

# Energy Levels and Spectrum of Hydrogenic Atoms

Michael Papasimeon

April 16, 1998

# Hamiltonian for Electron in Coulomb Potential

A hydrogenic atom can be thought of as an electron with charge  $(-e)$  moving around the spherically symmetric Coulomb potential of the nucleus which has charge  $Z$ .

$$H = \frac{\mathbf{p}^2}{2\mu} - \frac{Ze^2}{4\pi\epsilon_0 r}. \quad (1)$$

where  $\mu$  is the reduced mass...

$$\mu = \frac{Mm}{M + m} \quad (2)$$

# Time Independent Schrödinger Equation

The problem reduces to solving the time independent Schrödinger equation for the above Hamiltonian.

$$\left[ \frac{\mathbf{p}^2}{2\mu} - \frac{Ze^2}{4\pi r} \right] \psi_{nlm} = E_n \psi_{nlm} \quad (3)$$

- Solving for this equation gives the energy levels of hydrogenic atoms  $E_n$ , and corresponding wavefunctions  $\psi_{nlm}$ .
- The energy levels only depend on the principle quantum number  $n$ , whereas the wavefunctions also depend on the orbital angular momentum quantum number  $l$  ( $l = 0, 1, \dots, n-1$ ) and magnetic quantum number  $m$  ( $m = -l, -l+1, \dots, +l-1, +l$ ).

# Wave Functions for Hydrogenic Atoms

The wavefunctions of hydrogenic atoms are given by

$$\psi_{nlm}(\mathbf{r}) = \psi_{nlm}(r, \theta, \phi) = R_{nl}(r) Y_{lm}(\theta, \phi) \quad (4)$$

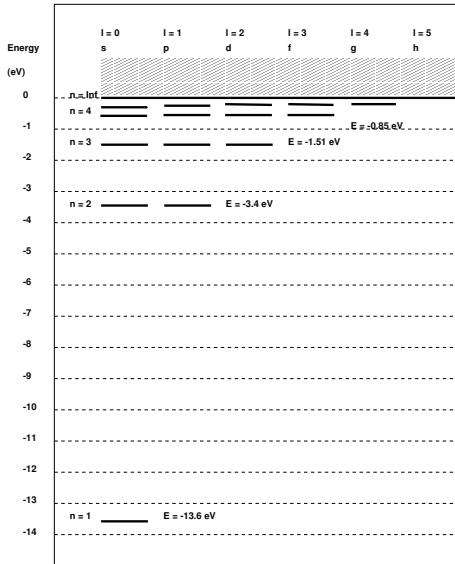
where  $R_{nl}(r)$  are the radial wavefunctions and  $Y_{lm}(\theta, \phi)$  are spherical harmonics. The corresponding energy levels of hydrogenic atoms are given by

$$E_n = -\frac{1}{2} \mu c^2 \frac{Z^2 \alpha^2}{n^2} \quad (5)$$

where  $\alpha$  is the fine structure constant:

$$\alpha = \frac{e^2}{4\pi\epsilon_0\hbar c}$$

# Energy Levels of Hydrogen $Z = 1$



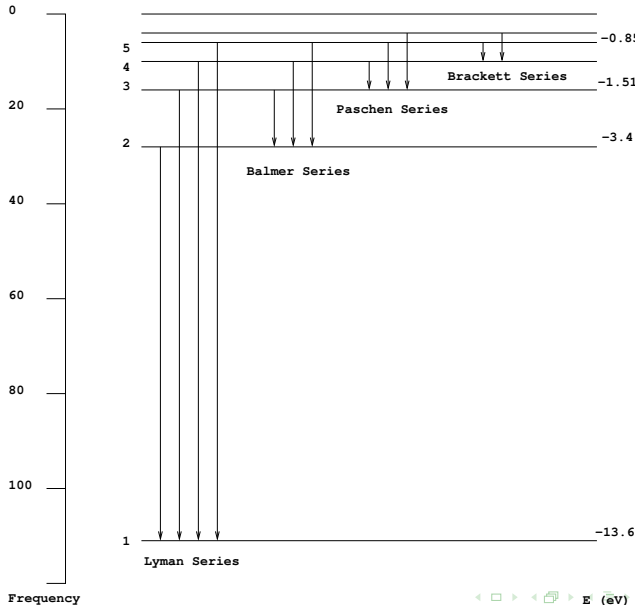
# Rydberg States

- A highly excited hydrogenic atom, that is one with a large principal quantum number  $n$  is said to be known as a *Rydberg Atom*, or to be in a *Rydberg State*.
- The Rydberg Atom with  $n = 100$ , is of an extremely large size – approximately the size of a simple bacteria. However, such an atom has a very small ionisation energy and hence the electron is very weakly bound.

