An easy computation shows that τ is a chain map and that the induced homomorphisms

$$\tau_*: H_*(X,A;\ G) \to H_*(\Delta(Y) \otimes M;\ G) \approx H_*(Y;G) \otimes M$$
$$\tau^*: H^*(Y;G) \otimes M^* \approx H^*(\operatorname{Hom}(\Delta(Y) \otimes M, G)) \to H^*(X,A;\ G)$$

equal Φ and Φ^*, respectively. Since Φ is assumed to be an isomorphism for $G = R$, the chain map τ induces an isomorphism of homology. The universal-coefficient theorems for homology and cohomology then imply that Φ and Φ^* are isomorphisms for all G. ∎

A *fiber-bundle pair* with base space B consists of a *total pair* $(E,\dot{E})$, a *fiber pair* $(F,\dot{F})$, and a *projection* $p: E \to B$ such that there exists an open covering $\{V\}$ of B and for each $V \in \{V\}$ a homeomorphism $\varphi_V: V \times (F,\dot{F}) \to (p^{-1}(V), p^{-1}(V) \cap \dot{E})$ such that the composite

$$V \times F \xrightarrow{\varphi_V} p^{-1}(V) \xrightarrow{p} V$$

is the projection to the first factor. If $A \subset B$, we let $E_A = p^{-1}(A)$ and $\dot{E}_A = p^{-1}(A) \cap \dot{E}$, and if $b \in B$, then $(E_b,\dot{E}_b)$ is the *fiber pair over b*.

Following are some examples.

2 For a space B and pair $(F,\dot{F})$ the *product-bundle pair* consists of the total pair $B \times (F,\dot{F})$ with projection to the first factor.

3 Given a bundle projection $\dot{p}: \dot{E} \to B$ with compact fiber $\dot{F}$, let E be the mapping cylinder of $\dot{p}$ and $p: E \to B$ the canonical retraction. Then $(E,\dot{E})$ is the total pair of a fiber-bundle pair over B with fiber $(F,\dot{F})$, where F is the cone over $\dot{F}$, and projection p.

4 If ξ is a q-sphere bundle over B, then $(E_\xi,\dot{E}_\xi)$ is the total pair of a fiber-bundle pair over B with fiber (E^{q+1},S^q) and projection $p_\xi: E_\xi \to B$.

Given a fiber-bundle pair with total pair $(E,\dot{E})$ and fiber pair $(F,\dot{F})$, a *cohomology extension of the fiber* is a homomorphism $\theta: H^*(F,\dot{F};\ R) \to H^*(E,\dot{E};\ R)$ of graded modules (of degree 0) such that for each $b \in B$ the composite

$$H^*(F,\dot{F};\ R) \xrightarrow{\theta} H^*(E,\dot{E};\ R) \to H^*(E_b,\dot{E}_b;\ R)$$

is an isomorphism. The following statements are easily verified.

5 *Let $\bar{p}: B \times (F,\dot{F}) \to (F,\dot{F})$ be the projection to the second factor. Then*

$$\theta = \bar{p}^*: H^*(F,\dot{F};\ R) \to H^*(B \times (F,\dot{F});\ R)$$

is a cohomology extension of the fiber of the product-bundle pair. ∎

6 *Let $\theta: H^*(F,\dot{F};\ R) \to H^*(E,\dot{E};\ R)$ be a cohomology extension of the fiber of a fiber-bundle pair over B and let $f: B' \to B$ be a map. There is an induced bundle pair over B', with total pair $(E',\dot{E}')$ and fiber $(F,\dot{F})$, and there is a map*

$\bar{f}$: $(E',\dot{E}') \to (E,\dot{E})$ commuting with projections. Then the composite

$$H^*(F,\dot{F};\, R) \xrightarrow{\theta} H^*(E,\dot{E};\, R) \xrightarrow{\bar{f}^*} H^*(E',\dot{E}';\, R)$$

is a cohomology extension of the fiber in the induced bundle. ■

7 *Given a fiber-bundle pair over B with total pair $(E,\dot{E})$, let the path components of B be $\{B_j\}$ and let $(E_j,\dot{E}_j)$ be the induced total pair over B_j. A cohomology extension θ of the fiber of the bundle pair over B corresponds to a family of cohomology extensions $\{\theta_j\}$ of the induced bundle pairs over B_j.* ■

We now establish the local form of the theorem toward which we are heading. It shows that any cohomology extension of the fiber in a product-bundle pair has homology properties as nice as the one given in statement 5 above.

8 LEMMA *Let $(F,\dot{F})$ be a pair such that $H_*(F,\dot{F};\, R)$ is free and finitely generated over R and let θ: $H^*(F,\dot{F};\, R) \to H^*(B \times (F,\dot{F});\, R)$ be a cohomology extension of the fiber of the product-bundle pair. Then the homomorphisms*

$$\Phi\colon H_*(B \times (F,\dot{F});\, G) \to H_*(B;G) \otimes H_*(F,\dot{F};\, R)$$
$$\Phi^*\colon H^*(B;G) \otimes H^*(F,\dot{F};\, R) \to H^*(B \times (F,\dot{F});\, G)$$

are isomorphisms for all R modules G.

