# WAFO Data Structures: Wave Spectrum and Covariance Function

Eva Sjö

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## 1 Introduction

The following types of wave spectra are available in WAFO:

- Directional spectrum (at fixed position). Defining $Cov\{\eta(0,0,0),\eta(t,x,y)\}$.
- Frequency spectrum (at fixed position). Defining $Cov\{\eta(0)\eta(t)\}$.
- Wave number spectrum in 2-D. Defining $Cov\{\eta(0,0,0),\eta(t,x,y)\}$.
- Wave number spectrum in 1-D. Defining $Cov\{\eta(0),\eta(x)\}$.
- Encountered directional spectrum (for given speed and direction). Defining $Cov\{\eta(0,0,0),\eta(t,x+vt\cos\phi,y+vt\sin\phi)\}$.
- Encountered frequency spectrum (for given speed and direction). Defining $Cov\{\eta(0,0,0),\eta(t,vt\cos\phi,vt\sin\phi)\}$.
- Rotated directional spectrum. Defining $\mathrm{Cov}\{\eta_\phi(0,0,0),\eta_\phi(t,x,y)\}$, $\eta_\phi(\cdot)$=wave field rotated angle $\phi$.
- Rotated wave number spectrum in 2-D. Defining $Cov\{\eta(0,0,0),\eta_\phi(t,x,y)\}$.
- Rotated wave number spectrum in 1-D. Defining $Cov\{\eta(0,0,0),\eta(0,x\cos\phi,x\sin\phi)\}$.

The idea of all these spectra is quite clear, but there are several possible definitions for each of them when it comes to units, interval of definition of parameters etc.

The suggestion here is to allow for different frequency units, but try to keep to only one definition for each type of spectrum (apart from the units).

Further, the spectra should not be assumed symmetric in $\omega$. This we can call the physical definition. The mathematical definition with (conjugate) symmetry will be used when needed internal in the routines.

With $g$ =constant of gravity$\approx 9.81$ and $\omega = 2\pi f$ we have

**Directional spectrum** $S(\omega, \theta)$, $0 < \omega < \infty$, $-\pi < \theta < \pi$. The main direction of waves is $\theta = 0$. Alternatively we have $S_f(f, \theta)$, $0 < f < \infty$, $-\pi < \theta < \pi$. Remark: The definition used in (Lindgren et al., 1999) is $-\infty < \omega < \infty$, $-\pi/2 < \theta < \pi/2$ with the motivation that it gives more transparent formulas for the calculation of encountered spectra.

**Frequency spectrum** $S(\omega)$, $0 \leq \omega < \infty$, or $S_f(f)$, $0 < f[Hz] < \infty$. Can be integrated out from the directional spectrum: $S(\omega) = \int_{-\pi}^{\pi} S(\omega, \theta) d\theta$.

**Wave number spectrum in 2-D** $S_k(\kappa_x, \kappa_y) = S_k(\boldsymbol{\kappa})$, $-\infty < \kappa_x, \kappa_y < \infty$. Related to the directional spectrum via

$$\begin{cases} \omega(\boldsymbol{\kappa}) = \sqrt{g||\boldsymbol{\kappa}|| \tanh\left(||\boldsymbol{\kappa}||h\right)} \\ \theta(\boldsymbol{\kappa}) = \arctan_2(\kappa_y/\kappa_x). \end{cases} \quad \text{or} \quad \begin{cases} \kappa_x(\omega, \theta) = k(\omega; h)\cos\theta \\ \kappa_y(\omega, \theta) = k(\omega; h)\sin\theta \end{cases}$$

where $\kappa(\omega; h)$ is the solution to the dispersion relation $\omega^2 = g\kappa\tanh(\kappa h)$, the dispersion relation for water depth $h$. The notation $\arctan_2$ means the four quadrant arctangent, and $||\boldsymbol{\kappa}|| = \sqrt{\kappa_x^2 + \kappa_y^2}$. For deep water $(h = \infty)$ we have $S_k(\kappa_x, \kappa_y) = \frac{g^2}{2\omega(\boldsymbol{\kappa})^3} S(\omega(\boldsymbol{\kappa}), \theta(\boldsymbol{\kappa}))$. Maple helped me to evaluate the Jacobian determinant for the change of variables from $\boldsymbol{\kappa}$ to $(\omega, \theta)$ for finite depth, but I do not think you would like to see that expression here... Similar relation can be derived to $S_f(f, \theta)$ (involving more $2\pi$'s ... )

