THE BORSUK-ULAM THEORM IN REAL-COHESIVE HOMOTOPY TYPE THEORY

DANIEL CICALA, CHANDRIKA SADANAND, MICHAEL SHULMAN, AND AMELIA TEBBE

Abstract. Borsuk-Ulam!

Writing Notes

Writing assignments:
- Amelia—section 5
- Chandrika—section 4
- Daniel—sections 2 and 3

Formalizing the cohomology proofs will be determined later.

1. Introduction

2. Overview of real-cohesive homotopy type theory

OUTLINE:
- HoTT as foundations
- Intepreting AlgTop theorems in HoTT is obsructed by discontinuous functions
- Relating continuous and discontinuous with flat and sharp, which are borrowed from cohesive topoi
- Formalizing flat and sharp in HoTT + axioms needed, e.g. Rflat
- Connecting sets used in AlgTop with HITs used in HoTT via shape

Homotopy type theory (HoTT) is an expression of a style of mathematics that expands the notion of "identity" to include logical identity, homotopy equivalence, and path connectedness. Experts call this style *Univalence foundations*. And as foundations, there is an ambitious program to encode all of mathematics in homotopy type theory. There is a growning community working to realize these ambitions and this paper belongs to this group.

Our present goal is to bring the classical theory of algebraic topology into the fray, and in particular the Borsuk-Ulam theorem. However, the HoTT approach to algebraic topology comes with on immediate challenge: the presence of so many fixed point theorems where, in the course of a proof, the fixed point must be specified precisely, not only up to homotopy. What is the problem with this? It is that homotopy type theory only works up to homotopy. Compare, for instance, the topological circle

$$\mathbb{S}^1 := \left\{(x,y) \in \mathbb{R}^2 | x^2 + y^2 = 1\right\}$$

with the homotopy type theoretical circle defined by a pair of constructors `base` and `loop`: `base` = `base`. One has infinitely many points that can be described exactly and the other has a single point. Brouwer's Fixed Point Theorem illustrates this problem nicely. We provide its statement and proof here for reference.

Theorem 2.1. *Let $\mathbb{D}^2$ denote the topological disk $\{(x, y) \in \mathbb{R}^2 | x^2 + y^2 \leq 1\}$. Any continuous map $f : \mathbb{D}^2 \to \mathbb{D}^2$ has a fixed point.*

Proof. Suppose that f is continuous but does not have a fixed point, hence $f(x) \neq x$ for all $x \in \mathbb{D}^2$. For each $x \in \mathbb{D}^2$, dray a ray from $f(x)$ to x. This ray intersects the circle in a point we denote by $s(x)$. This defines a continuous function $s : \mathbb{D}^2 \to \mathbb{S}^1$ with the property that $s(x) = x$ for all x on the boundary of $\mathbb{D}^2$. That implies that the identity on $\mathbb{S}^1$ factors as the inclusion $\mathbb{S}^1 \hookrightarrow \mathbb{D}^2$ followed by s. Appying the fundamental group function π_1 to this factorization gives that the identity on $\pi_1(\mathbb{S}^1) = \mathbb{Z}$ factors through $\pi_1(\mathbb{D}^2) = 1$ which is absurd. □

Note how this proof relied on our precise specification of the point $s(x)$ on the circle. This point cannot be specified precisely in HoTT. Even if we did work with the only homotopical point on the circle, that is with $\mathtt{S}^1$, there is no way to relate $\mathbb{S}^1$ to $\mathtt{S}^1$ inside of type theory. Semantically speaking, this involves comparing a topological space with ∞-groupoids. This is done using the fundamental ∞-groupiod construction. No such construction exists in HoTT. This is the problem that real-cohesive homotopy theory solves. It does so by proposing to combine two already existing, but previously unrelated, type semantics: topological and ∞-groupoidal. With this proposal, there are three puzzles to be solved.

(a) We need to define a model for a *topological ∞-groupoid*.
(b) What rules or axioms can we equip HoTT with so that we can compare, for example, $\mathbb{S}^1$ to $\mathtt{S}^1$.
(c) Topology is incompatible with the law of the excluded middle, which is required to prove these classic fixed point theorems. How can we resolve this?

Shulman's original paper on real-cohesive HoTT [**?**] discusses the solution to these puzzles in detail. Presently, we are content to simply say that the Lawvere's theory of cohesion provides a solution. Of course, we need to adapt cohesion to homotopy type theory and we leave the description of this to Shulman, but we do provide a high-level description of the role that cohesion plays.

