\begin{array}{ccccccccc}
\dots & 0 & 0 & 0 & | & \overrightarrow{0} & 0 & 0 & \dots \\
\dots & 0 & 0 & \overleftarrow{0} & | & 1 & 0 & 0 & \dots \\
\dots & 0 & 0 & 1 & | & \overrightarrow{1} & 0 & 0 & \dots \\
\dots & 0 & 0 & 1 & | & 2 & \overrightarrow{0} & 0 & \dots \\
\dots & 0 & 0 & 1 & | & \overleftarrow{2} & 1 & 0 & \dots \\
\dots & 0 & 0 & \overleftarrow{1} & | & \mathbf{0} & \mathbf{1} & 0 & \dots \\
\dots & 0 & \overleftarrow{0} & 2 & | & 0 & 1 & 0 & \dots \\
\dots & 0 & 1 & \overrightarrow{2} & | & 0 & 1 & 0 & \dots
\end{array}
$$

R is 2-complete

Example

$b = 01$

$$
\begin{array}{ccccc|ccccc}
\dots & 0 & 0 & 0 & & \overrightarrow{0} & 0 & 0 & \dots \\
\dots & 0 & 0 & \overleftarrow{0} & & 1 & 0 & 0 & \dots \\
\dots & 0 & 0 & 1 & & \overrightarrow{1} & 0 & 0 & \dots \\
\dots & 0 & 0 & 1 & & 2 & \overrightarrow{0} & 0 & \dots \\
\dots & 0 & 0 & 1 & & \overleftarrow{2} & 1 & 0 & \dots \\
\dots & 0 & 0 & \overleftarrow{1} & & \mathbf{0} & \mathbf{1} & 0 & \dots \\
\dots & 0 & \overleftarrow{0} & 2 & & 0 & 1 & 0 & \dots \\
\dots & 0 & 1 & \overrightarrow{2} & & 0 & 1 & 0 & \dots \\
\dots & 0 & \mathbf{1} & \mathbf{0} & & \overrightarrow{0} & 1 & 0 & \dots
\end{array}
$$

How fast is R?

After how many worlds do we see all binary sequences of length n?

How fast is R?

After how many worlds do we see all binary sequences of length n?

Surely $\Omega(2^n)$.

How fast is R?

Definition

T_n: minimum number worlds to see all finite binary sequences of length n.

T_1

Example

$T_1 = ?$

Looking for 0 and 1.

T_1

Example

$$\ldots \quad 0 \quad 0 \quad \mathbf{0} \quad | \quad \overrightarrow{\mathbf{0}} \quad 0 \quad 0 \quad \ldots$$

T_1

Example

$$\begin{array}{cccccccccc} \dots & 0 & 0 & \mathbf{0} & | & \overrightarrow{\mathbf{0}} & 0 & 0 & \dots \\ \dots & 0 & 0 & \overleftarrow{0} & | & \mathbf{1} & 0 & 0 & \dots \end{array}$$

T_1

Example

$$
\begin{array}{cccccccccc}
\dots & 0 & 0 & \mathbf{0} & | & \overrightarrow{\mathbf{0}} & 0 & 0 & \dots \\
\dots & 0 & 0 & \overleftarrow{0} & | & \mathbf{1} & 0 & 0 & \dots \\
\dots & 0 & 0 & \mathbf{1} & | & \overrightarrow{1} & 0 & 0 & \dots
\end{array}
\qquad T_1 = 2
$$

T_2

Example

$T_2 = ?$
Looking for 00, 01, 10, and 11.