**Wave number spectrum in 1-D** $S_k(\kappa)$, $0 \leq \kappa < \infty$. For deep water related to $S(\omega)$ through $S(w) = \frac{2\omega}{g} S_k(\omega^2/g)$. Or reversely, $S_k(\kappa) = \frac{1}{2}\sqrt{\frac{g}{\kappa}} S(\sqrt{g\kappa})$. For finite depth the dispersion relation is given by $\omega^2 = g\kappa\tanh(\kappa h)$, and from this a relation can be found.

**Encountered frequency spectrum** $S_e(\omega; v, \phi)$, for ships traveling with speed $v$ in the direction $\phi$. See (Lindgren et al., 1999) and (Podgórski et al., 2000) how it is derived from a directional spectrum.

**Encountered directional spectrum** $S(\omega, \theta; v, \phi)$. Same as previos, but without integrating out the angle.

**Rotated directional spectrum** $S_\phi(\omega, \theta) = S(\omega, \theta + \phi)$, where $S(\omega, \theta)$ is extended periodically for $\theta$ beyond $(-\pi, \pi)$.

**Rotated wave number spectrum** $S_k(\kappa;\phi)$, spectrum along a line in an angle $\phi$ with the main direction of waves. In 2-D: $S_k(\kappa_x,\kappa_y;\phi)$, $x$-axis in an angle $\phi$ with the main direction of waves. Can be derived from $S_\phi(\omega,\theta)$.

# 2 In MATLAB

All this is possible to describe in MATLAB with a so called 'Structured Array'. The fields needed to describe the spectra defined above are

**.type** Defining type of spectrum: 'dir', 'freq', 'k2D', 'k1D', 'encdir', 'enc', 'rotdir', 'rotk2D', 'rotk1D'. Default: 'freq'.

**.S** Values of spectrum in a matrix size $n_\theta \times n_f$.

**.w** Frequency lag when angular frequency, vector of length $n_f$. Equidistant values $\geq 0$. Should not be given when .f or .k (see below) is. Default: see .S.

**.f** Frequency lag when natural frequency ([Hz]), vector of length $n_f$. Equidistant values $\geq 0$. Should not be given when .w or .k (see below) is. Default: see .S.

**.k** Wave number lag (in first dimension), vector of length $n_f$. Equidistant values, $\geq 0$ when type 'k1D', negative values allowed when type 'k2D' etc. Only when some type of wave number spectrum. Default: see .S.

**.k2** Only for .type='k2D' or 'rotk2D'. Second dimension wave number lag, length $n_\theta$, equidistant. Default: equal to .k.

**.theta** Only for all types of dirctional spectra. Angular lags $-\pi < \theta[\text{rad}] < \pi$, equidistant vector of length $n_\theta$. Default: see .S

**.v** Only for .type='encdir' and 'enc'. Speed of ship, scalar, unit [m/s]. Default: 0

**.phi** Only for .type='encdir', 'enc' and 'rot ... '. Direction of ship or direction of '$x$-axis', scalar, unit [rad]. Defualt: 0

**.h** Water depth, positive scalar, unit [m]. Default: $\infty$

**.tr** Transformation funtion. Default: [] (none)

**.norm** Normalization flag. Logical 1 if variance normalized spectrum, 0 else. Default: 0

**.note** String for memorandum about the spectrum.

**.date** Date and time of creation or change of the spectrum.

Remark: The field .type in in most cases given by the other fields in the sense that the combination of fields, and the values in these, uniquely describes the .type. For example, if .v is set, then we have a encountered spectrum of some kind, if .theta also is set, then we have an encountered directional spectrum. A reason to keep the field .type anyway is that it provides an easy check of type of spectrum. Remark 2: The fields .w, .f and .k are disjoint.

