05/06/2016 11:46   SYRACUSE UNIVERSITY ANNEX PRODUCTION SYSTEM   PG 1
RPT# 00084089 ddeangue   cpyrept.

EDD RETRIEVAL RECEIPT

Order: 009810   Document Nbr: 905756
For: EDD
Copied: 05/06/2016
Shipped: 05/06/2016
Deliver To: Internal ILLIAD user
Item BarCode: 32911041496290
Item Title: European journal of operational research
Item Author:
Item Call Number: T57.6 .E92
Item Vol/Part:

Article Title: Modeling of building evacuation problems by netwo
Article Author: Choi, W. Choi, W.
Art Vol/Part: 35
Article Issue: 1
Beg Page: 98   End Page: 110   Total Pages: 13
Other Info: 905756
Notes: 3|Exempt

European Journal of Operational Research 35 (1988) 98-110
North-Holland

Theory and Methodology

Modeling of building evacuation problems by network flows with side constraints

W. CHOI, H.W. HAMACHER and S. TUFEKCI
Department of Industrial and Systems Engineering, University of Florida, Gainesville, FL, USA

Abstract: In this paper we model building evacuations by network flows with side constraints. Side constraints come from variable arc capacities on some arcs which are functions of flows in incident arcs. In this context we study maximum flow, minimum cost, and minimax objectives. For some special structured networks we propose 'greedy' algorithms for solving these problems. For more general network structures, solution procedures are recommended which take advantage of the network structures of the problems.

Keywords: Network programming, optimization

1. Introduction

Network flow modeling of building evacuation problems is given by Chalmlet et al. [3]. In this model the architectural design of a building is transformed into a network structure where the nodes of this network represent hallways, workplaces, and stairwells, while the arcs represent the connection between these places. In this model arc capacities, representing the number of people which can traverse a given connection per unit time, are assumed to be constant.

It is well known (Fruin [8,9], Pauls [23]), that capacities on such arcs are not constant. In fact the capacity on a given arc is a function of the number of people present in that place at a given time.

Incorporating these flow dependent capacities converts the corresponding network flow problems into network flow problems with side constraints.

In what follows we assume that a network representation of a building is available. For further details on obtaining a network representation of a building we refer to Chalmlet et al. [3].

Let $G=(V, E)$ represent the network of a building where V is the node set and E is the arc set. We call G the *static network*. Since each arc $e \in E$ in this network represents a passage from one component of a building to another, we associate with it a transit time $\tau(e)$.

Consider the network $G_T=(V_T, E_T)$, the *T-time expanded* network obtained from $G=(V, E)$ as follows:

$$V_T=\{x^i : x \in V, 0 \leqslant i \leqslant T\}.$$

Here x^i is the i-th time copy of node $x \in V$. Similarly, the arc set E_T is given by

$$\begin{aligned} E_T= \{(x^i, x^j) : e=(x, y) \in E \text{ and} \\ j=i+\tau(e) \leqslant T, \\ i=1, 2, \ldots, T\} \\ \cup\{(x^i, x^{i+1}) : x \in V, i=0, 1, 2, \ldots,\}. \end{aligned}$$

The network G_T is also called a *dynamic network*. The arcs $(x^i, x^{i+I}) \in E_T$ are called the *holdover*

Received November 1985; revised March 1987

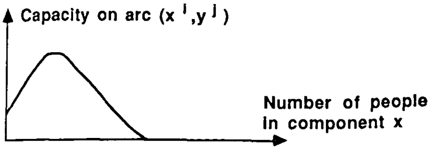

Figure 1. Capacity function on a typical movement arc

arcs. The content of such an arc represents the number of people still remaining in the building component represented by the node x. The arcs (x^i, y^j) are called the *movement arcs*. They represent the movements of people from one component of the building to another. This movement starts at time i in component x, and terminates in component y at time j.

Associated with each holdover arc (x^i, x^{i+1}) we have a constant capacity which represents the limiting capacity of this component. Associated with each movement arc (x^i, y^j) we have a capacity which represents the maximum number of people which can traverse that arc. The capacity of such a movement arc is a function of the density (people/ft^2) of people present at component x at time i. Given the constants area of the component x, the capacity of such an arc can be expressed as a function only of the number of people in that component at time i. Fruin [8,9] has shown that this function has the form of Figure 1.

Consider a building which is composed of a single floor with one exit to the outside. The static and dynamic network representation of this building is shown in Figure 2.

In this figure node D represents the 'outside' or the destination of all the evacuees. All arcs (x^j, D), $j = 1, 2, \ldots, T$, allow people to exit the building at

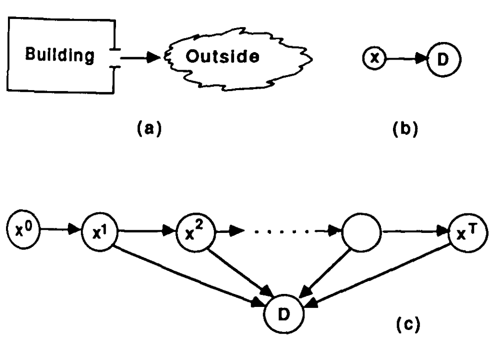

Figure 2. A one-floor building; (a) Graphical representation, (b) Static network representation, (c) Dynamic network representation

time j. The following problem has been studied by Choi et al. [5]: Given the flow dependent capacities on arcs (x^j, D), $j = 1, 2, \ldots, T$, what is the maximum number of people which can be accomodated in the building such that in case of an emergency evacuation all of them will exit the building within T time periods? The network flow formulation of this problem is given below:

$$\text{Maximize} \quad f(x^0, x^1), \tag{1.1}$$

subject to

$$f(x^i, x^{i+1}) - f(x^{i+1}, x^{i+2}) - f(x^{i+1}, D) = 0, \quad i = 0, 1, \ldots, T-1, \tag{1.2}$$

$$f(x^{T-1}, x^T) - f(x^T, D) = 0, \tag{1.3}$$

$$0 \leqslant f(x^i, D) \leqslant c_i\big(f(x^{i-1}, x^i)\big), \quad i = 1, 2, \ldots, T, \tag{1.4}$$

$$f(x^i, x^{i+1}) \geqslant 0, \quad i = 0, 1, 2, \ldots, T-1. \tag{1.5}$$

Note here that the constraints (1.2) and (1.3) are regular flow conservation equations and (1.4) are flow dependent capacity constraints on arcs (x^i, D), $i = 1, 2, \ldots, T$.

Choi et al. [5] have shown that a unique optimal solution exists if $c_i(\cdot)$ satisfies the following conditions:

(A1) $c_t(\cdot)$ is continuous over $[0, \infty)$,
(A2) $c_t(q) \leqslant 0$ for all $q \geqslant 0$ and $c_t(0) > 0$,
(A3) the function $\bar{c}_t(q) = q - c_t(q)$, which is called the *complement function*, is strictly increasing and unbounded above.