T_2

Example

$$\ldots \quad 0 \quad \mathbf{0} \quad \mathbf{0} \quad | \quad \overrightarrow{\mathbf{0}} \quad \mathbf{0} \quad 0 \quad \ldots$$

T_2

Example

$$
\begin{array}{cccccccccc}
\dots & 0 & \mathbf{0} & \mathbf{0} & | & \overrightarrow{\mathbf{0}} & \mathbf{0} & 0 & \dots \\
\dots & 0 & 0 & \overleftarrow{0} & | & \mathbf{1} & \mathbf{0} & 0 & \dots
\end{array}
$$

T_2

Example

$$
\begin{array}{cccccccccc}
\ldots & 0 & \mathbf{0} & \mathbf{0} & | & \overrightarrow{\mathbf{0}} & \mathbf{0} & 0 & \ldots \\
\ldots & 0 & 0 & \overleftarrow{0} & | & \mathbf{1} & \mathbf{0} & 0 & \ldots \\
\ldots & 0 & \mathbf{0} & \mathbf{1} & | & \overrightarrow{1} & 0 & 0 & \ldots
\end{array}
$$

T_2

Example

$$
\begin{array}{cccc|cccc}
\dots & 0 & \mathbf{0} & \mathbf{0} & \overrightarrow{\mathbf{0}} & \mathbf{0} & 0 & \dots \\
\dots & 0 & 0 & \overleftarrow{0} & \mathbf{1} & \mathbf{0} & 0 & \dots \\
\dots & 0 & \mathbf{0} & \mathbf{1} & \overrightarrow{1} & 0 & 0 & \dots \\
\dots & 0 & 0 & 1 & 2 & \overrightarrow{0} & 0 & \dots
\end{array}
$$

T_2

Example

$$
\begin{array}{cccc|cccc}
\dots & 0 & \mathbf{0} & \mathbf{0} & \overrightarrow{\mathbf{0}} & \mathbf{0} & 0 & \dots \\
\dots & 0 & 0 & \overleftarrow{0} & \mathbf{1} & \mathbf{0} & 0 & \dots \\
\dots & 0 & \mathbf{0} & \mathbf{1} & \overrightarrow{1} & 0 & 0 & \dots \\
\dots & 0 & 0 & 1 & 2 & \overrightarrow{0} & 0 & \dots \\
\dots & 0 & 0 & 1 & \overleftarrow{2} & 1 & 0 & \dots
\end{array}
$$

T_2

Example

$$
\begin{array}{cccccccccc}
\dots & 0 & \mathbf{0} & \mathbf{0} & | & \overrightarrow{\mathbf{0}} & \mathbf{0} & 0 & \dots \\
\dots & 0 & 0 & \overleftarrow{0} & | & \mathbf{1} & \mathbf{0} & 0 & \dots \\
\dots & 0 & \mathbf{0} & \mathbf{1} & | & \overrightarrow{1} & 0 & 0 & \dots \\
\dots & 0 & 0 & 1 & | & 2 & \overrightarrow{0} & 0 & \dots \\
\dots & 0 & 0 & 1 & | & \overleftarrow{2} & 1 & 0 & \dots \\
\dots & 0 & 0 & \overleftarrow{1} & | & \mathbf{0} & \mathbf{1} & 0 & \dots
\end{array}
$$

T_2

Example

$$
\begin{array}{cccc|cccc}
\dots & 0 & \mathbf{0} & \mathbf{0} & \overrightarrow{\mathbf{0}} & \mathbf{0} & 0 & \dots \\
\dots & 0 & 0 & \overleftarrow{0} & \mathbf{1} & \mathbf{0} & 0 & \dots \\
\dots & 0 & \mathbf{0} & \mathbf{1} & \overrightarrow{1} & 0 & 0 & \dots \\
\dots & 0 & 0 & 1 & 2 & \overrightarrow{0} & 0 & \dots \\
\dots & 0 & 0 & 1 & \overleftarrow{2} & 1 & 0 & \dots \\
\dots & 0 & 0 & \overleftarrow{1} & \mathbf{0} & \mathbf{1} & 0 & \dots \\
\dots & 0 & \overleftarrow{0} & 2 & 0 & 1 & 0 & \dots
\end{array}
$$

