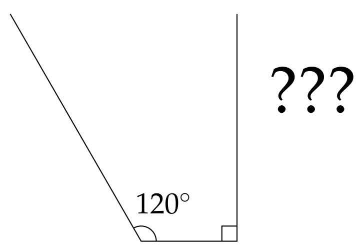
From Trigonometry to Calculus II in a Semester

Checkpoint 2: From Definition in Triangles to Definition in Circles

We've done so many things on SOHCAHTOA, but ... have you found any problem with it?

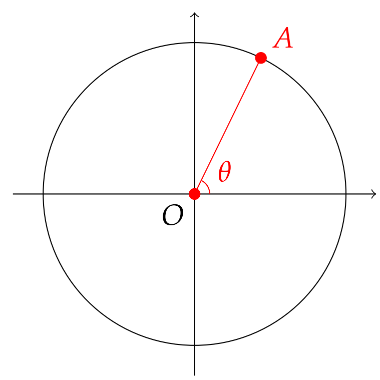
Well, the biggest problem that we're going to address today is: **How can we define trigonometric functions on angles greater than $90°$?**

Using our SOHCAHTOA definition, if we need to find $\sin 120°$, we need to draw a right triangle with a $120°$ angle, but ... IT'S IMPOSSIBLE!

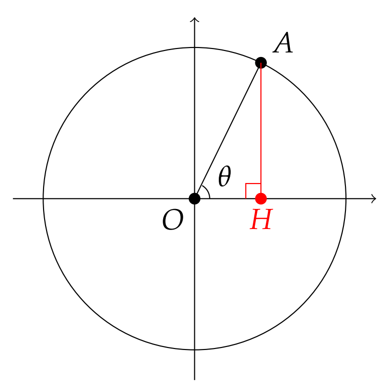


Therefore, it's time for a new definition. Read carefully, the story's gonna begin.

(1) Assume we have a *unit circle*, that is, a circle centered at the origin O at position $(0, 0)$ with radius 1, if we pick a point A on the circle and connect it with the origin, the segment will form an angle θ with the positive x-axis. To make it simple, we pick A only at the first quadrant for now.



(2) Now, we draw a segment AH perpendicular to the x-axis, where H is on the x-axis. Now, triangle AHO is a right triangle, where $\angle AHO = 90°$. Since the radius of the circle is 1, $AO = 1$.

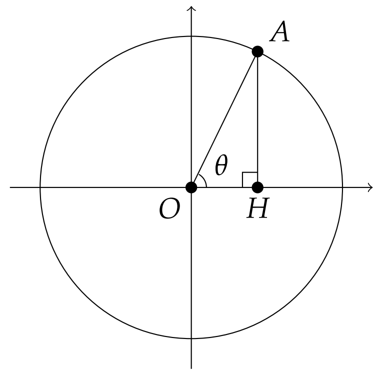


(3) By SOHCAHTOA,

$$\cos\theta = \frac{OH}{AO} = \frac{OH}{1} = OH$$

$$\sin\theta = \frac{AH}{AO} = \frac{AH}{1} = AH$$

Interestingly, we see that OH is the x-coordinate of point A, while AH is the y-coordinate of point B.



(4) Therefore,

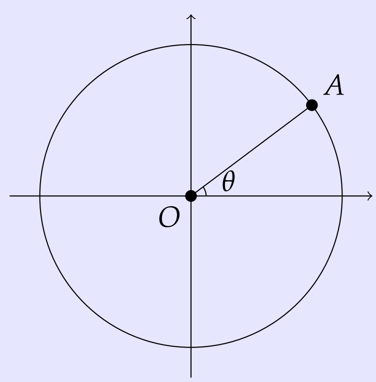
$$\cos\theta = OH = x, \quad \sin\theta = AH = y.$$

Putting everything together: **if we pick a point on the unit circle, the x-coordinate is $\cos\theta$, and the y-coordinate is $\sin\theta$.**

Oh yeah! We finally redefined trig functions! But, how is it useful?

