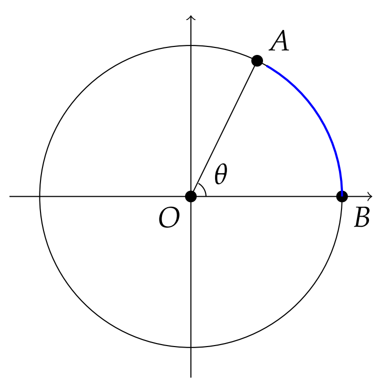
From Trigonometry to Calculus II in a Semester

Checkpoint 3: The Radian Measure, More Trig Functions and Typical Test Problems

In this checkpoint, we will go through a new way to measure angles. We've learned degrees way back in geometry, and the new way to measure angles is called **radians**. Then, we will introduce 3 new trigonometric functions: cot, sec and csc. After these, we will go through some test problems to review everything we learned.

1 Radians

Same thing as last time, we draw a circle centered at O with radius 1, and we name $\angle AOB$ as θ. Recall a geometry concept called **arc**: we call the blue part "arc AB".



What's the length of arc AB? We can first get that the circumference of the circle is $C = 2\pi r = 2\pi \cdot 1 = 2\pi$. Then, arc AB is $\frac{\theta}{360^\circ}$ of the circumference, so

$$\text{The length of arc } AB = \frac{\theta}{360^\circ} \cdot 2\pi.$$

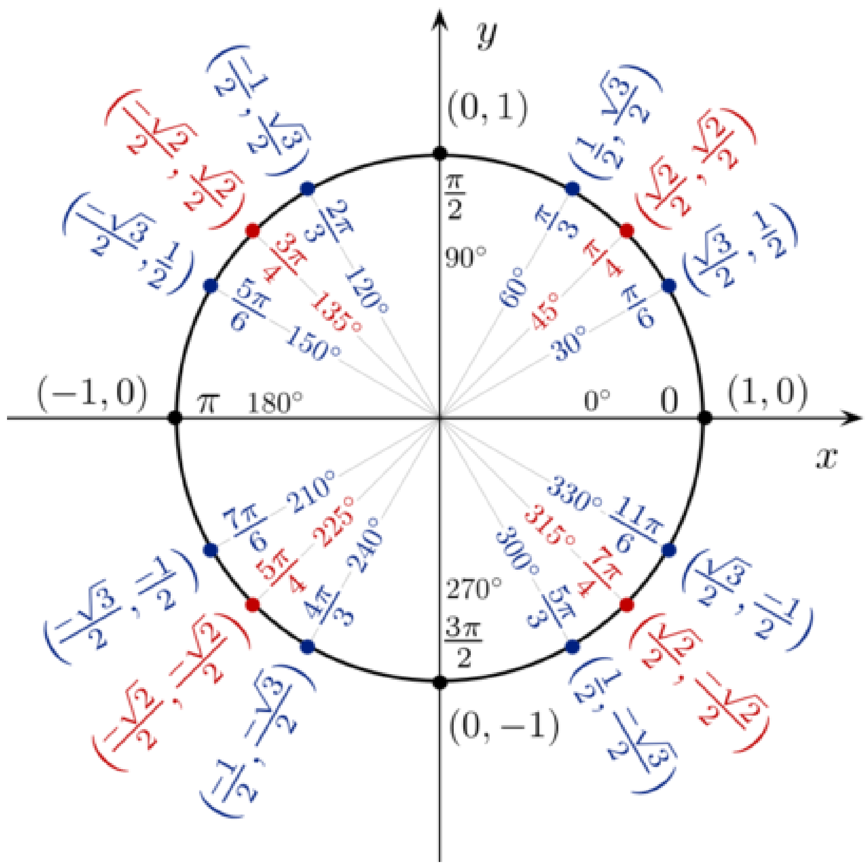
Now, you might be wondering: what's the point of doing all this? Here's a surprise for you: we've just defined **radians**! That's right. The length of arc AB is the radian measure of θ. More precisely, the formula is as follows:

$$\text{radian measure} = \frac{2\pi}{360^\circ} \times \text{degree measure}.$$

For example, the radian measure of 60° is

$$\frac{2\pi}{360^\circ} \times 60^\circ = \frac{2\pi}{6} = \frac{\pi}{3}.$$

Pull back the big unit circle you've seen in the last Checkpoint. You were probably curious about what does the numbers with π mean. Now you know that they are the radian measures of the corresponding degrees. For example, $30^\circ = \frac{\pi}{6}$, and $45^\circ = \frac{\pi}{4}$.