T_2

Example

$$
\begin{array}{ccccccccc}
\ldots & 0 & \mathbf{0} & \mathbf{0} & | & \overrightarrow{\mathbf{0}} & \mathbf{0} & 0 & \ldots \\
\ldots & 0 & 0 & \overleftarrow{0} & | & \mathbf{1} & \mathbf{0} & 0 & \ldots \\
\ldots & 0 & \mathbf{0} & \mathbf{1} & | & \overrightarrow{1} & 0 & 0 & \ldots \\
\ldots & 0 & 0 & 1 & | & 2 & \overrightarrow{0} & 0 & \ldots \\
\ldots & 0 & 0 & 1 & | & \overleftarrow{2} & 1 & 0 & \ldots \\
\ldots & 0 & 0 & \overleftarrow{1} & | & \mathbf{0} & \mathbf{1} & 0 & \ldots \\
\ldots & 0 & \overleftarrow{0} & 2 & | & 0 & 1 & 0 & \ldots \\
\ldots & 0 & 1 & \overrightarrow{2} & | & 0 & 1 & 0 & \ldots
\end{array}
$$

T_2

Example

$$
\begin{array}{ccccccccc}
\dots & 0 & \mathbf{0} & \mathbf{0} & | & \overrightarrow{\mathbf{0}} & \mathbf{0} & 0 & \dots \\
\dots & 0 & 0 & \overleftarrow{0} & | & \mathbf{1} & \mathbf{0} & 0 & \dots \\
\dots & 0 & \mathbf{0} & \mathbf{1} & | & \overrightarrow{1} & 0 & 0 & \dots \\
\dots & 0 & 0 & 1 & | & 2 & \overrightarrow{0} & 0 & \dots \\
\dots & 0 & 0 & 1 & | & \overleftarrow{2} & 1 & 0 & \dots \\
\dots & 0 & 0 & \overleftarrow{1} & | & \mathbf{0} & \mathbf{1} & 0 & \dots \\
\dots & 0 & \overleftarrow{0} & 2 & | & 0 & 1 & 0 & \dots \\
\dots & 0 & 1 & \overrightarrow{2} & | & 0 & 1 & 0 & \dots \\
\dots & 0 & \mathbf{1} & \mathbf{0} & | & \overrightarrow{0} & 1 & 0 & \dots
\end{array}
$$

T_2

Example

$$
\begin{array}{cccc|cccc}
\ldots & 0 & \mathbf{0} & \mathbf{0} & \overrightarrow{\mathbf{0}} & \mathbf{0} & 0 & \ldots \\
\ldots & 0 & 0 & \overleftarrow{0} & \mathbf{1} & \mathbf{0} & 0 & \ldots \\
\ldots & 0 & \mathbf{0} & \mathbf{1} & \overrightarrow{1} & 0 & 0 & \ldots \\
\ldots & 0 & 0 & 1 & 2 & \overrightarrow{0} & 0 & \ldots \\
\ldots & 0 & 0 & 1 & \overleftarrow{2} & 1 & 0 & \ldots \\
\ldots & 0 & 0 & \overleftarrow{1} & \mathbf{0} & \mathbf{1} & 0 & \ldots \\
\ldots & 0 & \overleftarrow{0} & 2 & 0 & 1 & 0 & \ldots \\
\ldots & 0 & 1 & \overrightarrow{2} & 0 & 1 & 0 & \ldots \\
\ldots & 0 & \mathbf{1} & \mathbf{0} & \overrightarrow{0} & 1 & 0 & \ldots \\
\ldots & 0 & 1 & \overleftarrow{0} & \mathbf{1} & \mathbf{1} & 0 & \ldots
\end{array}
$$

