# Problem Set #4: Even More Compartment Modeling/Python

1 Suppose a cup of hot coffee is brought into a room with temperature 70°F. The coffee will cool off, and it will cool off faster when the temperature difference between the coffee and the room is greater. The simplest assumption we can make is that the rate of cooling is proportional to this temperature difference (this is called Newton's law of cooling). Let $C$ denote the temperature of the coffee, in °F, and $C'$ the rate at which it is cooling, in °F per minute. The key here is that $C'$ is proportional, not to $C$, but to the difference between $C$ and the room temperature of 70°F.

(a) Write an equation that relates $C'$ and $C$. It will contain a proportionality constant; call it $k$. How did you indicate that the coffee is cooling and not heating up?

(b) When the coffee is at 180°F, it is cooling at a rate of 9°F per minute. What is the value of $k$?

(c) At what rate is the coffee cooling when its temperature is 120°F?

(d) By hand, estimate how long it takes the temperature to fall from 180°F to 120°F. Then make a better estimate, and explain why it is better.

(e) Use Python to generate a graph of the temperature of the cup of coffee over time.

2 Wine is made by yeast; yeast digests the sugars in grape juice and produces alcohol as a waste product. This process is called fermentation. The alcohol is toxic to the yeast, though, and the yeast is eventually killed by the alcohol. This stops fermentation, and the liquid has become wine, with about $8-12\%$ alcohol. Although alcohol isn't a"species," it acts like a predator on yeast. Unlike the other predator-prey problems we have considered, though, the yeast does not have an unlimited food supply.

(a) In our first model, assume that the sugar supply is not depleted, that no alcohol appears, and that the yeast simply grows logistically. Begin by adding 0.5 lbs of yeast to a large vat of grape juice whose carrying capacity is 10 lbs of yeast. Assume that the natural growth rate of the yeast is 0.2 lbs of yeast per hour, per pound of yeast. Let $Y(t)$ be the number of pounds of live yeast present after $t$ hours. What rate equation describes the growth of $Y$?

(b) Graph the solution $Y(t)$. Indicate on your graph approximately when the yeast reaches one-half the carrying capacity of the vat, and when it gets to within 1% of the carrying capacity.

(c) Suppose you use a second strain of yeast whose natural growth rate is only half that of the first strain of yeast. If you put 0.5 lbs of this yeast into the vat of grape juice, when will it reach one-half the carrying capacity of the vat, and when will it get to within 1% of the carrying capacity? Compare these values to the values produced by the first strain of yeast: are they larger, or smaller? Graph the solution $Y(t)$ for this strain of yeast.

(d) Now consider how the yeast produces alcohol. Suppose that waste products are generated at a rate proportional to the amount of yeast present; specifically, suppose each pound of yeast produces 0.05 lbs of alcohol per hour. (The other major waste product is carbon dioxide gas, which bubbles out of the liquid.) Let $A(t)$ denote the amount of alcohol generated after $t$ hours. Write a rate equation that describes the growth of $A$.

(e) Consider the toxic effect of the alcohol on the yeast. Assume that yeast cells die at a rate proportional to the amount of alcohol present, and also to the amount of yeast present. Specifically, assume that, in each pound of yeast, a pound of alcohol will kill 0.1 lbs of yeast per hour. Then, if there are $Y$ lbs of yeast and $A$ lbs of alcohol, how many pounds of yeast will die in one hour? Modify the original logistic equation for $Y$ (strain 1) to take this effect into account. The modification involves subtracting off a new term that describes the rate at which alcohol kills yeast. What is the new rate equation?

(f) You should now have two rate equations describing the rates of growth of yeast and alcohol. The equations are coupled, in the sense that the yeast equation involves alcohol, and the alcohol equation involves yeast. Assuming that the vat contains, initially, 0.5 lbs of yeast and no alcohol, describe by means of a graph what happens to the yeast. How close does the yeast get to carrying capacity, and when does this happen? Does the fermentation end? If so, when; and how much alcohol has been produced by that time? (Note that since $Y$ will never get all the way to 0, you will need to adopt some convention like $Y \leq 0.01$ to specify the end of fermentation.)

(g) What happens if the rates of toxicity and alcohol production are different? Specifically, increase the rate of alcohol production by a factor of five—from 0.05 to 0.25 lbs of alcohol per hour, per pound of yeast—and at the same time reduce the toxicity rate by the same factor—from 0.10 to 0.02 lbs of yeast per hour, per pound of alcohol and pound of yeast. How do these changes affect the time it takes for fermentation to end? How do they affect the amount of alcohol produced? What happens if only the rate of alcohol production is changed? What happens if only the toxicity rate is reduced?

(h) The third model will take into account that the sugar in the grape juice is consumed. Suppose the yeast consumes 0.15 lbs of sugar per hour, per lbs of yeast. Let $S(t)$ be the amount of sugar in the vat after $t$ hours. Write a rate equation that describes what happens to $S$ over time.

(i) Since the carrying capacity of the vat depends on the amount of sugar in it, the carrying capacity must now vary. Assume that the carrying capacity of $S$ lbs of sugar is 0.4 $S$ lbs of yeast. How much sugar is needed to maintain a carrying capacity of 10 lbs of yeast? How much is needed to maintain a carrying capacity of 1 lbs of yeast? Rewrite the logistic equation for yeast so that the carrying capacity is 0.4 $S$ lbs, instead of 10 lbs, of yeast. Retain the term you developed to reflect the toxic impact of alcohol on the yeast.

(j) There are now three rate equations. Using them, describe what happens to 0.5 lbs of yeast that is put into a vat of grape juice that contains 25 lbs of sugar at the start. Does all the sugar disappear? Does all the yeast disappear? How long does it take before there is only 0.01 lb of yeast? How much sugar is left then? How much alcohol has been produced by that time?