Engineering Calc II Summary

Mike Pierce · Math136 · Last tended 4 May 2023 · Hosted at `coloradomesa.edu/~mapierce2/136/summary`

Sometimes the purpose of a course becomes obfuscated over its duration; it can become hard to appreciate the overarching plot of a story when one spends months buried in fine exposition and occasional side-plots. This narrative overview, in its brevity, serves to summarize the plot of this course and to clarify our purpose, for the sake of keeping both student and instructor focused on what's important.

Transcendental Functions

In this second semester studying calculus we start by introducing the familiar transcendental functions $\ln(x)$ and e^x from a new perspective.

> Note, the trigonometric functions sine, arctangent, etc, are *also* transcendental functions, though the designers of most calculus textbooks subtitled "late transcendentals" don't appear to be bothered introducing them early.

We introduce these functions by *defining* the (natural) logarithm function as a definite integral

$$\ln(x) = \int_1^x \frac{1}{t}\,\mathrm{d}t\,,$$

and from there defining e^x as its inverse. Starting from this perspective though, it becomes pertinent to figure out how the derivative of the inverse of a function relates the function itself. Thankfully the *inverse function theorem* gives us a succinct formula for the relationship:

$$\left(f^{-1}\right)' = \frac{1}{f' \circ f^{-1}}\,.$$

- **Skill** · Be able to compute a formula for the derivative of a function given a formula for the derivative of its inverse.

We can get a lot of information from the inverse function theorem, not just about the derivative of e^x and about exponential/logarithmic function of non-natural bases, but also about the inverses of trigonometric functions.

- **Fact** · Know the derivatives and antiderivatives of exponential, logarithmic, and inverse trigonometric functions.

- **Skill** · Be able to calculate the derivative of any rational sum, product, or composite of exponential or logarithmic functions of any base, and of inverse trigonometric functions. Also be able to calculate the antiderivative of any "simple" rational sum or composite of such functions.

Beyond that core narrative we build two new tools with the help of logarithms.

- **Skill** · Know the technique of *logarithmic differentiation*, and be able to calculate the derivative of functions of the form $f(x)^{g(x)}$ for non-constant f and g. Explicitly as a formula,

$$\left(f(x)^{g(x)}\right)' = f(x)^{g(x)}\left(g(x)\frac{f'(x)}{f(x)} + g'(x)\ln\big(f(x)\big)\right).$$

- **Skill** · Be able to apply *L'Hospital's Rule* to evaluate limits: for suitable functions f and g such that either $\lim_{n\to a} f(x) = \lim_{n\to a} g(x) = 0$ or $\lim_{n\to a} f(x) = \lim_{n\to a} g(x) = \pm\infty$,

$$\lim_{n\to a}\frac{f(x)}{g(x)} = \lim_{n\to a}\frac{f'(x)}{g'(x)}$$

  if the right-most limit exists.

Before we conclude our exploration of transcendental functions, there is one last sight we really should see, just to be aware of.

- **Fact** · Be *familiar* with hyperbolic trigonometric functions: what they are, how they relate to exponential functions, and what their derivatives and antiderivatives look like.

The addition of these logarithmic, exponential, trigonometric, and hyperbolic functions completes the dictionary of differentiable functions we've been building in this class, and so we are now well-prepared to dive into the art and practice of evaluating integrals.

Techniques & Applications of Integration

We've seen already that it's more complicated to calculate formulas for antiderivatives than for derivatives. Now we dip our toes into this complexity. Now begins the gauntlet.

First remember integrals are linear: they break up across sums/differences, and constants may be factored out. We also know to recognize a few integrals on sight simply as being the derivatives of familiar functions. And we have a single tool for computing derivatives, *substitution*, which "undoes" the chain-rule of differentiation. The first new tool we'll develop is called *integration by-parts*, a way of "undoing" the product-rule of differentiation, the utility of which is captured by the formula

$$\int u\,\mathrm{d}v = uv - \int v\,\mathrm{d}u\,.$$

With this tool we'll finally be able to calculate formulas for the integrals of the functions $\ln(x)$ and $\sec(x)$ and move on to learning the tricks for dealing with more complicated trigonometric integrands. The "trick" is usually to cleverly wield some trigonometric identities to rewrite the integrands. We get extra mileage out of the Pythagorean identities in particular due to their ability to turn a sum/difference of squares into a single square term, a technique known as *trigonometric substitution*.

The last technique of integration we'll cover isn't so much about integration as it is about writing a rational expression as a sum of polynomials and rational expression with degree-1 or -2 denominators. This sum is called its *partial fraction decomposition*, and is straightforward to integrate using previously learned techniques.

