Change of Variables Exercise

HY-673, 2023-2024, Spring

Exercise 1 (2022-2023 Assignment #1)

(a) Let $X \sim \mathcal{N}(\mu, \sigma^2)$ and $Y = e^X =: g(X)$. Compute analytically the probability density function (PDF) of Y using the change of variables formula.

Solution:

According to the change of variables rule, if we apply a transformation $g(x) : \mathbb{R} \to \mathbb{R}$ to a random variable X of Probability Density Function (PDF), say $p_X(x)$, then the PDF of the transformed random variable Y that we are looking for, say $p_Y(y)$, can be computed as follows:

$$p_Y(y) = p_X\left(g^{-1}(y)\right) \left| \frac{d}{dy}\left(g^{-1}(y)\right) \right|, \tag{1}$$

as long as $g(x)$ is a monotonic function. In our case, it is given that our transformation is:

$$g(x) = e^x,\ x \in \mathbb{R}, \tag{2}$$

therefore, since:

$$\frac{d}{dx} g(x) = \frac{d}{dx} e^x = e^x > 0,\ \forall x \in \mathbb{R}, \tag{3}$$

we can indeed use the change of variables rule because $g(x) : \mathbb{R} \to \mathbb{R}^+$ is a monotonically increasing function. Firstly, the inverse of $g(x)$ is:

$$g^{-1}(x) = \ln(x),\ x \in \mathbb{R}^+, \tag{4}$$

as it holds that $g(g^{-1}(x)) = e^{\ln x} = x$. Now, we can calculate:

$$\left| \frac{d}{dy} g^{-1}(y) \right| = \left| \frac{d}{dy} \ln(y) \right| = \left| \frac{1}{y} \right| = \frac{1}{y},\ y \in \mathbb{R}^+, \tag{5}$$

and since it is given that $X \sim \mathcal{N}\left(\mu, \sigma^2\right)$, it will be:

$$p_X(g^{-1}(y)) = p_X(\ln y) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{\ln(y)-\mu}{\sigma}\right)^2},\ y \in \mathbb{R}^+. \tag{6}$$

So, the answer using eq. 1 is:

$$p_Y(y) = \frac{e^{-\frac{1}{2}\left(\frac{\ln(y)-\mu}{\sigma}\right)^2}}{y\sigma\sqrt{2\pi}} H(y),\ y \in \mathbb{R}, \tag{7}$$

where $H(y)$ is the Heaviside step function:

$$H(y) = \begin{cases} 1, & y > 0 \\ 0, & \text{otherwise.} \end{cases} \tag{8}$$

(b) Compute the histogram of the dataset $\{y_i = g(x_i) : x_i \sim \mathcal{N}(1,4)\}_{i=1}^{n}$, with $n = 10^2, 10^3, 10^4$. Compare the estimated histogram with $P_Y(y)$ from (a). Write down what you observe as n increases.

Solution:

The code is in `03c_change_of_variables.ipynb`, and the requested plots are presented below:

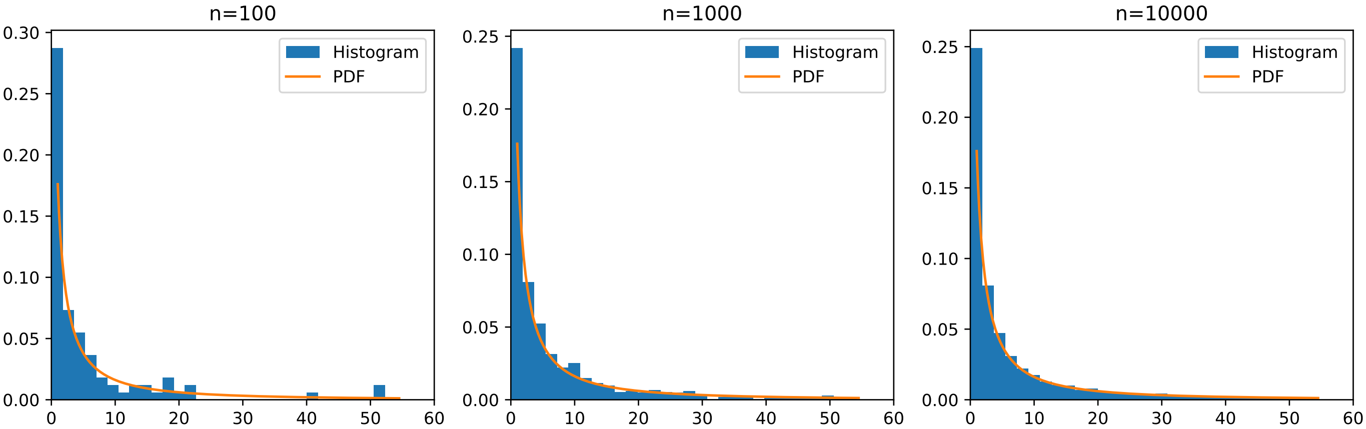

Figure 1: As n increases the histograms better approximate the actual PDF of Y.