The following routines are closly related to the spectrum structure: **check-spec(spec)** to check if spec.type is compatible with the rest of the fields in spec, or if any field has to be changed.
**spec2spec(spec,newtype)** that can translate from one type to another when possible, see Table 1.
**createspec(type, freqtype)** that sets up an empty structure. **w2k** and **k2w** translates between the frequency world and the wave number world in both one and two dimensions.

| .type | dir | freq | k2D | k1D | encdir | enc | rotdir | rotk2D | rotk1D |
|---|---|---|---|---|---|---|---|---|---|
| dir | $\bullet$ | $\Rightarrow$ | $\Rightarrow$ | $\rightarrow$ | $\Rightarrow$ | $\Rightarrow$ | $\Rightarrow$ | $\Rightarrow$ | $\rightarrow$ |
| freq | | $\bullet$ | | $\Rightarrow$ | | | | | |
| k2D | $\Rightarrow$ | $\rightarrow$ | $\bullet$ | $\Rightarrow$ | $\rightarrow$ | $\rightarrow$ | $\rightarrow$ | $\Rightarrow$ | $\rightarrow$ |
| k1D | | $\Rightarrow$ | | $\bullet$ | | ? | | | |
| encdir | ? | ? | ? | ? | $\bullet$ | $\Rightarrow$ | ? | ? | ? |
| enc | | ? | | | | $\bullet$ | | | ? |
| rotdir | $\Rightarrow$ | $\Rightarrow$ | $\rightarrow$ | $\rightarrow$ | $\rightarrow$ | $\rightarrow$ | $\bullet$ | $\Rightarrow$ | $\Rightarrow$ |
| rotk2D | $\rightarrow$ | $\rightarrow$ | $\Rightarrow$ | $\rightarrow$ | $\rightarrow$ | $\rightarrow$ | $\Rightarrow$ | $\bullet$ | $\Rightarrow$ |
| rotk1D | | $\Rightarrow$ | | | | ? | | | $\bullet$ |

Table 1: Scheme of possible type changes. An empty box means translation impossible, $\Rightarrow$ means direct translation, $\rightarrow$ means translaition via another type is most easy, and ? means that I had no time to find out if it is possible.

NB! All field names and file names above are only suggestions, any idea of improvement is welcome, as well as comments/corrections on the Table.

# 3 Covariance function data structure

To keep everything gathered, the covariance function is also stored in a structured array together with its derivatives and other thing that can be useful.

One idea with this structure is that although many fields are included in the list below, giving possibilities to cover a wide range of different situations, it is preferable to only include the fields actually needed. For example, we allow for any combination of the varables $x, y$ and $t$, and for up to four derivatives, but if we have the covariance function $R(t)$, then no $x$- or $y$-variables or -derivatives should be in the struct for this cvf. When we have more than one variable, the mutual order in the struct between them should always be $x, y, t$.

This may seem complicated, but to make it simpel there is a script: **createcov(nrofder,variables,type)** that sets up an empty struct with the fields given by the options: nrofder=number of derivatives, variables= a string of up to 3 characters of any combination of x,y and t, type='none','enc' or 'rot'. All the possible types for spectra are not interesting for covariance functions, but if we have *any* of the rotated types or *any* of the encountered types are relevant also for cvfs, all other types passes as 'none'.

Here follows a list of all possible fields (if I am right, worst case is 44 fields...) in recommended order:

**.R** Matrix/vector with covariance function. Size $n_x \times n_y \times n_t$, but no singleton dimesions. If no y, for example, then the size is $n_x \times n_t$.

**.x** Lag of first space dimension, vector of length $n_x$, unit [m].

**.y** Lag of second space dimension, vector of length $n_t$, unit [m].

**.t** Time lag, vector of length $n_t$, unit [s].

**.h** Water depth, scalar, unit [m].

**.tr** Transformation, see spectrum structure.

**.type** 'enc','rot' or 'none', see above.

**.v** Only if type 'enc'. Speed of ship, scalar, unit [m/s].

**.phi** Only if type 'enc' or 'rot'. Direction of ship or direction of '$x$-axis', scalar, unit [rad].

**.norm** Normalization flag. Logical 1 if autocorrelation function(= normalized cvf), 0 else.

**.Rx** ... **.Rtttt** Matrices with obvious derivatives of R.

**.note** String for memorandum about the cvf.

**.date** Date and time of creation or change of the cvf.

## References


Lindgren, G., Rychlik, I., and Prevosto, M. (1999). Stochastic doppler shift and encountered wave period distributions in Gaussian waves. *Ocean Engineering*, 26:507–518.

Podgórski, K., Rychlik, I., and Machado, U. (2000). Exact distributions for apparent waves in irregular seas. *Ocean Engineering*, 27(9).