Assume A Step Function Lorentz Distribution

In this work we assume $N_{\rm tot} = 5,000$ shells which have their Lorentz factors distributed as a step function. Each shell is launched with the same energy, so the mass of shells changes depending on the Lorentz factor assigned. We choose the two Lorentz factor values $\Gamma = 100$ and $\Gamma = 400$, where each section of step function has an equal mass. This results in 1,000 launched shells with $\Gamma = 100$ and 4,000 shells launched with $\Gamma = 400$. Each shell is launched $\Delta t_{\rm e} = 0.002$ sec after the previous.

We define the initial position of these shells as

$$R_{\rm i,0} = -c * \beta_{\rm i} * \Delta t_{\rm e} * (N_{\rm i} - 1) \tag{1}$$

Where $N_{\rm i}$ is the shell index and $\beta = \sqrt{1 - \frac{1}{\Gamma^2}}$. In this way, the first shell to be launched stars at $R_{0,0} = 0$ cm and all shells to be launched after have an initially negative radius.

For an observer, the arrival time of emission from a collision is given by,

$$t_{\rm a} = t_{\rm coll} - \frac{R_{\rm coll}}{c} \tag{2}$$

where $t_{\rm e}$ and $R_{\rm coll}$ are the time and radius of collision (in the jet frame), respectively.

The first collision must be between the final shell which has $\Gamma = 100$ (i.e., $N_{\rm i} = 1000$) and the first shell with $\Gamma = 400$ (i.e., $N_{\rm i} = 1001$). For simplicity, we will denote these with a subscript s (for slow) and r (for rapid), respectively. We can calculate when the first collision will be and when the emission will arrive at the observer.

$$t_{\rm coll,0} = \frac{(R_{\rm s,0} - R_{\rm r,0})}{c(\beta_{\rm r} - \beta_{\rm s})} \tag{3}$$

$$R_{\rm coll,0} = R_{\rm s,0} + (c\beta_{\rm s} * t_{\rm coll,0}) \tag{4}$$

Which can be used to find the time of first arrival:

$$t_{\rm a,0} = t_{\rm coll,0} - \frac{R_{\rm coll,0}}{c} \tag{5}$$

$$t_{\rm a,0} = t_{\rm coll,0} - \frac{R_{\rm s,0} + (c\beta_{\rm s} * t_{\rm coll,0})}{c} \tag{6}$$

$$t_{\rm a,0} = t_{\rm coll,0} - \frac{R_{\rm s,0}}{c} - \beta_{\rm s} t_{\rm coll,0} \tag{7}$$

$$t_{\rm a,0} = t_{\rm coll,0}(1 - \beta_{\rm s}) - \frac{R_{\rm s,0}}{c} \tag{8}$$

$$t_{\rm a,0} = \frac{(R_{\rm s,0} - R_{\rm r,0})}{c(\beta_{\rm r} - \beta_{\rm s})}(1 - \beta_{\rm s}) - \frac{R_{\rm s,0}}{c} \tag{9}$$

$$t_{\rm a,0} = \frac{([-c\beta_{\rm s}\Delta t_{\rm e}(1000 - 1)] - [-c\beta_{\rm r}\Delta t_{\rm e}(1001 - 1)])}{c(\beta_{\rm r} - \beta_{\rm s})}(1 - \beta_{\rm s}) - \frac{[-c\beta_{\rm s}\Delta t_{\rm e}(1000 - 1)]}{c} \tag{10}$$

$$t_{\rm a,0} = \frac{\Delta t_{\rm e}(1000\beta_{\rm r} - 999\beta_{\rm s})}{(\beta_{\rm r} - \beta_{\rm s})}(1 - \beta_{\rm s}) + 999\beta_{\rm s}\Delta t_{\rm e} \tag{11}$$

If we use the values of $\Delta t_{\mathrm{e}} = 0.002$ sec, $\beta_{\mathrm{s}} = \sqrt{1 - \frac{1}{100^2}}$, and $\beta_{\mathrm{r}} = \sqrt{1 - \frac{1}{400^2}}$, we find

$$t_{\mathrm{a},0} \approx 2.000133 \text{ sec} \tag{12}$$