(c) Let $U_1, U_2 \sim \mathcal{U}(0,1)$, two independent r.v.s, and define $Y = -\lambda_1^{-1}\log(U_1) - \lambda_2^{-1}\log(U_2)$, with $\lambda_1, \lambda_2 > 0$. Compute analytically the PDF of Y.

Solution #1:

We can start by observing that the transformation $Z = -\lambda^{-1}\log(U)$, $\lambda > 0$, $U \sim \mathcal{U}(0,1)$ follows an exponential distribution $Z \sim \text{Exp}(\lambda)$. This is a known probability fact, but, we can also prove it using the change of variable formula with $Z = g(U) = -\lambda \log(U)$. We first calculate the inverse $g^{-1}(z) = -\exp(-\lambda z)$. Also, notice that since the range of U is $(0,1)$, Z can never take negative values. So, using the change of variable formula:

$$p_Z(z) = \begin{cases} \lambda \exp(-\lambda z), & z \geq 0 \\ 0, & \text{otherwise} \end{cases}, \tag{9}$$

which is the PDF of an exponential distribution with parameter $\lambda > 0$. The r.v. $Y = -\lambda_1^{-1}\log(U_1) - \lambda_2^{-1}\log(U_2)$ is essentially the sum of two independent r.v.s $Z_1 \sim \text{Exp}(\lambda_1)$, and $Z_2 \sim \text{Exp}(\lambda_2)$. This can be computed as the convolution of the PDFs of Z_1 and Z_2:

$$p_Y(y) = (p_{Z_1} * p_{Z_2})(y) = \int_{-\infty}^{\infty} p_{Z_1}(z) p_{Z_2}(y-z)dz \tag{10}$$
$$= \int_0^y \lambda_1 \exp(-\lambda_1 z) \cdot \lambda_2 \exp(-\lambda_2(y-z))dz \tag{11}$$
$$= \lambda_1\lambda_2 \exp(-\lambda_2 y) \int_0^y \exp(-\lambda_1 z)\exp(\lambda_2 z)dz \tag{12}$$
$$= \lambda_1\lambda_2 \exp(-\lambda_2 y) \int_0^y \exp((\lambda_2 - \lambda_1)z)dz \tag{13}$$
$$= \frac{\lambda_1\lambda_2}{\lambda_1 - \lambda_2} \exp(-\lambda_2 y)(1 - \exp((\lambda_2 - \lambda_1)y)) \tag{14}$$
$$= \frac{\lambda_1\lambda_2}{\lambda_1 - \lambda_2} \left(\exp(-\lambda_2 y) - \exp(-\lambda_1 y)\right),\ y \geq 0. \tag{15}$$

Otherwise, for $y < 0$, the PDF of Y is 0. We can deduce this when calculating the extremes of integration, or, simply, noticing that the sum of two positive r.v.s Z_1, Z_2 must also be positive.

Solution #2:

We can start by computing the PDF of a new random variable Z, where $Z = -\frac{1}{\lambda}\ln(U)$, $\lambda > 0$, and $U \sim \mathcal{U}(0,1)$. The transformation in this case will be:

$$g(x) = -\frac{1}{\lambda}\ln(x),\ x \in \mathbb{R}^+. \tag{16}$$

Similarly, we can indeed use the change of variables formula, as it holds that:

$$\frac{d}{dx}g(x) = \frac{d}{dx}\left(-\frac{1}{\lambda}\ln(x)\right) = -\frac{1}{\lambda x} < 0,\ x \in \mathbb{R}^+, \tag{17}$$

proving that $g(x) : \mathbb{R}^+ \to \mathbb{R}$ is a monotonically decreasing function. Likewise, for the inverse of $g(x)$:

$$g^{-1}(x) = e^{-\lambda x},\ x \in \mathbb{R}, \tag{18}$$

because $g(g^{-1}(x)) = -\frac{1}{\lambda}\ln\left(e^{-\lambda x}\right) = -\lambda x/(-\lambda) = x$. Following on:

$$\left|\frac{d}{dz}g^{-1}(z)\right| = \left|\frac{d}{dz}e^{-\lambda z}\right| = \left|-\lambda e^{-\lambda z}\right| = \lambda e^{-\lambda z},\ x \in \mathbb{R}, \tag{19}$$

and since $U \sim \mathcal{U}(0,1)$:

$$p_U(g^{-1}(z)) = p_U\left(e^{-\lambda z}\right) = \begin{cases} 1, & 0 < e^{-\lambda z} < 1 \\ 0, & \text{otherwise} \end{cases} = \begin{cases} 1, & z > 0 \\ 0, & \text{otherwise} \end{cases} = H(z). \tag{20}$$

