Analysis of Lotka-Volterra Predator-Prey Model

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1 Steady States

We start with a basic ODE model, where x is the level of the prey, and y is the level of the predators.

$$\frac{dx}{dt} = \dot{x} = ax - bxy \tag{1}$$

$$\frac{dy}{dt} = \dot{y} = cxy - hy \tag{2}$$

To find the steady states, we simply set the ODEs equal to 0.

$$0 = ax - bxy \tag{3}$$

$$0 = cxy - hy \tag{4}$$

We'll refer to the steady state values of x and y as $\hat{x}$ and $\hat{y}$. First, we note that there is a trivial steady state at $(\hat{x} = 0, \hat{y} = 0)$. This makes sense – if there are no prey and no predators, then there will be no change happening in the system. We can also do a bit of algebra to solve for a non-trivial steady state.

$$\hat{x} = \frac{h}{c} \tag{5}$$

$$\hat{y} = \frac{a}{b} \tag{6}$$

2 Linearization

WARNING: LINEAR ALGEBRA AHEAD!!!!

$$A = \begin{bmatrix} x \\ y \end{bmatrix} \tag{7}$$

$$\hat{A} = \begin{bmatrix} \hat{x} \\ \hat{y} \end{bmatrix} \tag{8}$$

$$\dot{A} = \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} \tag{9}$$

$$\dot{A}_l = \begin{bmatrix} \dot{x}_l \\ \dot{y}_l \end{bmatrix} \tag{10}$$

Reminder of the basic point-slope formula:

$$y - y_1 = m(x - x_1) \rightarrow y = m(x - x_1) + y_1 \tag{11}$$

Now, this is the formula we're going to plug things into to find out linearized equations:

$$\dot{A}_l = \nabla f(A)|_{A=\hat{A}}(A - \hat{A}) + f(\hat{A}) \tag{12}$$

Ok, so let's start trying to fill in some of these terms. First, the easy part – we know $f(\hat{A})$ must be 0, since this is just the values of the ODEs at the steady states. But as a sanity-check, we can prove it to ourselves anyway.

$$f(\hat{A}) = \begin{bmatrix} a\hat{x} - b\hat{x}\hat{y} \\ c\hat{x}\hat{y} - h\hat{y} \end{bmatrix} = \begin{bmatrix} a\frac{h}{c} - b\frac{h}{c}\frac{a}{b} \\ c\frac{h}{c}\frac{a}{b} - h\frac{a}{b} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{13}$$

Next, let's calculate the gradient term $\nabla f(A)$:

$$\nabla f(A) = \begin{bmatrix} \frac{d\dot{x}}{dx} & \frac{d\dot{x}}{dy} \\ \frac{d\dot{y}}{dx} & \frac{d\dot{y}}{dy} \end{bmatrix} = \begin{bmatrix} a - by & -bx \\ cy & cx - h \end{bmatrix} \tag{14}$$

Remeber, we want to gradient evaluated at the steady state $\nabla f(A)|_{A=\hat{A}}$. To get this, we just plug the steady state values $\hat{x}$ and $\hat{y}$ into our gradient equation.

$$\nabla f(A)|_{A=\hat{A}} = \begin{bmatrix} a - b\hat{y} & -b\hat{x} \\ c\hat{y} & c\hat{x} - h \end{bmatrix} = \begin{bmatrix} a - b\frac{a}{b} & -b\frac{h}{c} \\ c\frac{a}{b} & c\frac{h}{c} - h \end{bmatrix} = \begin{bmatrix} 0 & -b\frac{h}{c} \\ c\frac{a}{b} & 0 \end{bmatrix} \tag{15}$$

Ok, so we've solved for $f(\hat{A})$ and $\nabla f(A)|_{A=\hat{A}}$. Next is this term $(A - \hat{A})$. All we need to do here is again, plug in the steady state values.

$$(A - \hat{A}) = \begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} \hat{x} \\ \hat{y} \end{bmatrix} = \begin{bmatrix} x - \hat{x} \\ y - \hat{y} \end{bmatrix} = \begin{bmatrix} x - \frac{h}{c} \\ y - \frac{a}{b} \end{bmatrix} \tag{16}$$

Now, we just need to plug everything back in. Remeber that $f(\hat{A}) = 0$, so really we're just plugging in $\nabla f(A)|_{A=\hat{A}}$ and $(A - \hat{A})$.

$$\dot{A}_l = \nabla f(A)|_{A=\hat{A}}(A - \hat{A}) \tag{17}$$

$$\dot{A}_l = \begin{bmatrix} 0 & -b\frac{h}{c} \\ c\frac{a}{b} & 0 \end{bmatrix} \begin{bmatrix} x - \frac{h}{c} \\ y - \frac{a}{b} \end{bmatrix} \tag{18}$$

This is now just a matrix multiplication problem.

$$\dot{A}_l = \begin{bmatrix} -\frac{bh}{c}y + \frac{ah}{c} \\ \frac{ac}{b}x - \frac{ah}{b} \end{bmatrix} \tag{19}$$

Finally, we just unpack this matrix to get our linearized ODEs.

$$\dot{x}_l = -\frac{bh}{c}y + \frac{ah}{c} \tag{20}$$

$$\dot{y}_l = \frac{ac}{b}x - \frac{ah}{b} \tag{21}$$

At this point we've solved the problem and can stop here if we want. However, in order to make further analysis easier, we may want to rewrite the linearized ODE system in matrix form so that it looks like:

$$\begin{bmatrix} \dot{x}_l \\ \dot{y}_l \end{bmatrix} = \begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} ? \\ ? \end{bmatrix} \tag{22}$$

If we've done things correctly, the multiplication matrix should be the same as what we found above, and the addition component we can get from the solution ODEs.

$$\begin{bmatrix} \dot{x}_l \\ \dot{y}_l \end{bmatrix} = \begin{bmatrix} 0 & -b\frac{h}{c} \\ c\frac{a}{b} & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \frac{ah}{c} \\ -\frac{ah}{b} \end{bmatrix} \tag{23}$$