- **Skill** • Be able to employ any of the previous tools/techniques to find formulas for indefinite integrals, or to calculate values of definite integrals,

This concludes the gauntlet of learning to manually compute certain integrals, so we should step back and discuss the broader context. Not every function's antiderivative is computable from its formula. I.e. not every indefinite integral can be expressed as a formula in terms of elementary functions. In practice when we need to compute the value of a definite integral we can use a computer to approximate it to arbitrary precision. You already know about Riemann sums, but there are other methods.

- **Concepts** • Be *familiar* with techniques of numerically computing the value of definite integrals, particularly with the *trapezoid rule*, but also with the *Newton–Cotes formulas* of higher degree, and with the bounds on their error.

In recent history there's been advances in symbolically computing the formulas for *in*definite integrals using sufficiently sophisticated symbolic computer algebra systems (CAS). You can get a small taste of this sophistication by asking WolframAlpha to calculate a formula for an integral for you.

Before moving on to applications of integration we should address a geometric quirk that we've ignored so far, but will become crucial to think about later when discussing series: unbounded regions *may* contain a finite amount of area. Considering unbounded regions "under" curves and the integrals that compute their area, this means our bounds of integration may be $\pm\infty$, or that our curve may have an asymptote (pole) between its bounds of integration. Such integrals are called *improper*, and, computationally, we handle improper integrals with limits. If our bounds of integration are infinite,

$$\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = \lim_{L\to-\infty}\int_{L}^{0} f(x)\,\mathrm{d}x + \lim_{R\to\infty}\int_{0}^{R} f(x)\,\mathrm{d}x\,,$$

or if f has a pole at p between a and b,

$$\int_{a}^{b} f(x)\,\mathrm{d}x = \lim_{\ell\to p^-}\int_{a}^{\ell} f(x)\,\mathrm{d}x + \lim_{r\to p^+}\int_{r}^{b} f(x)\,\mathrm{d}x\,.$$

- **Skill** • Be able to navigate the nuances of improper integrals, compute their value if they converge, and determining whether they converge or diverge by comparing their integrands to simpler functions whose convergence/divergence is known.

After going over these techniques and becoming comfortable with improper integrals, we'll talk about a few "applications" of integration. The first are applications within mathematics, specifically within geometry, whereas the others are sincere applications,

- **Skill** • Be able to write down an integral that computes the arclength of a planar curve. Be able to write down an integral that computes the surface area of a solid generated by revolving a planar region about an axis.
- **Skill** • Be able to write down integrals that compute the coordinates of the center of mass (centroid) of a given planar region.
- **Concept** • Be *familiar* with other applications of integration to engineering, biology, economics, and probability.

Then since we're talking about applications of calculus anyways, now is a good time to introduce the main character from the field of applied mathematics: the differential equation.

Differential Equations & Coordinate Geometry

Before moving on the final major topic of the course, we should take a quick foray into these two topics which, though they feel rather tangential to the narrative of this course, are still important to cover since each serves as the focus of math classes you'll likely take after this one.

First, having developed some fluency with integration, we can now talk about the topic of solving differential equations, a practice so ubiquitous in research and industry that there are many college courses dedicated to it exclusively.

- **Concept** • Know the definition of a *differential equation*, what it means for a function to be a *solution* to a differential equation, the difference between an *implicit* and an *explicit* solution, the difference between a *general* and a *particular* solution, and how *initial conditions* determine a particular solution.
- **Skill** • Be able to sketch the *direction field* of a differential equation of the form $\dot{y} = f(x, y)$ and be able to sketch a particular solution based on the direction field.
- **Skill** • Be able to solve any sufficiently reasonable *separable* differential equation.

Second, we've developed the theory of calculus quite thoroughly up to this point, but only from a limited perspective: the perspective of single-variable real-valued functions and their graphs plotted in rectangular (Cartesian) coordinates. The topic of *multi*-variable functions is a broad one reserved for another class, but we have time to explore beyond the scope of functions and their graphs in rectangular coordinates in this class. For example, what about curves in rectangular coordinates that aren't the graph of a function at all? What about curves that are the graph of a function plotted in polar coordinates instead of rectangular? The calculus knowledge we've developed transfers over to these settings.

The graph $y = f(x)$ of a function f is the set of all points $\big(x, f(x)\big)$ in the (x, y)-plane; the y-coordinate is a function of the x-coordinate. But we can certainly define curves as the set of all points $\big(x(t), y(t)\big)$, where x and y are functions of some new parameter t. Such a curve is said to be *parametrically-defined*, and if the functions x and y are differentiable/integrable, we can calculate geometric measures of this curve using the tools we've developed with calculus.