The optimal solution to this problem is given by

$$f^*(x^{T-1}, x^T) = \bar{c}_T^{-1}(0), \tag{1.6}$$

$$f^*(x^{t-1}, x^t) = \bar{c}_t^{-1}\big(f^*(x^t, x^{t+1})\big), \quad t = T-1, T-2, \ldots, 1, \tag{1.7}$$

$$f^*(x^t, D) = c_t\big(f^*(x^{t-1}, x^t)\big), \quad t = 1, 2, \ldots, T. \tag{1.8}$$

Here, $\bar{c}_t^{-1}$ is the inverse function of $\bar{c}_t$.

An example of such a function $c_t(\cdot)$ satisfying (A1)–(A3) is the linear function

$$c_t(q) = Aq + B, \quad t = 1, 2, \ldots, T, \tag{1.9}$$

where $0 < A < 1$, and $B > 0$. For such a capacity function the unique optimum solution is given by

$$f^*(x^t, D) = \frac{B}{(1-A)^{T+1-t}}, \quad t = 1, 2, \dots, T,$$

and

$$f^*(x^0, x^1) = (B/A)\frac{1}{(1-A)^T - 1}.$$

Francis [7] considered the continuous version of this problem for the linear capacity functions given in (1.9). In this model $g(t)$, $0 \leqslant t \leqslant T$, is a differentiable exit function such that

$$\int_0^t g(s)\,\mathrm{d}s$$

represents the number of people exiting the lobby in the period $[0, t]$. Given a unit time capacity function $C(x) = Ax + B$ as in (1.9) the problem is formulated as

$$\text{Maximize} \quad F = \int_0^T g(s)\,\mathrm{d}s, \tag{1.10}$$

subject to

$$0 \leqslant g(t) \leqslant A\left[\int_t^T g(s)\,\mathrm{d}s\right] + B, \quad 0 \leqslant t \leqslant T. \tag{1.11}$$

The unique optimal exit function to this problem is given by

$$g^*(t) = B\exp(A(T-t)), \quad 0 \leqslant t \leqslant T.$$

Further, the maximum flow is given by

$$f^* = (B/A)(\exp AT - 1).$$

It is also shown that there exists a dual problem to (1.10)–(1.11) given by

$$\text{Maximize} \quad B\int_0^T \theta(r)\,\mathrm{d}r, \tag{1.12}$$

subject to

$$\theta(t) - A\int_0^t \theta(r)\,\mathrm{d}r \geqslant 1, \quad 0 \leqslant t \leqslant T, \tag{1.13}$$

$$\theta(t) \geqslant 0, \qquad 0 \leqslant t \leqslant T. \tag{1.14}$$

This primal–dual pair satisfies the weak theorem of complementary slackness. Moreover, at optimality both objective functions are equal.

Choi et. al [6] have extended Francis' work to a piece-wise linear capacity function which is a linear approximation of the function given in Figure 1. This function is given by

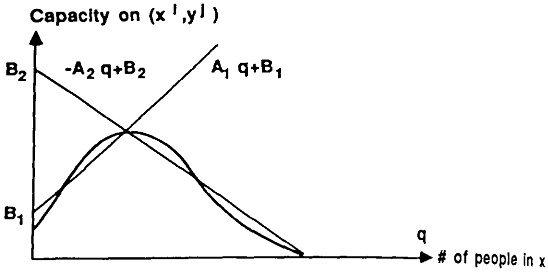

Figure 3. A piece-wise linear approximation for $c(q)$

$$c(q) = \min\{A_1 q + B_1, \ -A_2 q + B_2\},$$

where $0 < A_1 < 1$, $A_2 > 0$, and $(1 + A_2)B_1 < (1 - A_1)B_2$. Such an approximation is shown in Figure 3.

The continuous time formulation of this problem is given by

$$\text{Maximize} \quad F = \int_0^T g(s)\,\mathrm{d}s, \tag{1.15}$$

subject to

$$g(t) - A_1\int_t^T g(s)\,\mathrm{d}s \leqslant B_1, \quad 0 \leqslant t \leqslant T, \tag{1.16}$$

$$g(t) + A_2\int_t^T g(s)\,\mathrm{d}s \leqslant B_2, \quad 0 \leqslant t \leqslant T, \tag{1.17}$$

$$g(t) \geqslant 0, \qquad 0 \leqslant t \leqslant T. \tag{1.18}$$

The corresponding dual problem is

$$\text{Minimize } B_1\int_0^T \theta_1(r)\,\mathrm{d}r + B_2\int_0^T \theta_2(r)\,\mathrm{d}r, \tag{1.19}$$

subject to

$$\theta_1(t) + \theta_2(t) - A_1\int_0^t \theta_1(r)\,\mathrm{d}r + A_2\int_0^t \theta_2(r)\,\mathrm{d}r \geqslant 1, \quad 0 \leqslant t \leqslant T, \tag{1.20}$$

$$\theta_1(t),\ \theta_2(t_2) \geqslant 0, \quad 0 \leqslant t \leqslant T. \tag{1.21}$$

It is also shown in [6] that a similar duality relationship exists between the primal and the dual problem. The optimal exit rate function $g^*(t)$ is given by

$$g^*(t) = \begin{cases} KB_1\exp(-A_2(T-t)) \\ \quad \text{for } 0 \leqslant t \leqslant T - \hat{t}, \\ B_1\exp(A_1(T-t)) \\ \quad \text{for } T - \hat{t} \leqslant t \leqslant T, \end{cases}$$

where

$$K = [(A_2B_1 + A_1B_2)/B_1(A_1 + A_2)]^{(A_2/A_1+1)}$$

and

$$\hat{t} = (1/A_1)\log((A_2B_1 + A_1B_2)/(B_1(A_1 + A_2))) \leqslant T.$$

If $\hat{t} > T$ then the optimal exit rate function is the same as the one given in Francis [7].

Consider now a building whose static network representation contains more than one node and one arc. The corresponding building evacuation problem has been modeled in [3] as a minimum cost network flow problem with *constant* arc capacities. In this formulation all the holdover arcs between the building components are assigned zero costs. The movement arcs (x^t, D), from a component to the outside are assigned a cost t, which is the time this exit takes place. This cost is termed the 'turnstile' cost indicating that any person leaving the building at time t must pay a turnstile cost of value t. The minimum cost network flow problem is then formulated as

$$\text{Minimize} \quad \sum_{t=1}^{T} f(x^t, D)t, \tag{1.22}$$

subject to

$$f \text{ satisfies the flow conservation equations,} \tag{1.23}$$

$$0 \leqslant f(e) \leqslant u(e), \quad e \in E_T. \tag{1.24}$$

Here $u(e)$ is the *constant* capacity of arc $e \in E_T$.

In this paper we will study multiple-floor buildings with *flow dependent* arc capacities. In this context we assume that each floor may have its own exit to the outside.

We pose the following three questions associated with such a building.