T_2

Example

$$
\begin{array}{cccc|cccc}
\dots & 0 & \mathbf{0} & \mathbf{0} & \overrightarrow{\mathbf{0}} & \mathbf{0} & 0 & \dots \\
\dots & 0 & 0 & \overleftarrow{0} & \mathbf{1} & \mathbf{0} & 0 & \dots \\
\dots & 0 & \mathbf{0} & \mathbf{1} & \overrightarrow{1} & 0 & 0 & \dots \\
\dots & 0 & 0 & 1 & 2 & \overrightarrow{0} & 0 & \dots \\
\dots & 0 & 0 & 1 & \overleftarrow{2} & 1 & 0 & \dots \\
\dots & 0 & 0 & \overleftarrow{1} & \mathbf{0} & \mathbf{1} & 0 & \dots \\
\dots & 0 & \overleftarrow{0} & 2 & 0 & 1 & 0 & \dots \\
\dots & 0 & 1 & \overrightarrow{2} & 0 & 1 & 0 & \dots \\
\dots & 0 & \mathbf{1} & \mathbf{0} & \overrightarrow{0} & 1 & 0 & \dots \\
\dots & 0 & 1 & \overleftarrow{0} & \mathbf{1} & \mathbf{1} & 0 & \dots \\
\dots & 0 & \mathbf{1} & \mathbf{1} & \overrightarrow{1} & 1 & 0 & \dots
\end{array}
$$

$T_2 = 10$

T_n

Goal: a recurrence for T_n.

T_n

Goal: a recurrence for T_n.

Method: study the progression from W_0 to W_{T_n} in $E(R)$.

Describing W_{T_n}

Lemma

1. *The world W_{T_n} has the form:* $\ldots 01^n|\overrightarrow{1}1^{n-1}0\ldots$

Describing W_{T_n}

Lemma

1. *The world W_{T_n} has the form: $\ldots 01^n \overrightarrow{|1} 1^{n-1} 0 \ldots$*
2. *The progression from W_0 to W_{T_n} does not step outside of the locations between $-n$ and $n-1$.*

W_{T_1}

$$\begin{array}{cccccccc}
\dots & 0 & 0 & 0 & | & \overrightarrow{0} & 0 & 0 & \dots \\
\dots & 0 & 0 & \overleftarrow{0} & | & 1 & 0 & 0 & \dots \\
\dots & 0 & 0 & 1 & | & \overrightarrow{1} & 0 & 0 & \dots
\end{array}$$

W_{T_2}

$$
\begin{array}{ccccc|cccc}
\dots & 0 & 0 & 0 & & \overrightarrow{0} & 0 & 0 & \dots \\
\dots & 0 & 0 & \overleftarrow{0} & & 1 & 0 & 0 & \dots \\
\dots & 0 & 0 & 1 & & \overrightarrow{1} & 0 & 0 & \dots \\
\dots & 0 & 0 & 1 & & 2 & \overrightarrow{0} & 0 & \dots \\
\dots & 0 & 0 & 1 & & \overleftarrow{2} & 1 & 0 & \dots \\
\dots & 0 & 0 & \overleftarrow{1} & & 0 & 1 & 0 & \dots \\
\dots & 0 & \overleftarrow{0} & 2 & & 0 & 1 & 0 & \dots \\
\dots & 0 & 1 & \overrightarrow{2} & & 0 & 1 & 0 & \dots \\
\dots & 0 & 1 & 0 & & \overrightarrow{0} & 1 & 0 & \dots \\
\dots & 0 & 1 & \overleftarrow{0} & & 1 & 1 & 0 & \dots \\
\dots & 0 & 1 & 1 & & \overrightarrow{1} & 1 & 0 & \dots
\end{array}
$$

Proof of Lemma: Base Case

Proof.

By induction on n.

Proof of Lemma: Base Case

Proof.

By induction on n.

Base case: we have already seen W_{T_1} (and W_{T_2}). □

Proof of Lemma: Inductive Step

Assume:

1. The world $W_{T_{n-1}}$ has the form: $\dots 01^{n-1}\overrightarrow{|1}1^{n-2}0\dots$
2. The progression from W_0 to $W_{T_{n-1}}$ does not step outside of the locations between $-(n-1)$ and $n-2$.