Radians are another way to show the size of angles.

> **¿1** What is the value of $\cos \frac{3\pi}{4}$?

Solution: First, we need to turn the radian measure $\frac{3\pi}{4}$ into degrees. Using our formula,

$$\frac{3\pi}{4} = \frac{2\pi}{360^\circ} \times \text{degree measure},$$

$$\text{degree measure} = \frac{3\pi}{4} \times \frac{360^\circ}{2\pi} = 135^\circ.$$

Therefore,

$$\cos \frac{3\pi}{4} = \cos 135^\circ = -\frac{\sqrt{2}}{2}.$$

A trick to turn radians into degrees is to replace π with 180°, because π is 180°. For example,

$$\frac{11\pi}{6} = \frac{11 \cdot 180^\circ}{6} = 330^\circ.$$

¿2 What is the value of $\sin \frac{5\pi}{3}$?

Solution: First, turn $\frac{5\pi}{3}$ into degrees:

$$\frac{5\pi}{3} = \frac{5 \cdot 180^\circ}{3} = 300^\circ.$$

Then,

$$\sin \frac{5\pi}{3} = \sin 300^\circ = -\frac{\sqrt{3}}{2}.$$

2 More Trig Functions

Time for more trig functions. Let's welcome our 3 new characters: cot (cotangent), sec (secant) and csc (cosecant)!

First, let's begin with cot. It is defined as $\cot\theta = \frac{1}{\tan\theta}$. For example,

$$\cot 60^\circ = \frac{1}{\tan 60^\circ} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}.$$

Next up, we have sec. It is defined as $\sec\theta = \frac{1}{\cos\theta}$. For example,

$$\sec 30^\circ = \frac{1}{\cos 30^\circ} = \frac{1}{\frac{\sqrt{3}}{2}} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}.$$

Finally, csc. It is defined as $\csc\theta = \frac{1}{\sin\theta}$. For example,

$$\csc 45^\circ = \frac{1}{\sin 45^\circ} = \frac{1}{\frac{\sqrt{2}}{2}} = \frac{2}{\sqrt{2}} = \sqrt{2}.$$

3 Test Problems

We've done a lot so far. It's time for a test to check your progress. Start a timer on your own for 20 minutes an try to finish within the time frame.

1. In a right triangle, angle θ is an acute angle. What does "$\sin\theta$" equal?

   A. $\dfrac{\text{opposite}}{\text{adjacent}}$ B. $\dfrac{\text{adjacent}}{\text{hypotenuse}}$ C. $\dfrac{\text{adjacent}}{\text{opposite}}$ D. $\dfrac{\text{opposite}}{\text{hypotenuse}}$

2. What is the value of $\tan 60°$?

   A. $\dfrac{\sqrt{3}}{3}$ B. $\dfrac{\sqrt{3}}{2}$ C. $\sqrt{3}$ D. $\dfrac{1}{2}$

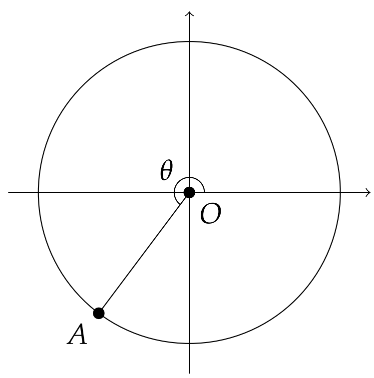
3. As shown in the right figure, the circle is a unit circle centered at point O at position $(0,0)$ with radius 1. Point A is at position $(-\frac{11}{20}, -\frac{3\sqrt{31}}{20})$. What is the value of $\cos\theta$?