(P.1) *Maximum flow problem (MFP)*

Given flow dependent arc capacities, what is the maximum number of people which can be accomodated in the building such that all these people will be able to exit the building within T time periods?

(P.2) *Turnstile cost problem (TP)*

Given flow dependent arc capacities, and the numbers and locations of the occupants in the building, what is the evacuation pattern which minimizes the turnstile objective? We note here that any flow pattern which minimizes the turnstile objective also minimizes the building evacuation time and maximizes the cumulative number of people exiting the building for all time periods t, $0 \leqslant t \leqslant T$, if all arc capacities are constant. This property is called the 'triple optimization' by Jarvis and Ratliff [15]. This property may not be true in nonconstant arc capacity cases.

(P.3) *Minimax (bottleneck) problem (MP)*

Given flow dependent arc capacities and numbers and locations of occupants in the building, what is the flow pattern which minimizes T, the time the last person exits the building?

In this paper we will show that for some simple static network structures and some nonconstant capacity functions the optimal solution can be obtained in a greedy fashion. For more complex networks and capacity functions we also recommend solution procedures which take advantage of the underlying network structure.

2. The maximum flow problem

Let $G = (V, E)$ be the static network with transit times $\tau(e) = 1$, $e \in E$. Also let $X \subset V$ be the set of nodes representing the building components occupied by people. The corresponding maximum flow problem can be formulated as

$$\text{Maximize} \quad \sum_{x \in X} f(x^0, x^1), \tag{2.1}$$

subject to

$$f \text{ is a conserving flow in } G_T, \tag{2.2}$$

$$0 \leqslant f(e) \leqslant c(e), \quad e \in E_T. \tag{2.3}$$

Here, $f(x^0, x^1)$, $x \in X$, represents the initial content of component x, and c(e) is the flow dependent capacity function.

An example of a three-floor building is given in Figure 4. Note that we included the exit arcs (x^i, D), $x \in V$, $i = 1, 2, \ldots, T$, for the completeness of the picture. If there is no exit from floor $x \in V$ to the outside then all arcs (x^i, D), $i = 1, 2, \ldots, T$, should be dropped.

We will analyze the maximum flow problem for linear flow dependent capacity function $c(e)$ and

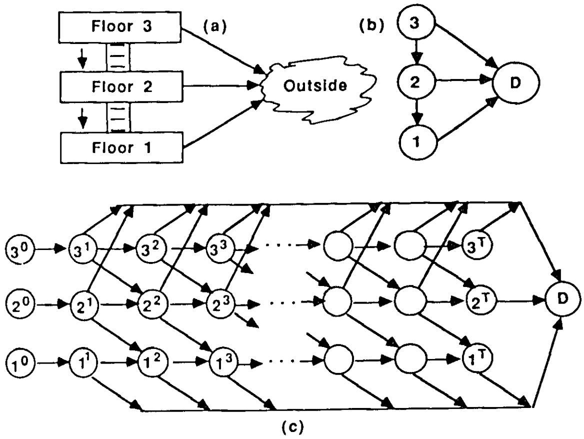

Figure 4. A three-floor building; (a) Graphical representation, (b) Static network, (c) Dynamic network

provide some insight into the optimal solution.

In what follows the holdover arc capacities are assumed to be sufficiently large. Let the static network structure of an M-floor building be as shown in Figure 4 (for $M=3$). Moreover, we assume that each component (floor) has its own exit to the outside. If only component (floor) 1 had an exit to the outside the problem becomes trivial. In such a case the optimal solution is given by solving a one-floor problem for floor 1. Note here that all exit arcs from floor 1 are at their capacities in such a solution.

We assume the following form of capacity functions for each movement arc and for each exit arc:

$$C_{x,D}(x^i, D) = A_x\big(f(V_T, x^i)\big) + B_x = A_x\Big(f(x^{i-1}, x^i) + \sum_{y\neq x}\sum_{j<i} f(y^j, x^i)\Big) + B_x \tag{2.4}$$

and

$$C_{x,s(x)}\big(x^i, s(x)^{i+1}\big) = A_{x,s(x)}\big(f(V_T, x^i)\big) + B_{x,s(x)} = A_{x,s(x)}\Big(f(x^{i-1}, x^i) + \sum_{y\neq x}\sum_{j<i} f(y^j, x^i)\Big) + B_{x,s(x)}. \tag{2.5}$$

Here, $s(x)$ is the successor of node x in G toward the first floor, and $0<A_x<1$, $B_x>0$, $0<A_{x,s(x)}<1$, $B_{x,s(x)}>0$ for all $x\in V$. Note here that the capacity functions defined in (2.5) assume $\tau(e)=1$, $e\in E$. If $\tau(e)>1$, then the necessary adjustments are obvious.

Assumption 2.1. In what follows it is assumed that $A_x + A_{x,s(x)} < 1$.

If $A_x + A_{x,s(x)} \geqslant 1$, then regardless of the number of people reaching component x they can be evacuated in one period which is a physical impossibility. For simplicity of notations let $\gamma_x = 1 - A_x$, and $\gamma_{x,s(x)} = 1 - A_{x,s(x)}$. With these preliminaries the (MFP) can be formulated as

$$\text{Maximize} \quad \sum_{x\in V} f(x^0, x^1), \tag{2.6}$$

subject to

$$f \text{ is a conserving flow in } G_T, \tag{2.7}$$

$$f(x^i, D) \leqslant c_{x,D}(x^i, D), \quad x\in V, \quad i=1,2,\ldots,T, \tag{2.8}$$

$$f\big(x^i, s(x)^{i+1}\big) \leqslant c_{x,s(x)}\big(x^i, s(x)^{i+1}\big), \quad x\in V, \quad i=1,2,\ldots,T, \tag{2.9}$$

$$f(e)\geqslant 0, \quad e\in E_T. \tag{2.10}$$

The problem (MFP) in this form is a maximum flow problem with linear side constraints, and thus, can be solved by appropriate techniques [4,10,11,16]. However, we will further analyze this problem in more detail.

Lemma 2.1. *For an arbitrary static network $G=(V, E)$ constraints* (2.8) *are tight at optimality.*

Proof. Suppose $f^*(e)$, $e\in E_T$, be an optimal solution of (MFP). Let $f^*(x^t, D) < c_{x,D}(x^t, D)$ for some $x\in V$ and $t\leqslant T$. Consider the following changes:

$$f'(x^t, D) = f^*(x^t, D) + \varepsilon,$$
$$f'(x^{t-j}, x^{t-j+1}) = f^*(x^{t-j}, x^{t-j+1}) + \varepsilon, \quad j=1,2,\ldots,t,$$

and all other $f'(e) = f^*(e)$, $e\in E_T$.