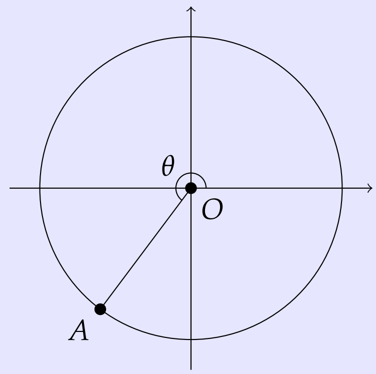
Recall our problem at the beginning, if we use this definition, even when θ is greater than $90°$, we can still calculate the trig values by finding a point on the unit circle.

¿1 The x-coordinate of point A is 0.8, and the y-coordinate of point A is 0.6. What is $\sin\theta$?



Recall, for example, if a point is at position $(3,4)$, the x-coordinate of the point is 3 and the y-coordinate of the point is 4.

¿2 Point A is at position $(-0.6, -0.8)$. What is $\cos\theta$?

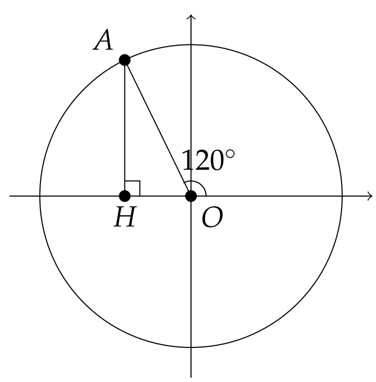


Through the definition on the unit circle, we can find more special trig values.

¿3 Find $\cos 120^\circ$ by drawing a unit circle.

Solution: As shown in the figure below, segment AO forms an angle of 120° with the positive x-axis. If we draw $AH \perp OH$ ($\perp$ means perpendicular to), then $\angle AOH = 60^\circ$.

Using what we got in the last Checkpoint, we have $\cos 60^\circ = \frac{1}{2}$, so $\frac{OH}{AO} = \frac{1}{2}$. Since $AO = 1$, $OH = \frac{1}{2}$. This means that the x-coordinate of A is $-\frac{1}{2}$ since A is at the left of the origin. Therefore, $\cos 120^\circ = \boxed{-\dfrac{1}{2}}$.



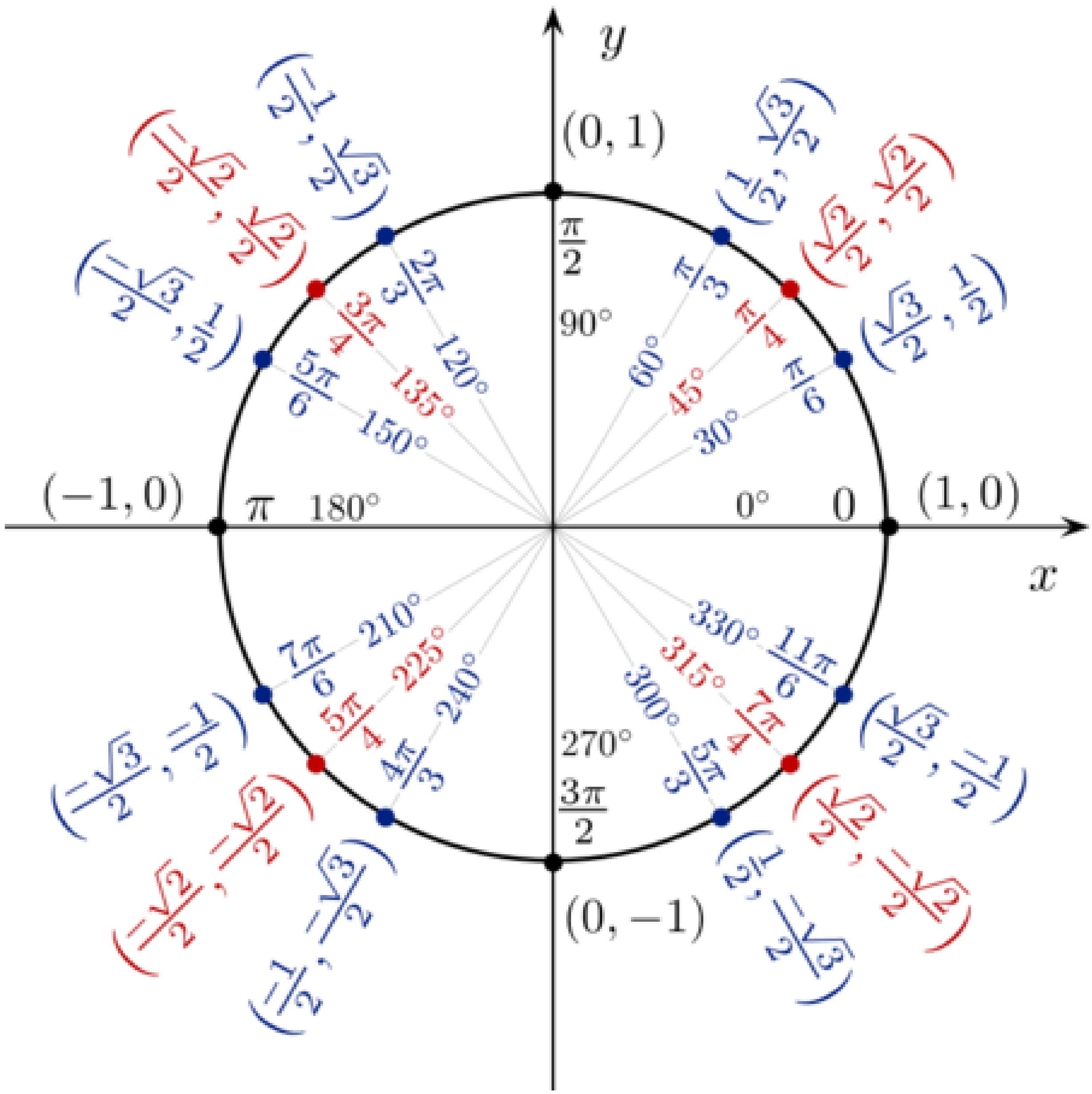
Using this way, we can find a lot more special value of trigonometric functions. Try to use the same way to do some more, and make problems to test your friends! Some example problems are:

$$\sin 315^\circ, \quad \cos 210^\circ, \quad \sin 150^\circ, \quad \cos 135^\circ.$$

In a typical trigonometry textbook, you will see a big unit circle with all the special points indicating the special values of trig functions.

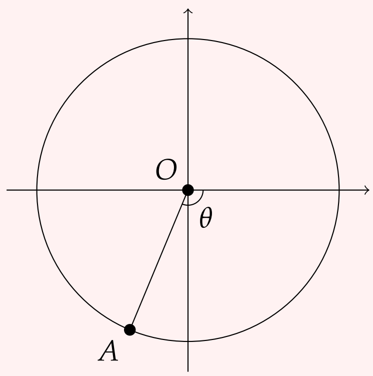
Use what we learned, the x is the cos and the y is the sin. For example, the $240°$ corresponds to position $\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$, so $\cos 240° = -\frac{1}{2}$, and $\sin 240° = -\frac{\sqrt{3}}{2}$.

Some tips for following our speedrun progress: Don't memorize all of it. Try to find some points and use SOHCAHTOA to show why the position is what's shown on the figure.

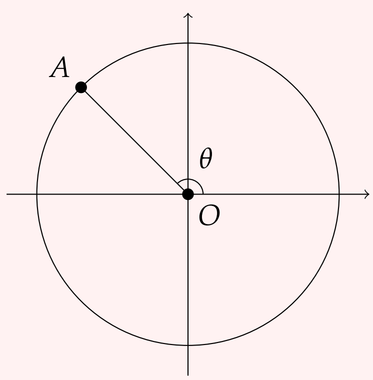


Same thing, below are some problems that can expand your thought on the material and train your logical reasoning ability, but it's okay to skip them as none of them will appear on a typical trigonometry test.

★1 Point A is at position $\left(-\frac{5}{13}, -\frac{12}{13}\right)$. What is $\sin\theta$? Notice that the θ is not in a usual position as we've seen. (*Hint: convert what you haven't seen into what you have*)



★2 The circle is centered at $(0,0)$ with **radius 5**, and point A is at position $\left(\frac{35}{11}, \frac{30\sqrt{2}}{11}\right)$. What is the value of $\cos\theta$?



★3 The circle is centered at point P at $(1,0)$ with radius 1, and point A is at position $\left(\frac{4}{3}, \frac{2\sqrt{2}}{3}\right)$. The circle and the x-axis intersects at point C at position $(2,0)$. What is the value of $\cos\angle APC$?