   A. $-\dfrac{11}{20}$

   B. $-\dfrac{3\sqrt{31}}{20}$

   C. $-\dfrac{3\sqrt{31}}{11}$

   D. $\dfrac{3\sqrt{31}}{20}$



4. What is the value of $\sin\frac{7\pi}{4}$? (*Hint: this is in radians, converting to degrees might be more familiar to you*)

   A. $\dfrac{\sqrt{2}}{2}$ B. $-\dfrac{\sqrt{2}}{2}$ C. -1 D. $-\sqrt{2}$

5. What is the value of $\csc 210°$?

   A. $-\dfrac{\sqrt{3}}{2}$ B. $-\dfrac{1}{2}$ C. $-\dfrac{2\sqrt{3}}{3}$ D. -2

(Continue on the next page for problem 6-7)

6. If $f(x) = \sin 2x$, what is $f\left(\frac{5\pi}{6}\right)$?

   A. $-\frac{\sqrt{2}}{2}$ B. $\frac{\sqrt{3}}{2}$ C. $-\frac{\sqrt{3}}{2}$ D. $-\frac{1}{2}$

7. $\cos\left(\frac{\pi}{2} - \theta\right) =?$

   A. $\cos(-\theta)$ B. $\cos\theta$ C. $\sin(-\theta)$ D. $\sin\theta$

(You have finished. You can use the remaining space for scratch works)

You might completely have no idea on problem 5-7 and that's okay. I put difficult problems here to train your thinking ability, and also there's a small detail about $\sin(\theta + 2\pi) = \sin\theta$ that will be mentioned in the next Checkpoint.

The answer for the test is on the website. Problem 1 to 4 are straightforward. You can find the solution on previous Checkpoints. The following is the solution for problem 5-7.

5. By the definition of csc, we have

$$\csc 210^\circ = \frac{1}{\sin 210^\circ} = \frac{1}{-\frac{1}{2}} = -2,$$

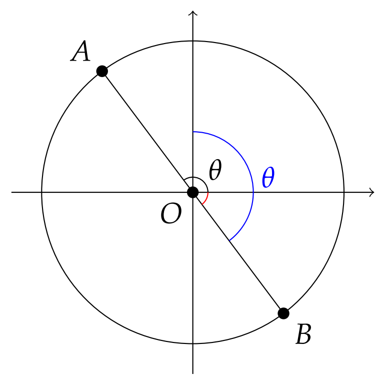
so $\boxed{D}$ is the correct answer.

6. Since $f(x) = \sin 2x$, we can plug in $x = \frac{5\pi}{6}$ to this function. That is, **replace** x with $\frac{5\pi}{6}$.

$$f\left(\frac{5\pi}{6}\right) = \sin 2\left(\frac{5\pi}{6}\right) = \sin\left(\frac{5\pi}{3}\right) = -\frac{\sqrt{3}}{2},$$

so $\boxed{C}$ is the correct answer.

7. As shown in the figure, if point A represents θ, then point B represents $\frac{\pi}{2} - \theta$. If we draw multiple such graphs and find patters, we can find that point A and B are symmetric about the line $y = x$. That is, the x-coordinate of A is the y-coordinate of B, and vice versa. Therefore,



$$x_B = \cos\left(\frac{\pi}{2} - \theta\right) = y_A = \sin\theta,$$

so $\boxed{D}$ is the correct answer.

It's okay if you don't know what's going on, because most people don't at this stage, and we will go through this in detail in the next several Checkpoints. If that is the case, one trick is to plug in a special value, say $\theta = \frac{\pi}{6}$. Then,

$$\cos\left(\frac{\pi}{2} - \theta\right) = \cos\frac{\pi}{3} = \frac{1}{2}.$$

Analyze the choice one by one,

A. $\cos(-\theta) = \cos(-\frac{\pi}{6}) = \frac{\sqrt{3}}{2}$.

B. $\cos\theta = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$

C. $\sin(-\theta) = \sin(-\frac{\pi}{6}) = -\frac{1}{2}$

D. $\sin\theta = \sin\frac{\pi}{6} = \frac{1}{2}$