It is easy to see that for $\varepsilon>0$ (and small enough) f' is a feasible flow since $c(\cdot)$ is a monotone increasing function. Moreover, $f'(x^0, x^1) = f^*(x^0, x^1) + \varepsilon > f^*(x^0, x^1)$, a contradiction to the optimality of $f^*(e)$. □

Lemma 2.1 can be used to eliminate $|V|T$ side constraints given by (2.8) from the (MFP). This can be accomplished by replacing the variables $f(x^i, D)$ in (2.7) by the right hand sides of (2.8).

We note here that in the proof of Lemma 2.1 we only used the monotonicity of $c(\cdot)$. Therefore, the lemma is correct for any monotone increasing capacity function $c(\cdot)$.

In the following lemma we describe a condition where the optimal solution of the (MFP) can be obtained in a simple way.

Lemma 2.2. *Let $G=(V, E)$ be an acyclic network. The nodes $x \in V$ are indexed such that if $(x, y) \in E$ then $x > y$. For each $x \in V$ if $A_x \leqslant A_y$, then there exists an optimal solution $f'(e)$, $e \in E_T$ such that $f'(e)=0$ for each movement arc $e=(x^i, y^{i+1}) \in E_T$.*

Proof. Let $f^*(e)$, $e \in E_T$ be an optimal solution such that some movement arcs have positive flows. Let $x \in V$ be the largest indexed node such that $f^*(x^t, y^{t+1}) > 0$ for at least one $t \leqslant T$, and $e=(x, y) \in E$. Let k be the smallest index such that $f^*(x^k, y^{k+1}) > 0$. Note that with this selection we have either $k=1$ or $f^*(x^j, y^{j+1})=0$, $(x, y) \in E$, for $j<k$. Now consider the following flow changes for each arc $(x, y) \in E$.

$$f'(x^k, y^{k+1}) = 0 \quad \text{for each } (x, y) \in E,$$
$$f'(x^{k-1}, x^k) = \gamma_x^{-1}\big(f^*(x^k, x^{k+1}) + B_x\big),$$
$$f'(x^{k-j}, x^{k-j+1}) = \gamma_x^{-1}\big(f'(x^{k-j+1}, x^{k-j+2}) + B_x\big), \quad j=2, 3, \ldots, k,$$
$$f'(y^k, y^{k+1}) = f^*(y^k, y^{k+1}) + f^*(x^k, y^{k+1}) \quad \text{for each } (x, y) \in E,$$
$$f'(y^{k-j}, y^{k-j+1}) = f^*(y^{k-j}, y^{k-j+1}) + \gamma_y^{-1} f^*(x^k, y^{k+1}), \quad j=1, \ldots, k, \ (x, y) \in E,$$
$$f'(x^j, D) = A_x f'(V_T, x^j) + B_x, \quad j=1, 2, \ldots, k,$$
$$f'(y^j, D) = A_y f'(V_T, y^j) + B_y, \quad j=1, 2, \ldots, k; \ (x, y) \in E,$$

and all other $f'(e)=f^*(e)$, $e \in E_T$.

It can be shown that f^* capacitates all the exit arcs and also satisfies the flow conservation equations. Therefore f is feasible. Moreover,

$$\begin{aligned} f'(x^0, x^1) + f'(y^0, y^1) &= f^*(x^0, x^1) + f^*(y^0, y^1) \\ &\quad + f^*(x^k, y^{k+1})\gamma_y^{-k} - f^*(x^k, y^{k+1})\gamma_x^{-k} \\ &\geqslant f^*(x^0, x^1) + f^*(y^0, y^1), \end{aligned}$$

since

$$\gamma_y^{-k} \geqslant \gamma_x^{-k}, \quad y \in s(x).$$

This argument may be repeated at a new index $k' > k$. Thus, we have just shown that after at most T such flow changes we obtain a maximum flow $f'(e)$, $e \in E_t$, such that $f'(x^t, y^{t+1})=0$ for $t=1, 2, \ldots, T-1$. We now can carry this argument at a new node index $z < x$. Thus, after at most $T|V|$ many flows changes we get a maximum flow f' satisfying $f'(e)=0$ for all movement arcs $e \in E_T$. $\square$

The significance of this lemma is that if $A_x < A_y$ for all $e=(x, y) \in E$ then the problem decomposes into M 1-floor problems. The optimal solution for 1-floor problems is given in equations (1.6)–(1.8).

If the (MFP) does not satisfy the conditions given in Lemma 2.2, then we recommend solving this problem by using the network with side constraints technique after approximating the capacity functions by piecewise linear functions, if the capacity functions are not already linear. We note here that the finer the approximations the larger the number of side constraints in the problem. If the static network had Q movement and exit arcs and if each capacity function is approximated by P line segments, we will have QPT side constraints in the problem.

3. The turnstile cost problem (TP)

Given the static network $G=(V, E)$, its corresponding dynamic network $G_T=(V_T, E_T)$, and the numbers a_x of occupants at each node $x \in V$, the (TP) is formulated as

$$\text{Minimize } \sum_{t=1}^{T} f(x^t, D)t, \tag{3.1}$$

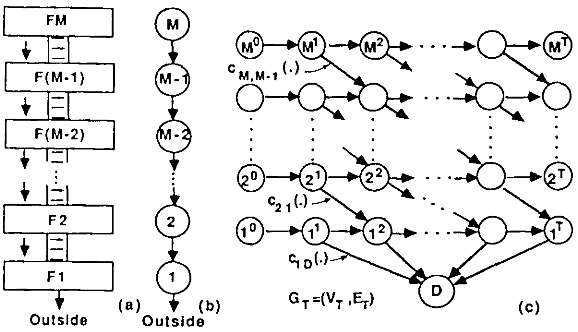

Figure 5. An M-floor path structured building. a_x is the initial content of floor $x \in V$ and $c(\cdot)$ is the capacity function of a movement arc or an exit arc

subject to

$$f(x^0, x^1) = a_x, \quad x \in V, \text{ and} \tag{3.2}$$

f is a conserving flow in G_T,

$$0 \leqslant f(e) \leqslant c(e), \quad e \in E_T. \tag{3.3}$$

Here $c(e)$ is a flow dependent capacity function. In this section we will study some special structures of G and some special forms of $c(e)$. For these cases we will provide 'greedy' algorithms for obtaining optimal solutions.

Path structured network

Consider an M-floor building where each floor is represented by a single node and only descending from a floor i to floor $(i-1)$ is allowed, for $i = M, M-1, \ldots, 2$. The only exit of the building to the outside is at the first floor. Figure 5 shows such a building and its corresponding networks.