**Table:** Bohr orbit radius and binding energy for Hydrogen atoms in the ground state  $n = 1$  and Rydberg state  $n = 100$

Quantity	$n = 1$	$n = 100$
Bohr Orbit Radius [m]	$5.3 \times 10^{-11}$	$5.3 \times 10^{-7}$
Binding Energy $ E_n $ [eV]	$-13.6 \times 10^0$	$1.36 \times 10^{-3}$

# Hydrogenic Spectrum



# Lyman, Balmer and Paschen Spectral Lines

## Lyman Series (Ultra Violet Region)

Spectral Line	Wavelength (Å)
$\text{Ly}_\alpha$	1216
$\text{Ly}_\beta$	1026
$\text{Ly}_\gamma$	972.5
$\text{Ly}_\delta$	949.7

## Balmer Series (Visible–Ultra Violet Region)

Spectral Line	Wavelength (Å)
$\text{H}_\alpha$	6563
$\text{H}_\beta$	4861
$\text{H}_\gamma$	4340

## Paschen Series (Infra Red Region)

Spectral Line	Wavelength (Å)
$\text{P}_\alpha$	18751
$\text{P}_\beta$	12818



# Fine Structure

- The fine structure of the energy levels of hydrogenic atoms are due to relativistic effects and the spin of the electron which were not taken into account in the previous section.
- We can obtain an estimate on how much the energy levels are shifted, by using perturbation theory if we assume the relativistic and spin effects are small. We can treat the Hamiltonian, energy eigenfunctions and energy levels as the unperturbed case and use first order perturbation theory for the perturbing components of the Hamiltonian.

# Perturbed Hamiltonian for Fine Structure

The Schrödinger equation is then of the following form:

$$[H_0 + H'_1 + H'_2 + H'_3] \psi_{nlm} = E \psi_{nlm} \quad (6)$$

where  $H_0$  corresponds to the unperturbed Hamiltonian used in the previous section, and  $H'_1$ ,  $H'_2$  and  $H'_3$  correspond to small perturbations involving relativistic corrections to the kinetic energy, the effect of spin-orbit coupling and the Darwin interaction.

# Modification to Unperturbed Hamiltonian

The unperturbed Schrödinger equation needs to be slightly modified in order to treat perturbing Hamiltonians which take the spin of the electron into account. The modified 'unperturbed' equation is given by:

$$H_0 \psi_{nlm_l m_s}^{(0)} = E_n^{(0)} \psi_{nlm_l m_s}^{(0)} \quad (7)$$

where  $E_n^{(0)}$  are the Schrödinger energy eigenvalues (with  $\mu = m$ ) and the zero order wave functions  $\psi_{nlm_l m_s}^{(0)}$  are modified two component wave functions (also known as Pauli wave functions or spin-orbitals).

# Pauli Wave Functions or Spin-Orbitals

$$\psi_{nlm_l m_s}^{(0)}(\mathbf{q}) = \psi_{nlm_l}^{(0)}(\mathbf{r}) \chi_{\frac{1}{2}, m_l} \quad (8)$$

The parameter  $q$  represents the combined spin and space coordinate, and  $\chi_{\frac{1}{2}, m_l}$  are two component spinors (spin eigenfunctions for particle of spin one half).

# Normalised Spinors

For spin-up ( $m_s = +1/2$ ) and spin-down ( $m_s = -1/2$ ), the normalised spinors are denoted by

$$\chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (9)$$

- Since the unperturbed Hamiltonian does not depend on the spin variable, the Pauli wavefunctions are separable in the spin and coordinate variables.
- To describe the state of an electron in a hydrogenic atom, we now have the four quantum number  $n$ ,  $l$ ,  $m_l$  and  $m_s$ .
- As a result of this, each of the unperturbed energy levels are  $E_n$  are  $2n^2$  degenerate.

# Perturbed Energy Levels

Finally the new perturbed energy levels to first order are given by

$$E = E_n^{(0)} + \Delta E'_1 + \Delta E'_2 + \Delta E'_3 \quad (10)$$

where  $\Delta E'_{(k)}$  is the energy shift resulting from the  $k$ 'th perturbation.