Proof of Lemma: Inductive Step (cont.)

Proof.

$$W_0: \qquad \ldots \quad 0 \quad 0 \quad 0^{n-1} \,\Big|\, \overrightarrow{0} \quad 0^{n-2} \quad 0 \quad 0 \quad \ldots$$

Proof of Lemma: Inductive Step (cont.)

Proof.

$$\begin{array}{lcccccccccc}
W_0: & \dots & 0 & 0 & 0^{n-1} & \big| \overrightarrow{0} & 0^{n-2} & 0 & 0 & \dots \\
\downarrow & & & & & & & & & \\
W_{T_{n-1}}: & \dots & 0 & 0 & 1^{n-1} & \big| \overrightarrow{1} & 1^{n-2} & 0 & 0 & \dots
\end{array}$$

Proof of Lemma: Inductive Step (cont.)

Proof.

$$
\begin{array}{lcccc|ccccc}
W_0 : & \dots & 0 & 0 & 0^{n-1} & \overrightarrow{0} & 0^{n-2} & 0 & 0 & \dots \\
\downarrow & & & & & & & & & \\
W_{T_{n-1}} : & \dots & 0 & 0 & 1^{n-1} & \overrightarrow{1} & 1^{n-2} & 0 & 0 & \dots \\
\downarrow & & & & & & & & & \\
W_{T_{n-1}+n-1} : & \dots & 0 & 0 & 1^{n-1} & 2 & 2^{n-2} & \overrightarrow{0} & 0 & \dots
\end{array}
$$

Proof of Lemma: Inductive Step (cont.)

Proof.

$$
\begin{array}{lcccccccccc}
W_0 : & \ldots & 0 & 0 & 0^{n-1} & \vert\ \overrightarrow{0} & 0^{n-2} & 0 & 0 & \ldots \\
\downarrow & & & & & & & & & \\
W_{T_{n-1}} : & \ldots & 0 & 0 & 1^{n-1} & \vert\ \overrightarrow{1} & 1^{n-2} & 0 & 0 & \ldots \\
\downarrow & & & & & & & & & \\
W_{T_{n-1}+n-1} : & \ldots & 0 & 0 & 1^{n-1} & \vert\ 2 & 2^{n-2} & \overrightarrow{0} & 0 & \ldots \\
\downarrow & & & & & & & & & \\
W_{T_{n-1}+3n-2} : & \ldots & 0 & \overleftarrow{0} & 2^{n-1} & \vert\ 0 & 0^{n-2} & 1 & 0 & \ldots
\end{array}
$$

Proof of Lemma: Inductive Step (cont.)

Proof.

$$
\begin{array}{lccccc|cccccc}
W_0 : & \ldots & 0 & 0 & 0^{n-1} & & \overrightarrow{0} & 0^{n-2} & 0 & 0 & \ldots \\
\downarrow & & & & & & & & & & \\
W_{T_{n-1}} : & \ldots & 0 & 0 & 1^{n-1} & & \overrightarrow{1} & 1^{n-2} & 0 & 0 & \ldots \\
\downarrow & & & & & & & & & & \\
W_{T_{n-1}+n-1} : & \ldots & 0 & 0 & 1^{n-1} & & 2 & 2^{n-2} & \overrightarrow{0} & 0 & \ldots \\
\downarrow & & & & & & & & & & \\
W_{T_{n-1}+3n-2} : & \ldots & 0 & \overleftarrow{0} & 2^{n-1} & & 0 & 0^{n-2} & 1 & 0 & \ldots \\
\downarrow & & & & & & & & & & \\
W_{T_{n-1}+4n-2} : & \ldots & 0 & 1 & 0^{n-1} & & \overrightarrow{0} & 0^{n-2} & 1 & 0 & \ldots
\end{array}
$$

Proof of Lemma: Inductive Step (cont.)