Figure 5 and the following formulation of the (TP) assumes $\tau(e) = 1$. If $\tau(e) > 1$ for some $e \in E$, then the necessary adjustments must be made in the formulation.

$$\text{Minimize} \quad \sum_{t=1}^{T} f(1^t, D)t, \tag{3.4}$$

subject to

$$f(x^0, x^1) = a_x, \quad x \in V, \text{ and} \tag{3.5}$$

$$f \text{ is a conserving flow in } G_T, \tag{3.6}$$

$$0 \leqslant f\big(x^t, (x-1)^{t+1}\big) \leqslant C_{x,x-1}\big(f(x^{t-1}, x^t) + f\big((x+1)^{t-1}, x^t\big)\big), \tag{3.7}$$

$$x = M, M-1, \ldots, 2; \quad t = 1, 2, \ldots, T,$$

$$0 \leqslant f(1^t, D) \leqslant c_{1,D}\big(f(1^{t-1}, 1^t) + f(2^{t-1}, 1^t)\big), \quad t = 1, 2, \ldots, T. \tag{3.8}$$

In this formulation we assumed that the holdover arcs have sufficiently large capacities such that they will not be binding in the optimal solution. We also make the following assumptions on the capacity function $c_{x,x-1}(q) \equiv \min\{q, c_{x,x-1}(q)\}$, for each movement arc $(x, x-1) \in E$. Note that this definition of $c_{x,x-1}$ does not affect the real form of $c_{x,x-1}$.

(A1′) $c_{x,x-1}(\cdot)$ is continuous on $[0, \infty)$, $x \in V$,

(A2′) $c_{x,x-1}(q) \geqslant 0$, for all $q \geqslant 0$, and $c_{x,x-1}(0) = 0$,

(A3′) the complement function $\bar{c}_{x,x-1}(q) = q - c_{x,x-1}(q)$ is monotone increasing.

Now consider a flow f and abbreviate $f_i = f(1^i, D)$, $i = 1, \ldots, T$. With $\boldsymbol{f}$ we denote the vector $\boldsymbol{f} = (f_1, f_2, \ldots, f_T)$. A partial ordering can be defined on the set of all flows by

$$\boldsymbol{f} \succcurlyeq \boldsymbol{g} \quad \Leftrightarrow \quad \sum_{j=1}^{t} f_j \geqslant \sum_{j=1}^{t} g_j \quad \text{for all } t = 1, 2, \ldots, T.$$

Obviously $\succcurlyeq$ is not a total ordering. If $\boldsymbol{f}^* = \{\max \boldsymbol{f} \mid \text{f is a feasible flow}\}$ exists, we call $\boldsymbol{f}^*$ an *earliest arrival flow* (EAF). Here 'max' is the maximum with respect to the ordering '$\succcurlyeq$'.

Lemma 3.1. *Let f be a flow satisfying* (3.2)–(3.3) *such that $\boldsymbol{f}^*$ is an EAF. Then f is an optimal solution of the turnstile cost problem.*

Proof. Let f be any flow satisfying (3.2)–(3.3). Since f^* is an EAF we get

$$\begin{aligned}
&\sum_{j=1}^{T} f_j^*(T+1-j) \\
&\quad = f_1^* + (f_1^* + f_2^*) + \cdots + \left(\sum_{j=1}^{T-1} f_j^*\right) + \left(\sum_{j=1}^{T} f_j^*\right) \\
&\quad \geqslant f_1 + (f_1 + f_2) + \cdots + \left(\sum_{j=1}^{T-1} f_j\right) + \left(\sum_{j=1}^{T} f_j\right) \\
&\quad = \sum_{j=1}^{T} f_j(T+1-j).
\end{aligned}$$

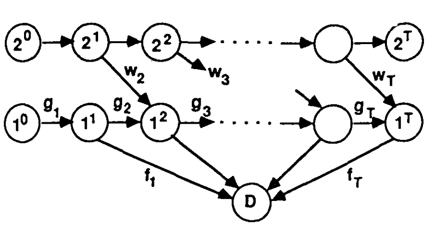

Figure 6. First floor situation with given initial content a_1 and arrivals w_j, $j = 1, 2, \ldots, T$

Since

$$f_j^* = f^*(x^j, D), \qquad f_j = f(x^j, D)$$

and

$$(T+1)\sum_{j=1}^{T} f_j^* = (T+1)\sum_{j=1}^{T} f_j = (T+1)\sum_{x \in V} a_x$$

by (3.5), we get

$$\sum_{j=1}^{T} f^*(x^j, D)j = \sum_{j=1}^{T} f_j^* j \leqslant \sum_{j=1}^{T} f_j j = \sum_{j=1}^{T} f(x^j, D)j.$$

Thus, f^* is an optimal solution of the turnstile cost problem. It should be noted that Lemma 3.1 has been shown by Jarvis and Ratliff [15] for dynamic flows with constant arc capacities. □

By Lemma 3.1 it is sufficient to find a flow f^* such that f^* is an EAF. We will show this by first studying the following (see Figure 6). Let a_1 be the initial content of floor 1, and suppose that the arrivals $w_j = f(2^{j-1}, 1^j)$, $j = 2, \ldots, T$, from the second floor are known. To simplify the exposition we denote $g_j = f(1^{j-1}, 1^j)$, $j = 1, 2, \ldots, T$.

In the next lemma it is shown that we can easily find an earliest arrival flow f^* and thus, by Lemma 3.1, an optimal solution of the turnstile cost problem.

Lemma 3.2. *Let f^* be a flow satisfying*

(1) $g_1^* = f^*(1^0, 1^1) = a_1$,

(2) $w_j^* = f^*(2^{j-1}, 1^j) = w_j, \quad j = 2, \ldots, T$,

(3) $f_j^* = c_{1,D}(g_j^* + w_j^*), \quad j = 1, 2, \ldots, T$,

(4) $g_j^* = g_{j-1}^* + w_{j-1}^* - f_{j-1}^*, \quad j = 2, 3, \ldots, T$.

Then f^* is an earliest arrival flow of all flows satisfying (1) *and* (2).

Proof. Let f be an arbitrary flow satisfying $f(1^0, 1^1) = a_1$ and $f(2^{j-1}, 1^j) = w_j$, $j = 2, \ldots, T$. We first show that $g_j^* \leqslant g_j$, $j = 1, 2, \ldots, T$. It is obvious that f^* and g^* are both feasible. We have $g_1^* = g_1 = a_1$. Suppose $g_j^* \leqslant g_j$ for $j = 1, 2, \ldots, i$. Then

$$g_i^* + w_i \leqslant q_i + w_i. \tag{3.9}$$

By (3) and (4) we have

$$g_{i+1}^* = g_i^* + w_i - f_i^* = g_i^* + w_i - c_{i,D}(g_i^* + w_i),$$

or

$$g_{i+1}^* = \bar{c}_{i,D}(g_i^* + w_i). \tag{3.10}$$

Morever, from the flow conservation

$$g_{i+1} = g_i + w_i - f_i. \tag{3.11}$$

Since $f_i \leqslant c_{i,D}(g_i + w_i)$, we get by using (3.11),

$$\begin{aligned} g_{i+1} &= g_i + w_i - f_i \\ &\geqslant g_i + w_i - c_{i,D}(g_i + w_i) \\ &= \bar{c}_{i,D}(g_i + w_i). \end{aligned} \tag{3.12}$$