# Relativistic Correction to the Kinetic Energy

$$H'_1 = -\frac{p^4}{8m^3c^2} \quad (11)$$

Since relativistic effects were not taken into account previously, the perturbing Hamiltonian  $H'_1$  represents the relativistic correction to the kinetic energy of the electron. Using first order perturbation theory the energy shift is given by

$$\Delta E'_1 = \left\langle nlm_l m_s \left| \frac{-p^4}{8m^3c^2} \right| nlm_l m_s \right\rangle \quad (12)$$

$$\Delta E'_1 = -E_n \frac{Z^2 \alpha^2}{n^2} \left[ \frac{3}{4} - \frac{n}{l + \frac{1}{2}} \right] \quad (13)$$

# Spin Orbit Interaction

$$H'_2 = \frac{1}{2m^2c^2} \frac{1}{r} \frac{dV}{dr} \mathbf{L} \cdot \mathbf{S} \quad (14)$$

This perturbation corresponds to the shift in the energy as a result of the interaction between the internal spin of the electron and its orbital angular momentum as it orbits the nucleus. Since the potential  $V(r) = -Ze^2/(4\pi\epsilon_0 r)$  for a hydrogenic atom, the perturbing Hamiltonian becomes

$$H'_2 = \frac{1}{2m^2c^2} \frac{Ze^2}{4\pi\epsilon_0 r^3} \mathbf{L} \cdot \mathbf{S} \quad (15)$$



# Energy Shift for Perturbed Hamiltonian

The energy shift for this perturbing Hamiltonian using first order perturbation theory is

$$\Delta E'_2 = -E_n \frac{Z^2 \alpha^2}{2nl(l + \frac{1}{2})(l + 1)} \begin{cases} l, & j = l + \frac{1}{2} \\ -l - 1, & j = l - \frac{1}{2} \end{cases} \quad (16)$$

# Darwin Interaction

$$H'_3 = \frac{\pi \hbar^2}{2m^2 c^2} \left( \frac{Ze^2}{4\pi\epsilon_0} \right) \delta(\mathbf{r}) \quad (17)$$

The Darwin term does not act on the spin variable and only applies when the orbital angular momentum is zero ( $l = 0$ ). The shift energy from the Darwin term is given by

$$\Delta E'_3 = -E_n \frac{Z^2 \alpha^2}{n}, \quad l = 0 \quad (18)$$

# Fine Structure Energy Levels to First Order

Based on the energy levels obtained for hydrogenic atoms from the Schrödinger equation, and the energy shifts obtained from the three perturbation when relativistic and spin effects are taken into account we get

$$E_{nj} = E_n + \Delta E'_1 + \Delta E'_2 + \Delta E'_3. \quad (19)$$

Now subtracting from the binding energy  $E = mc^2 - E_{nj}$

$$E = mc^2 \left[ 1 - \frac{(Z\alpha)^2}{2n^2} - \frac{(Z\alpha)^4}{2n^3} \left( \frac{n}{j + \frac{1}{2}} - \frac{3}{4} \right) \right] \quad (20)$$

This result is valid if the perturbation is small, and therefore it begins to break down for hydrogenic atoms with large atomic numbers  $Z$ .

# Dirac Equation

The equation which provides the correct energy eigenvalues and eigenstates for the fine structure of hydrogenic atoms is the Dirac equation. For a Hydrogenic atom that Dirac equation can be written as:

$$\left[ c\boldsymbol{\alpha} \cdot \mathbf{p} + \beta mc^2 - \frac{Ze^2}{4\pi\epsilon_0 r} \right] \psi = E\psi \quad (21)$$

where  $\psi$  is a 4-component spinor.

# Dirac Energy Levels

The Dirac equation takes special relativistic effects and the spin of the electron (or any spin- $\frac{1}{2}$  particle) into account, and gives the following solutions for the energy eigenvalues for hydrogenic atoms.

$$E_{nj}^{Dirac} = \frac{mc^2}{\sqrt{1 + \frac{(Z\alpha)^2}{n-j-\frac{1}{2} + \sqrt{(j+\frac{1}{2})^2 - (Z\alpha)^2}}}} \quad (22)$$

Expanding the above equation in a series we obtain

$$E = mc^2 \left[ 1 - \frac{(Z\alpha)^2}{2n^2} - \frac{(Z\alpha)^4}{2n^2} \left( \frac{1}{j + \frac{1}{2}} - \frac{3}{4n} \right) - \dots \right] \quad (23)$$

This can be seen to agree with the result obtained previously using the perturbation theory for the first few terms.