Proof.

$$\begin{array}{rcccccccccc}
W_0 : & \dots & 0 & 0 & 0^{n-1} & \big| \overrightarrow{0} & 0^{n-2} & 0 & 0 & \dots \\
\downarrow & & & & & & & & & \\
W_{T_{n-1}} : & \dots & 0 & 0 & 1^{n-1} & \big| \overrightarrow{1} & 1^{n-2} & 0 & 0 & \dots \\
\downarrow & & & & & & & & & \\
W_{T_{n-1}+n-1} : & \dots & 0 & 0 & 1^{n-1} & \big| \; 2 & 2^{n-2} & \overrightarrow{0} & 0 & \dots \\
\downarrow & & & & & & & & & \\
W_{T_{n-1}+3n-2} : & \dots & 0 & \overleftarrow{0} & 2^{n-1} & \big| \; 0 & 0^{n-2} & 1 & 0 & \dots \\
\downarrow & & & & & & & & & \\
W_{T_{n-1}+4n-2} : & \dots & 0 & 1 & 0^{n-1} & \big| \overrightarrow{0} & 0^{n-2} & 1 & 0 & \dots \\
\downarrow & & & & & & & & & \\
W_{2T_{n-1}+4n-2} : & \dots & 0 & 1 & 1^{n-1} & \big| \overrightarrow{1} & 1^{n-2} & 1 & 0 & \dots
\end{array}$$

Proof of Lemma: Inductive Step (cont.)

Proof.

$$
\begin{array}{rcccccccccc}
W_0 : & \dots & 0 & 0 & 0^{n-1} & | & \overrightarrow{0} & 0^{n-2} & 0 & 0 & \dots \\
\downarrow & & & & & & & & & & \\
W_{T_{n-1}} : & \dots & 0 & 0 & 1^{n-1} & | & \overrightarrow{1} & 1^{n-2} & 0 & 0 & \dots \\
\downarrow & & & & & & & & & & \\
W_{T_{n-1}+n-1} : & \dots & 0 & 0 & 1^{n-1} & | & 2 & 2^{n-2} & \overrightarrow{0} & 0 & \dots \\
\downarrow & & & & & & & & & & \\
W_{T_{n-1}+3n-2} : & \dots & 0 & \overleftarrow{0} & 2^{n-1} & | & 0 & 0^{n-2} & 1 & 0 & \dots \\
\downarrow & & & & & & & & & & \\
W_{T_{n-1}+4n-2} : & \dots & 0 & 1 & 0^{n-1} & | & \overrightarrow{0} & 0^{n-2} & 1 & 0 & \dots \\
\downarrow & & & & & & & & & & \\
W_{2T_{n-1}+4n-2} : & \dots & 0 & 1 & 1^{n-1} & | & \overrightarrow{1} & 1^{n-2} & 1 & 0 & \dots \\
= W_{T_n} & & & & & & & & & &
\end{array}
$$

□

Recurrence for T_n

Corollary

$T_n = 2T_{n-1} + 4n - 2.$

Solving the Recurrence

$$T_n = 2T_{n-1} + 4n - 2$$

Solving the Recurrence

$T_n = 2T_{n-1} + 4n - 2$

Assume $T_n = x \cdot 2^n + yn + z$ for some x, y, z and solve.

Solving the Recurrence

$$T_n = 2T_{n-1} + 4n - 2$$

Assume $T_n = x \cdot 2^n + yn + z$ for some x, y, z and solve.

$$T_n = 6 \cdot 2^n - 4n - 6$$
$$T_n = \Theta(2^n).$$

Further work

- Other complete rules

Further work

- Other complete rules
  - k-complete k-rule?

Further work

- Other complete rules
    - k-complete k-rule?
- Boundedness

Further work

- Other complete rules
  - k-complete k-rule?
- Boundedness