- **Skill** • Be able to plot a parametrically-defined curve.

- **Skill** • If one of the coordinates of a parametrically-defined curve can be written as a function of the other coordinate, be able to do so. Authors usually call this *elimination of a parameter*.

- **Skill** • Be able to calculate the length of a parametrically-defined curve segment, the area of a region bound by parametrically-defined curves, and the equation of a line tangent to a parametrically-defined curve.

Note that these parametric curves are still defined in rectangular space, the (x, y)-plane. Instead of generalizing how we describe the coordinates of points, we could also plot points in a different space entirely: polar space, the (r, θ)-plane, where every points is described by its distance r from the origin, the angle θ by which is it's inclined from the positive x-axis. Explicitly, the coordinates transformation is given by these equations:

$$\begin{cases} x = r\cos\theta \\ y = r\cos\theta \end{cases} \iff \begin{cases} r = \sqrt{x^2 + y^2} \\ \theta = \operatorname{atan2}(y, x) \end{cases}$$

Typically you see the r coordinate written as a function of θ and plot the graph $r = f(\theta)$ in polar space. It's easy to plot the graph of a function f in rectangular space to appear as if it's plotted in polar space though. For a parameter θ, the curve defined as all points $\big(f(\theta)\cos\theta, f(\theta)\sin\theta\big)$ in rectangular space will be the same curve as if $r = f(\theta)$ were plotted in polar coordinates. Using this, we can just refer to the calculus skills we developed in terms of parametric equations to build new tools for the graphs of functions in polar coordinates.

- **Skill** • Be able to plot the graph of a function in polar coordinates.

- **Skill** • Be able to calculate the arclength of a segment of a function's graph in polar space, the area of a region bound by graphs in polar space, and the equation of a line tangent to a graph in polar space.

We could describe curves in polar space parametrically too, as the set of all points $\big(r(t), \theta(t)\big)$ for some parameter t, ... but I've never heard anyone talk about this.

Taylor Series

Every function we've dealt with so far has been a real-valued functions with real inputs. This last segment of the course pivots towards real-valued functions with positive integer inputs, *sequences*, with the goal of defining and exploring Taylor series. One of the important questions we'll want to answer about Taylor series is when the sum of their coefficients, the *series* that their coefficients comprise, converges and sincerely defines a function, so first we'll spend some time studying the convergence of series.

- **Skill** • Know what it means for a series to *converge absolutely*, to *converge conditionally*, and to diverge, and how to determine a series' convergence using various tests.

- **Skill** • Be able to compute the limit/value of certain convergent sequences and series (e.g. geometric series).

Once we're trained to tell when a series converges, we can apply this skill to decide where one defines a function. The series $\sum_{n=0}^{\infty} c_n(x-a)^n$ is called *power series* centered at $x = a$, and can be thought of a polynomial with infinitely many terms. A power series defines a function on some *interval of convergence* centered at $x = a$ consisting of all x for which the power series $\sum c_n(x-a)^n$ converges. This interval then serves as the domain of the function $f(x) = \sum c_n(x-a)^n$.

- **Skill** • Know how to calculate the interval of convergence and *radius of convergence* of a power series.

We must also consider the opposite question, not when a power series describes a function, but how to write a given function as a power series. The answer to this question: any smooth function f is equal to its *Taylor series* centered at $x = a$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

for some interval of convergence centered at $x = a$. This to say any smooth function, on some interval, can be expressed as the limit of a convergent sequence of polynomial functions.

- **Skill** • Given a smooth function and some point, be able to write down the Taylor series that represents the function centered at that point, and determine its interval of convergence.

This ability to express any smooth function as a series gives us a means of calculating approximations to the outputs of functions, like trigonometric or exponential/logarithmic functions, that are otherwise difficult to calculate. By truncating a Taylor series after the Nth term, we define the Nth *Taylor polynomial* (centered at $x = a$) of a function f as

$$T_N(x) = \sum_{n=0}^{N} \frac{f^{(n)}(a)}{n!}(x-a)^n,$$

and note that this polynomial approximates f around $x = a$. Being a polynomial it is much easier to calculate, and the approximation improves as $N \to \infty$.

- **Skill** • Understand how to quantify how well a smooth function is approximated by a Taylor polynomial on some subset of its domain. In particular, be familiar with *Taylor's inequality*.

This revelation that every smooth function, nearly every function we've studied in a math class before, is secretly the limit of polynomial functions (on some domain) serves as the climax and conclusion of this course.