# Hyperfine Structure

- Until now, the nucleus of hydrogenic atoms has been treated as a point charge, effectively of infinite mass.
- However, precise spectroscopic measurements of hydrogenic atoms reveal some very small effects on the energy levels which cannot be explained if the nucleus is treated in this way.
- These effects are known as *hyperfine effects* because they are much smaller than the fine structure effects predicted by the Dirac equation.
- The hyperfine effects can be grouped into two types.
  - 1 Hyperfine structure effects give rise to splittings in energy levels
  - 2 Isotope shifts slightly shift the energy levels and can usually be detected by observing the differences between two or more different isotopes.

# Electric Multipole Moments of the Nucleus

- Hyperfine structure effects arise from the electric multipole moments of the nucleus which can interact with the electromagnetic field produced at the nucleus by the electrons.
- The two main multipole moments are the magnetic dipole moment associated with the spin of the nucleus and the electric quadrupole moment due to the departure of the spherical charge distribution in the nucleus.
- Perturbation theory can be used to determine the shift in the energy these two multipole effects will have.

# Magnetic Dipole Interaction

The perturbation which describes the interaction of the nuclear magnetic dipole moment consists of two expressions depending on the orbital angular momentum  $l$ . For the case of  $l \neq 0$  we have

$$H'_{MD} = \frac{2\mu_0}{4\pi\hbar} g_I \mu_B \mu_N \frac{1}{r^3} \mathbf{G} \cdot \mathbf{I} \quad (24)$$

where

$$\mathbf{G} = \mathbf{L} - \mathbf{S} + 3 \frac{(\mathbf{S} \cdot \mathbf{r})\mathbf{r}}{r^2} \quad (25)$$

and the total spin of the atom (nucleus and electron) is given by

$$\mathbf{F} = \mathbf{I} + \mathbf{J}. \quad (26)$$

For the case when  $l = 0$ , the perturbation is given by

$$H'_{MD} = \frac{\mu_0}{4\pi} 2g_I \mu_B \mu_N \frac{8\pi}{3} \delta(\mathbf{r}) \mathbf{S} \cdot \mathbf{I}. \quad (27)$$



# Electric Quadrupole Interaction

The interaction Hamiltonian between the electric quadrupole moment of the nucleus and the electrostatic potential created by an electron at the nucleus is given (in atomic units) by

$$H'_{EQ} = B \frac{\frac{3}{2} \mathbf{I} \cdot \mathbf{J} (2\mathbf{I} \cdot \mathbf{J} + 1) - I^2 \mathbf{J}^2}{2I(2I-1)j(2j-1)} \quad (28)$$

# Hyperfine Spectrum

The magnetic dipole and electric quadrupole interactions above correspond to a total hyperfine energy shift of

$$\Delta E = \frac{C}{2}K + \frac{B}{4} \frac{\frac{3}{2}K(K+1) - 2I(I+1)j(j+1)}{I(2I-1)j(2j-1)} \quad (29)$$

where  $B$  is the quadrupole coupling constant given by

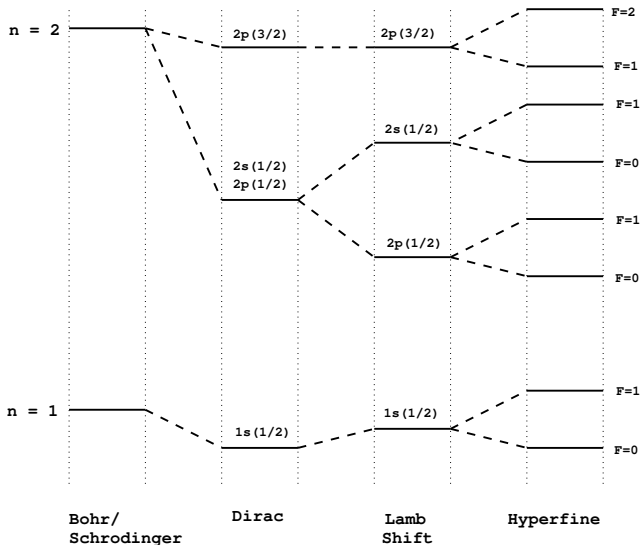
$$B = Q \left\langle \frac{\partial^2 V}{\partial z^2} \right\rangle \quad (30)$$

and

$$K = F(F+1) - I(I+1) - j(j+1) \quad (31)$$

$$C = \frac{\mu_0}{4\pi} 2g_I \mu_B \mu_N \frac{I(I+1)}{j(j+1)} \frac{Z^3}{a_\mu^3 n^3 I(I+1/2)(I+1)} \quad (32)$$