A *cohesive topos* is a pair of topoi equipped with a string of adjunctions between them. Lawever initially called these topos

3. Translating Borsuk-Ulam to homotopy type theory

OUTLINE:

- **Subsection 1.** Give statements for BU-classic, BU-odd, BU-retract) a la wikipedia. The proof strategy: show BU-retract implies BU-odd which is equivalent to BU-classic, then prove BU-retract. Give the proof for BU-retract.
- **Subsection 2.** Translate the classical statement into propositions as types. We want to model classical proof. The failure of contrpositive rule in constructive logic—(not q implies not p) is (p implies not not q)—means our proof strategy is BU-retract implies not not BU-odd which is equivalent to not not BU-classic. But not not BU-classic is sharp BU-classic. Prove BU-retract.
- To close out the section, list the ingredients we need to prove BU-retract.

4. Topological and homotopical real projective spaces

OUTLINE:

- Define n-disks as both sets and types, the latter which is simply 1, since they're contractible. Show that $\int \mathbb{D} = \mathtt{D}$
- Define n-spheres as sets. Use pushouts to glue disks together. Explain why we need to glue with a collar—i.e. the "topology" (as encoded by continuous paths $\mathbb{R} \to \mathtt{X}$ of a type $\mathtt{X}$. Show, via Shulman, that $\int \mathbb{S}^n = \mathtt{S}^n$
- Define $\mathbb{RP}^n$ as sets using pushouts and collaring. Recall Bulcholtz and Egbert's definition of HIT $\mathtt{RP}^n$. Prove that $\int \mathbb{RP}^n = \mathtt{RP}^n$

4.1. Defining $\mathbb{RP}^n$.

We define $\mathbb{RP}^n$ using push outs, tautological bundles, spheres, and a inductive process, following the work of Rijike [].

$\mathbb{RP}^1$ is defined as the following push out, where $a(-1,x) = \frac{x}{4}$ and $a(1,x) = \frac{x}{4} + \frac{3}{4}$, and $b(\pm 1, x) := (0, a(\pm 1, x))$.

$$\begin{tikzcd}
\mathbb{S}^0 \times \mathbb{I} \arrow[r, hook, "a"] \arrow[d, hook, "b"] & \mathbb{I} \arrow[d] \\
\{0\} \times \mathbb{I} \arrow[r] & \mathbb{RP}^1
\end{tikzcd}$$

Given $\mathbb{RP}^{n-1}$, the tautological bundle

5. Cohomology

OUTLINE:

- **Subsection 1.** Define cohomology for $\mathbb{Z}/2\mathbb{Z}$ coefficients and the EM-spaces for $\mathbb{RP}^n$
- **Subsection 2.** Show that we get a commutative graded ring structure for cohomology of any type $\mathtt{X}$ with $\mathbb{Z}/2\mathbb{Z}$-coefficients. Follow Brunerie's thesis.
- **Subsection 3.** Compute $\mathbb{Z}/2\mathbb{Z}$-cohomology ring for $\mathbb{RP}^n$ using Mayer-Vietoris. This needs us to first compute cohomology for disks and spheres.

5.1. Cohomology and EM-spaces for $\mathbb{RP}^n$.

We follow a similar construction for cohomology as found in [**?**], modifying their construction with $\mathbb{Z}$ coefficients to have coefficients in $\mathbb{Z}/2\mathbb{Z}$. In order to define cohomology, we must first define Eilenberg-MacLane spaces $K(\mathbb{Z}/2\mathbb{Z}, n)$. Eilenberg-MacLane spaces $K(G, n)$ were defined for arbitrary group G by Finster and Licata in [**?**], and so we follow their construction.

Definition 5.1. For $n : \mathbb{N}$, the type **Eilenberg-MacLane space** $K(\mathbb{Z}/2\mathbb{Z}, n)$ is the n-truncated and $(n-1)$-connected pointed type defined by

$$K(\mathbb{Z}/2\mathbb{Z}, n) := \begin{cases} \mathbb{Z}/2\mathbb{Z} & \text{for } n = 0 \\ \left|\left|\Sigma^{n-1}\mathbb{R}\mathrm{P}^2\right|\right|_n & \text{for } n \geq 1, \end{cases}$$

where Σ^{n-1} indicates the reduced suspension.

AMELIA: G showed equivalence of K_n and ΩK_{n+1} at this point. I don't think that's necessary given Finster and Licata.

Given this construction of the EM spaces, we define cohomology in the following way.

Definition 5.2. For a type $\mathtt{X}$ and $n : N$, the **n-th cohomology group of $\mathtt{X}$** is the type

$$H^n(\mathtt{X}; \mathbb{Z}/2\mathbb{Z}) := ||\mathtt{X} \to K(\mathbb{Z}/2\mathbb{Z}, n)||_0 .$$

AMELIA: B also defines reduced cohomology as well, not i'm not sure we need that.

5.2. Commutative Graded Ring Structure.

5.3. Computing the Cohomology Ring of $\mathbb{R}\mathbf{P}^n$.

6. The Borsuk-Ulam theorem

OUTLINE:

- The proof is done by this point. Just put it all together and reconnect the dots for the reader.