By (3.9), (3.10), (3.12) and the assumption (A3′) we get

$$g_{i+1}^* = \bar{c}_{i,D}(g_i^* + w_i) \leqslant \bar{c}_{i,D}(g_i + w_i) \leqslant g_{i+1}.$$

Thus, by the induction we have

$$g_i^* \leqslant g_i, \quad i = 1, 2, \ldots, T. \tag{3.13}$$

By the flow conservation

$$g_{i+1}^* = g_1^* + \sum_{j=1}^{i} w_j - \sum_{j=1}^{i} f_j^*, \quad i = 1, 2, \ldots, T-1,$$

and

$$g_{i+1} = g_1 + \sum_{j=1}^{i} w_j - \sum_{j=1}^{i} f_j, \quad j = 1, 2, \ldots, T-1.$$

Since $g_1^* = g_1 = a_1$ and from (3.13), we get

$$\sum_{j=1}^{i} f_j^* \geqslant \sum_{j=1}^{i} f_j, \quad i = 1, 2, \ldots, T-1 \tag{3.14}$$

and

$$\sum_{j=1}^{T} f_j^* = \sum_{j=1}^{T} f_j = a_1. \tag{3.15}$$

Thus we get $f^* \succcurlyeq f$ from (3.14) and (3.15). □

Note that if $\bar{c}_{1,D}(\cdot)$ is strictly monotone then for a given $W=(w_1, w_2,\ldots, w_n)$, f^* is unique from Lemma 3.2 and partial ordering '$\succcurlyeq$'.

We have shown in Lemma 3.2 that for given inputs a_1, and $W=(w_1,\ldots, w_T)$ the flow $f^*(W)$ defined by (1)–(4) in Lemma 3.1 yields an earliest arrival flow among all flows with the same input. Next we show how to choose W such that $f^*(W)$ yields an overall earliest arrival flow. Analogous to the partial ordering of $f=(f_1,\ldots, f_T)$ we let

$$W^*=(w_1^*,\ldots, w_T^*) \succcurlyeq (w_1,\ldots, w_T) = W$$

if

$$\sum_{i=1}^{j} w_i^* \geqslant \sum_{i=1}^{j} w_i \quad \text{for all } j=1, 2,\ldots, T.$$

In order to show the following result we need the additional assumption:

(A4′) $c_{1,D}(\cdot)$ is a nondecreasing function.

Lemma 3.3.. *Let $W^* \succcurlyeq W$ and let $f^*(W^*)$ and $f(W)$ be the corresponding earliest arrival flows defined by* (1)–(4) *in Lemma* 3.1.
Then

$$f^*(W^*)=(f_1^*,\ldots, f_T^*) \succcurlyeq (f_1,\ldots, f_T) = f(W).$$

Consequently, $f^*(W^*)$ is an optimal solution of the turnstile cost problem if f^* is defined by (1)–(4) *of Lemma* 3.2 *and if W^* is a maximum with respect to the partial ordering '$\succcurlyeq$'.*

Proof. Consider the flow f' where for $j=1, 2,\ldots, T$,

$$w_j' := f'(2^{j-1}, 1^j) := f^*(2^{j-1}, 1^j) = w_j^*,$$

$$f_j' := f'(1^j, D) := f(1^j, D) = f_j,$$

$$g_j' := f'(1^{j-1}, 1^j) := a_1 + \sum_{i=1}^{j-1} w_i^* - \sum_{i=1}^{j-1} f_i,$$

Notice that f' is defined in such a way that it is a conserving flow with inputs a_1, w_j^* and output $f_j'=f_j$, $j=1,\ldots, T$. Moreover,

$$g_j' + w_j^* = a_1 + \sum_{i=1}^{j} w_i^* - \sum_{i=1}^{j-1} f_i, \quad j=1, 2,\ldots, T, \tag{3.16}$$

and by the conservation of flow for f,

$$g_j + w_j = a_j + \sum_{i=1}^{j} w_i - \sum_{i=1}^{j-1}, \quad j=1, 2,\ldots, T. \tag{3.17}$$

Since $W^* \succcurlyeq W$, we have

$$\sum_{i=1}^{j} w_i^* \geqslant \sum_{i=1}^{j} w_j, \quad j=1, 2,\ldots, T, \tag{3.18}$$

thus from (3.16)–(3.18) we get

$$g_j' + w_j^* \geqslant g_j + w_j, \quad j=1, 2,\ldots, T. \tag{3.19}$$

Thus, the monotonicity of $c_{1,D}(\cdot)$ (Assumption (A4′)), and the definition of f_j in (4) of Lemma 3.2 yield

$$f_j' = f_j = c_{1,D}(g_j + w_j) \leqslant c_{1,D}(g_j' + w_j^*), \quad j=1, 2,\ldots, T.$$

Hence, f' is a feasible flow with input w_j^* and output f_j, and Lemma 3.1 yields:

$$(f_1^*,\ldots, f_T^*) \succcurlyeq (f_1',\ldots, f_T') = (f_1,\ldots, f_T)$$

as claimed in Lemma 3.3. □

By iteratively using Lemma 3.2 and 3.3 we can solve the turnstile cost problem with the following algorithm.

Greedy algorithm for (TP) in path structured networks

The following algorithm solves the M-floor turnstile problem in a greedy fashion starting from the top floor and evacuating people from each floor in an EAF manner.

Input: M, a_i, $i=1, 2,\ldots, M$, $c_{x,x-1}(\cdot)$, $x=M, M-1,\ldots,2$, $c_{1,D}(\cdot)$, satisfying assumptions (A1′)–(A4′).
Output: Minimum turnstile cost flow $f^*(e)$, $e \in E_T$.

begin for $i=M$ to 1 **do**
  begin
  Given the flow on the movement arcs from floor $(i+1)$ to i for each time period use the method described in Lemma 3.1 to obtain the EAF's from floor i to $(i-1)$.

end
end for
end

Note that for $i = M$ the flow from (an imaginary) floor $(M+1)$ to M is assumed to be zero for all values of $t \leqslant T$. Also note that at each iteration we assume that all movement arcs from floor k lead into an imaginary destination D'. Naturally, at floor 1 all movement arcs lead into the outside node D.

Theorem 3.1 *The algorithm produces an EAF on each floor.*

Proof. From Lemmas 3.2 and 3.3.

Theorem 3.2. *The algorithm produces an optimal solution to* (TP).

Proof. From Theorem 3.1 and Lemma 3.3.

We note in passing that the optimal solution obtained by the algorithm is unique if all the capacity functions are strictly increasing.

Tree structured networks

In this section we will analyze *convergent-tree structured* static networks (all arcs are directed towards a root node) which have a single exit to the outside. Note that in such a tree each node $x \in V$ has only one outgoing arc in G. It is this property of G which allows us to provide a simple algorithm for solving (TP).