# Energy Levels for Hydrogen Atom for $n = 1$ and $n = 2$



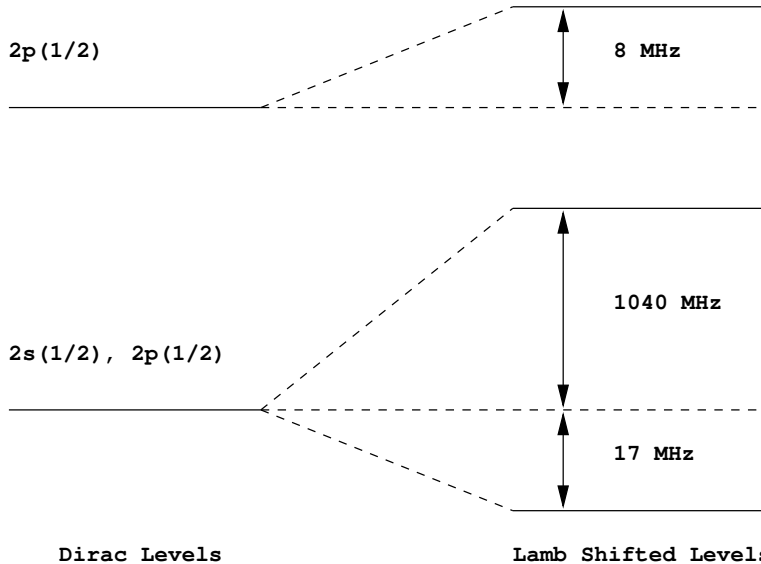
# 21cm Line of Atomic Hydrogen

- The energy level splitting diagram in the previous figure shows that the ground state of the hydrogen atom splits into two hyperfine levels with the total angular momentum of the atom being  $F = 0$  and  $F = 1$ .
- The difference in energy between these two levels is 1420 MHz which corresponds to a wavelength of  $\lambda \approx 21$  cm.
- The probability of a transition occurring between these two levels is very low and occurs on average only once every few million years.
- However, there is a large amount of hydrogen gas in the galaxy allowing radio telescopes easily detect this 21 cm transition and therefore allow the mapping of hydrogen in the galaxy.

# Lamb Shift

- Investigation of the fine structure of hydrogenic atoms using spectroscopic techniques in the 1930's showed that there were small differences between the observed spectra and the theoretical predictions made by the Dirac equation.
- For example, according to the Dirac equation the  $2s_{1/2}$  and the  $2p_{1/2}$  states coincide at the same energy level. Observations showed that the  $2s_{1/2}$  was shifted slightly upwards by about  $0.03 \text{ cm}^{-1}$ .
- A very accurate measurement of the shift was made in 1947 by Lamb and Retherford using microwave techniques to stimulate a direct radio-frequency transition between the  $2s_{1/2}$  and the  $2p_{1/2}$  levels.
- This small shift of energy levels became known as the Lamb shift.

# Visualising the Lamb Shift



# Quantum Electrodynamics

- The physics of the Lamb shift are described in the theory of quantum electrodynamics, in which radiative corrections to the Dirac equation are obtained by taking into account the interaction of a quantised electromagnetic field with an electron.
- The Lamb shift arises because of the zero point energy of a quantised electromagnetic field is non zero, similar to the zero point energy of a quantum harmonic oscillator. In a vacuum, fluctuations of the zero point energy of the quantised radiation field act on the electron.

# Quantum Electrodynamics (2)

- The effect of the electron is to cause it to oscillate rapidly about some equilibrium position. As a result of this oscillatory motion, the electron does not appear to be point charge – instead the electron charge is slightly smeared at in a sphere of some small radius.
- When the electron is bound by an electric field as it is in an atom, the potential it experiences is slightly different to that experienced by the electron in it's mean position.
- Therefore electrons which are most sensitive to short distance modifications such as those in the ground state are raised in energy with respect to other states to which the shift is much smaller.



# References

- 1 *Physics of Atoms and Molecules*, Bransden and Joachain
- 2 *Quantum Mechanics of One- and Two-Electron Atoms*, Bethe and Salpeter
- 3 *Intermediate Quantum Mechanics*, Bethe
- 4 *Advanced Quantum Mechanics*, Sakurai
- 5 *The Spectrum of Atomic Hydrogen - Advances*, Series
- 6 *Quantum Mechanics*, Merzbacher
- 7 *Modern Physics*, Serway, Moses and Moyer