Thus, according to eq. 1, the PDF of Z will be the exponential distribution:

$$p_Z(z;\lambda) = \lambda e^{-\lambda z} H(z),\ z \in \mathbb{R}. \tag{21}$$

From theory, we know that the PDF of the sum of two random variables, say X and Y, that are independent, i.e., $X \perp\!\!\!\perp Y$, is given by the convolution of the their individual PDFs:

$$p_{X+Y}(z) = \int_{-\infty}^{\infty} p_X(x) p_Y(z-x)dx = p_X(x) * p_Y(x). \tag{22}$$

For that reason, we can compute the PDF of Y, where $Y = -\frac{1}{\lambda_1}\ln(U_1) - \frac{1}{\lambda_2}\ln(U_2)$ and $U_1, U_2 \sim \mathcal{U}(0,1)$, $U_1 \perp\!\!\!\perp U_2$, using the PDF of Z (eq. 21) that we just calculated like so:

$$p_Y(y) = p_{U_1+U_2}(y) = p_Z(y; \lambda = \lambda_1) * p_Z(y; \lambda = \lambda_2) = \left(\lambda_1 e^{-\lambda_1 y} H(y)\right) * \left(\lambda_2 e^{-\lambda_2 y} H(y)\right). \tag{23}$$

Instead of computing the convolution via its definition (eq. 22), we can go through the Laplace domain:

$$\begin{aligned}\mathcal{L}\{p_Y(y)\} &= \lambda_1 \mathcal{L}\left\{e^{-\lambda_1 y} H(y)\right\} \lambda_2 \mathcal{L}\left\{e^{-\lambda_2 y} H(y)\right\} = \frac{\lambda_1\lambda_2}{(s+\lambda_1)(s+\lambda_2)} &(24)\\ &= \frac{\lambda_1\lambda_2}{\lambda_2-\lambda_1}\left(\frac{1}{s+\lambda_1} - \frac{1}{s+\lambda_2}\right),\ \mathfrak{Re}\{s\} > -\min(\lambda_1, \lambda_2). &(25)\end{aligned}$$

Lastly, inverting the Laplace transform of the PDF of Y concludes to the answer:

$$\begin{aligned}p_Y(y) = \mathcal{L}^{-1}\{\mathcal{L}\{p_Y(y)\}\} &= \frac{\lambda_1\lambda_2}{\lambda_2-\lambda_1}\left(\mathcal{L}^{-1}\left\{\frac{1}{s+\lambda_1}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s+\lambda_2}\right\}\right) &(26)\\ &= \frac{\lambda_1\lambda_2}{\lambda_2-\lambda_1}\left(e^{-\lambda_1 y} - e^{-\lambda_2 y}\right) H(y),\ y \in \mathbb{R}. &(27)\end{aligned}$$

(d) Repeat (b) for (c), but, for $\lambda_1 = 1, \lambda_2 = 2$.

Solution:

The code is in `03c_change_of_variables.ipynb` and the requested plots are presented below:

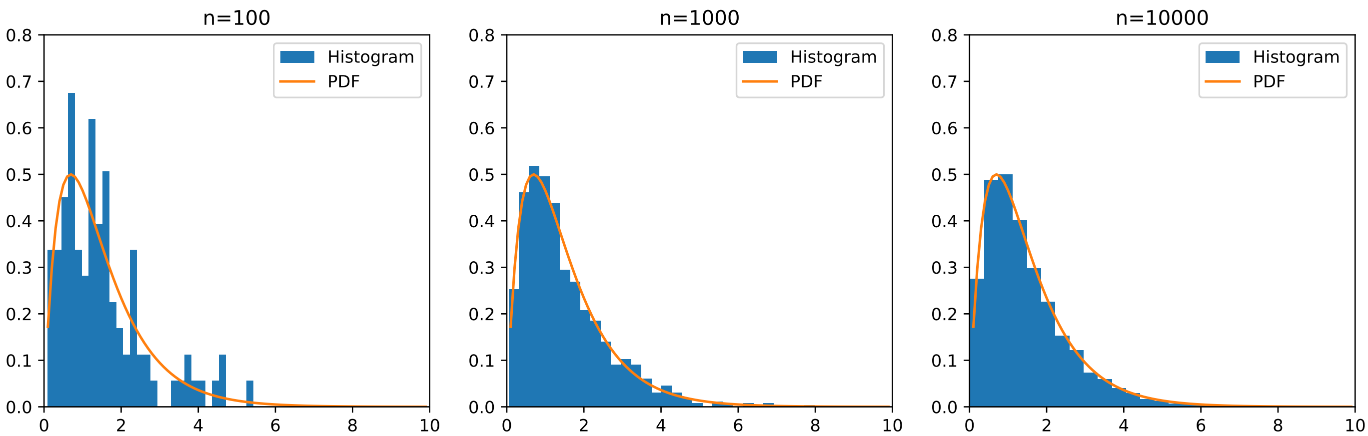

Figure 2: In like manner, the PDF of Y is better approximated by the histograms as n increases.