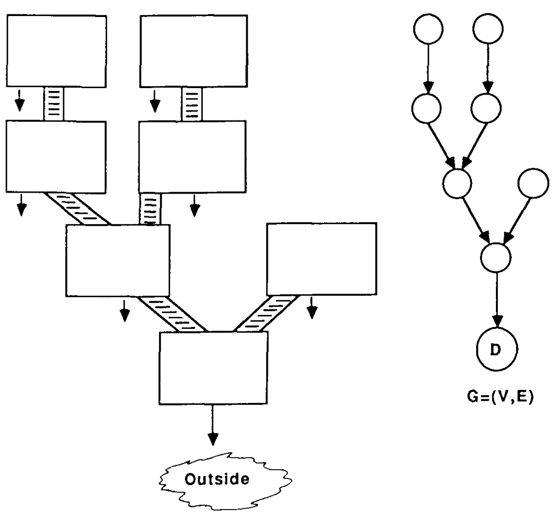

Figure 7. A building which has a convergent-tree structured static network representation

A convergent-tree structured building and its static network representation are given in Figure 7.

In what follows we assume that all capacity functions satisfy assumptions (A1′)–(A4′).

A greedy algorithm for the convergent-tree structured networks

Input: The static network $G = (V, E)$, the successor relationship $s(x)$, $x \in V$, initial contents a_x of each component $x \in V$, and capacity functions $c_{x,s(x)}(\cdot)$ satisfying (A1′)–(A4′).
Output: The EAF $f^*(e)$, $e \in E_T$.

begin
call procedure renumber
for $i = M, M-1, \ldots, 1$ **do**
begin
Given the flow on the movement arcs from static nodes $M, M-1, \ldots, i+1$ to node i for each time period use the method described in Lemma 3.2 to obtain the EAF for node i.
end
end for
end
procedure renumber
begin
Renumber the nodes of G such that if $(i, j) \in E$, then $i > j$.
end

Theorem 3.2. *The algorithm produces an optimal solution for convergent-tree structured static networks.*

Proof. All the arguments in Lemmas 3.2 and 3.3 as well as Theorem 3.1 remain valid by combining all the incoming arcs into a node by a single arc.

Remark 3.1. If the capacity functions are *not* monotone increasing then the above algorithm generally *does not* yield an optimal solution.

The following example shows that when the capacity functions are *not* nondecreasing the al-

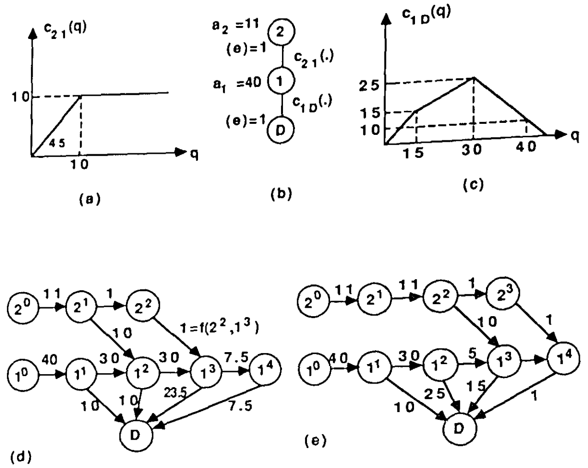

Figure 8. An example where the greedy algorithm fails to obtain the optimal solution: (a) capacity function $c_{2,1}(\cdot)$, (b) static network $G=(V, E)$, (c) capacity function $c_{1,D}(\cdot)$, (d) solution obtained by the greedy algorithm, (e) an improved solution

gorithm provides a nonoptimal solution (Figure 8). In this example we have a path structured static network with one capacity function violating the nondecreasing property.

In closing this section we remark that if the capacity function is a linear approximation such as the one given in Figure 3, the problem becomes a minimum cost flow problem with side constraints. Therefore it can be solved by the efficient solution procedures developed for such problems [4,10,11,16]. It is shown in [10,11] that a specialization of simplex method can be quite efficient as compared to regular simplex method for network flow problems with side constraints. Glover and Klingman [11] report that this specialization in one case solves a 30,000 variable problem 20 times faster than the machine tailored regular simplex code MPSX/370. The advantage of the specialization used in [11] is that the basis matrix is partitioned in such a way that only the inverse of a square submatrix of it (called working basis) has to be maintained at each iteration. The dual variables associated with flow conservation equations are updated by using tree operations. The size of the working basis is equal to the number of side constraints.

4. Minimax (bottleneck) problem

Given a static network $G=(V, E)$ and initial content a_x, of each node $x \in V$, let $f(e)$, $e \in E_T$ be a feasible flow pattern in $G_T=(V_T, E_T)$. Also let $T_x(f)$, $x \in V$ be the last time the building component represented by node x in G is occupied with respect to a given feasible flow f. The minimax problem (MP) is given by

$$\text{Minimize} \quad \left\{ \max_{x \in V} \{ T_x(f) \} \right\}, \tag{4.1}$$

subject to

f is a feasible flow in G_T;

i.e. it satisfies (3.2) and (3.3). (4.2)

Notice that the optimal objective value represents the shortest evacuation time of a building.

Let $f'(e)$ and $f^*(e)$, $e \in E_T$ represent the optimal solutions for the (MP) and (TP) on G_T, respectively. Since $f^*(e)$ is a feasible flow for (TP) we have

$$\max_{x \in V} \{ T_x(f') \} \leqslant \max_{x} \{ T_x(f^*) \}, \tag{4.3}$$

and

$$\sum_{t=1}^{T} \sum_{(x', D) \in E} t f'(x', D) \geqslant \sum_{t=1}^{T} \sum_{(x', D) \in E_T} t f^*(x', D). \tag{4.4}$$

The following lemma shows the equivalence of (MP) and (TP) for static networks with a single exit to the outside.

Lemma 4.1. *Given a convergent-tree structured static network $G=(V, E)$ with a single exit and its corresponding optimal turnstile cost solution, $f^*(e)$, $e \in E_T$, given by the greedy algorithm of Section 3, this solution is also an optimal solution for the minimax problem.*

Proof. Let node $1 \in V$ be the only node with an exit to the outside. Therefore, all flows must reach to node 1 and thus,

$T_x(f) \leqslant T_1(f)$ for all $x \in V$

and any feasible flow f. Thus the minimax objective function reduces to

Min $T_1(f)$.

Since

$$T_1(f) = \min\left\{ t \mid \sum_{j=1}^{t} f(1^j, D) = \sum_{x \in V} a_x \right\},$$

and f^* is an EAF, the proof follows.

Consider now a static network which is either a path or a convergent-tree as given in Figures 5 and 7. Moreover we assume that each node $x \in V$ has an exit to the outside. Again we assume that the nodes of the static network are numbered in such a way that if $(i, j) \in E$ then $i > j$. Finally, the exit capacity functions $c_{x,D}(\cdot)$, $x \in V$, and the movement arc capacity functions $c_{x,s(x)}(\cdot)$ are assumed to satisfy assumptions (A1′)–(A3′). Consider the solution f obtained by evacuating each node $x \in V$ through its own exit only. This corresponds to solving M independent 1-floor turnstile problems.

Lemma 4.2. *If* $T_x(\hat{f}) \leqslant T_{s(x)}(\hat{f})$, $x \in V$, *then* $\hat{f}$ *is an optimal solution for the* (MP).

Proof. Diverting any flow from a node $x \in V$ to $s(x) \in V$ can only worsen $T_{s(x)}$. Since $T_1 = \min_{x \in V}\{\max(T_x(\hat{f}))\}$ is the value of the objective function this solution is optimal for the (MP).

If the assumption of Lemma 4.2 is not satisfied we use a threshold approach to solve the (MP). We first find the optimal solution $\hat{f}$ of the M independent 1-floor problems. The resulting objective value $T = \max_{x \in V}\{T_x(\hat{f})\}$ is an upper bound for the optimal value of (MP). We now solve a turnstile cost problem in G_T by changing the cost on the exit arcs (x^j, D), $j \geqslant T$ to ∞. This problem can be solved with any network with side constraints technique [4,10,11,16]. If the optimal objective function value of the latter problem is ∞, we know that we cannot improve the upper bound. In this case T is the optimal value of (MP). Otherwise, we let f be the solution of the (TP) and iterate the procedure. After at most T applications of the network with side constraints algorithm the procedure terminates with an optimal solution to (MP). Note that we can use sensitivity analysis to reoptimize the flows f in each iteration.

5. Conclusion

We presented some models of building evacuation problems in this paper. All the problems analyzed become networks with linear or nonlinear side constraints. In some specially structured buildings (and their corresponding static networks) such as path or convergent-trees we have provided greedy algorithms for solving maximum flow, turnstile cost, and minimax problems, if certain assumptions about the capacity functions are met. In more general cases we proposed possible solution procedures which use efficient networks with side constraints algorithms, such as the one given in [11].

Still an open question is if there exist efficient network based algorithms (not using simplex) to find optimal solutions to those problems. In the maximum flow problem an approach using a generalization of flow augmentations and in the turnstile problems an approach which computes the dual variables without explicitly computing the basis inverse will be desirable. Until more efficient algorithms emerge, we recommend the use of network flows with side constraints algorithms such as the one provided in [11] for solving these problems efficiently.

References

[1] Bazaraa, M.S., and Shetty, C.M., *Nonlinear Programming*, Wiley, New York, 1977.

[2] Bennet, S.M., "An approach to some structured linear programming problems", *Operations Research* 14/4, 1966.

[3] Chalmet, L.G., Francis, R.L., and Saunders, P.B., "Network models for building evacuation", *Management Science* 28/1 (1982) 86–105.

[4] Chen, S., and Saigal, R., "A primal algorithm for solving a capacitated network flow problem with additional linear constraints", *Networks* 7/1, 1977.

[5] Choi, W., Francis, R.L., Hamacher, H.W., and Tufekci, S., "Network models of building evacuation problems with flow dependent exit capacities", *Operational Research '84*, North-Holland, Amsterdam 1984, pp. 1047–1059.

[6] Choi, W., and Francis, R.L., "Some optimization models of building evacuation problems with flow dependent arc capacities", Research Report #85-8, Department of Industrial and Systems Engineering, University of Florida, August 1985.

[7] Francis, R.L., "A negative exponential solution to an evacuation problem", Research Report #84-36, Department of Industrial and Systems Engineering, University of Florida, October 1984.

[8] Fruin, John J., *Designing for Pedestrians — A Level-of-*

Service Concept, Ph.D. dissertation, The Polytechnic Institute of Brooklyn, June, 1970.

[9] Fruin, John J., *Pedestrian Planning and Design*, Metropolitan Association of Urban Designers and Environmental Planners, New York, 1971 (out of print).

[10] Glover, F., and Klingman, D., "The simplex son algorithm for LP/embedded network problems", Technical Report CCS 317, Center for Cybernetics Studies, The University of Texas, Austin, TX, 1977.

[11] Glover, F., and Klingman, D., "Basis exchange characterizations for the simplex son algorithm for LP/embedded networks", *Mathematical Programming Study* 24 (1985) 141–157.

[12] Glover, F., Karney, D., Klingman, D., and Russel, R., "Solving singly constrained transshipment problems", *Transportation Science* 12/4, 1978.

[13] Graves, G.W., and McBride, R.D., "The factorization approach to large scale linear programming", *Mathematical Programming* 10/1, 1972.

[14] Hartman, J.K., and Lasdon, L.S., "A generalized upper bounding method for doubly coupled linear programs", *Naval Research Logistics Quarterly* 17/4, 1970.

[15] Jarvis, J.J., and Ratliff, H.D., "Some equivalent objectives for dynamic network flow problems", *Management Science* 28/1, 1982.

[16] Kennington, J.L. and Helgason, R.V., *Algorithm for Network Programming*, Wiley, New York, 1980.

[17] Maxwell, W.L., and Wilson, R.C., "Dynamic network flow modelling of fixed path material handling systems", AIIE Transactions, 13/1, 1981.

[18] Minieka, E.L., "Maximal, lexicographic and dynamic network flows", *Operations Research* 12/2, 1973.

[19] Pauls, J., and Jones, B.K., "Case studies of building evacuations", and "Building evacuation findings and recommendations", in D. Canter (ed.), *Behavior in Fires*, Wiley, New York, 1978.

[20] Pauls, J., "Evacuation of high rise office buildings", *Buildings* 72/5, (1978) 84–88.

[21] Pauls, J., "Building evacuation: Research findings and recommendations", in: J. Canter, (ed.), *Fires and Human Behavior*, Wiley, New York, 1980, pp. 251–275.

[22] Pauls, J. "Building design for egress", *Journal of Architectural Education*, Summer, 1980, 38–42.

[23] Pauls, J., "Effective-width model for crowd evacuation flow on stairs", *Sixth International Fire Protection Seminar*, Vol. 1, Karlsruhe, West Germany, September 21–24, 1982, pp. 295–306.

[24] Predtetschenski, W.M., and Milinski, A.I., Personenströme in Gebäuden, Rudolf Möller, Köln–Braunsfeld, 1971.

[25] Roytman, M.Y., *Principles of Fire Safety Standards for Building Construction*, published for the National Bureau of Standards, Washington, DC, by Amerind Publishing Col. Pvt. Ltd., New Delhi, 1975.

[26] Stahl, F. and Archea, J., "An assessment of the technical literature on emergency egress from buildings", NBSIR 77-1313, National Bureau of Standards, Washington, DC, 1977.

[27] Templer, J., *Stair Shape and Human Movement*, Ph.D. dissertation, Columbia University, New York, 1974. (A text, *Designing for Pedestrians*, to be published by the Whitney Library of Design, is in preparation.)

[28] Truemper, K., "Unimodular matrices of flow problems with additional constraints", *Networks* 7/4, 1977.