PROOF By lemma 1, it suffices to prove that Φ is an isomorphism for $G = R$. If $\{B_j\}$ is the set of path components of B, then

$$H_*(B \times (F,\dot{F});\, R) \approx \bigoplus_j H_*(B_j \times (F,\dot{F});\, R)$$

and

$$H_*(B;R) \otimes H_*(F,\dot{F};\, R) \approx \bigoplus_j H_*(B_j;R) \otimes H_*(F,\dot{F};\, R)$$

Therefore it suffices to prove the result for a path-connected space B. For such a B, $R \approx H^0(B;R)$.

By the Künneth formula, $H_*(B \times (F,\dot{F});\, R) \approx H_*(B;R) \otimes H_*(F,\dot{F};\, R)$. We define graded submodules N_s of $H_*(B;R) \otimes H_*(F,\dot{F};\, R)$ by

$$(N_s)_q = \bigoplus_{i+j=q,\, j\geq s} H_i(B;R) \otimes H_j(F,\dot{F};\, R)$$

Then

$$H_*(B;R) \otimes H_*(F,\dot{F};\, R) = N_0 \supset N_1 \supset \cdots \supset N_s \supset N_{s+1}$$

and $N_s = 0$ for large enough s. If $u \in H^s(F,\dot{F};\, R)$, then $\theta(u) = 1 \times \lambda(u) + \bar{u}$, where $\bar{u} \in \bigoplus_{i+j=s,j<s} H^i(B;R) \otimes H^j(F,\dot{F};\, R)$ and $\theta(u) \mid [b \times (F,\dot{F})] = 1 \times \lambda(u)$. Because θ is a cohomology extension of the fiber, λ is an automorphism of $H^*(F,\dot{F};\, R)$. Let $z' \in H_s(F,\dot{F};\, R)$ and consider $z \times z' \in N_s$. Then

$$\Phi(z \times z') = \sum_i p_*(\theta(m_i^*) \frown (z \times z')) \otimes m_i$$

and if $\deg m_i < s$, then $\theta(m_i^*) \frown (z \times z') \in N_1$ and $p_*(N_1) = 0$. Therefore

$\Phi(z \times z') \in N_s$, and so Φ maps N_s into itself for all s. Because of the short exact sequences

$$0 \to N_{s+1} \to N_s \to N_s/N_{s+1} \to 0$$

and the five lemma, it follows by downward induction on s that Φ is an isomorphism if and only if it induces an isomorphism of N_s/N_{s+1} onto itself for all s. For $z' \in H_s(F,\dot F;\, R)$, computing $\Phi(z \times z')$ in N_s/N_{s+1}, we obtain

$$\begin{aligned}\Phi(z \times z') &= \sum_{\deg m_i \geq s} p_*[(1 \times \lambda(m_i^*) + \bar m_i^*) \frown (z \times z')] \otimes m_i \\ &= \sum_{\deg m_i = s} p^*[1 \times \lambda(m_i^*) \frown (z \times z')] \otimes m_i\end{aligned}$$

because $\bar m_i^* \frown (z \times z') \in N_1$ and $p_*(N_1) = 0$. Now, by properties 5.6.21, 5.6.19, and 5.6.17,

$$\sum_{\deg m_i = s} p_*[1 \times \lambda(m_i^*) \frown (z \times z')] \otimes m_i$$
$$= \sum_{\deg m_i = s} z \otimes \langle \lambda(m_i^*), z' \rangle m_i = z \otimes \lambda_*(z')$$

where $\lambda_*: H_*(F,\dot F;\, R) \to H_*(F,\dot F;\, R)$ is the automorphism dual to λ. Hence $\Phi(z \times z') = z \times \lambda_*(z')$ in N_s/N_{s+1}, showing that Φ induces an isomorphism of N_s/N_{s+1} for all s. ■

The following *Leray-Hirsch theorem* shows that fiber-bundle pairs with cohomology extensions of the fiber have homology and cohomology modules isomorphic to those of the product of the fiber pair and the base.

9 THEOREM *Let $(E,\dot E)$ be the total pair of a fiber-bundle pair with base B and fiber pair $(F,\dot F)$. Assume that $H_*(F,\dot F;\, R)$ is free and finitely generated over R and that θ is a cohomology extension of the fiber. Then the homomorphisms*

$$\Phi: H_*(E,\dot E;\, G) \to H_*(B;G) \otimes H_*(F,\dot F;\, R) \qquad \Phi(z) = \sum_i p_*(\theta(m_i^*) \frown z) \otimes m_i$$

$$\Phi^*: H^*(B;G) \otimes H^*(F,\dot F;\, R) \to H^*(E,\dot E;\, G) \qquad \Phi^*(u \otimes v) = p^*(u) \smile \theta(v)$$

are isomorphisms (of graded modules) for all R modules G.

PROOF By lemma 1, it suffices to prove the result for the map Φ in the case $G = R$. For any subset $A \subset B$ let θ_A be the composite

$$H^*(F,\dot F;\, R) \xrightarrow{\theta} H^*(E,\dot E;\, R) \to H^*(E_A,\dot E_A;\, R)$$

Then θ_A is a cohomology extension of the fiber in the induced bundle over A. It follows from lemma 8 that if the induced bundle over A is homeomorphic to the product-bundle pair $A \times (F,\dot F)$, then

$$\Phi_A: H_*(E_A,\dot E_A;\, R) \approx H_*(A;R) \otimes H_*(F,\dot F;\, R)$$

Hence Φ_V is an isomorphism for all sufficiently small open sets V.

If V and V' are open sets in B, then $\{(E_V,\dot E_V), (E_{V'},\dot E_{V'})\}$ is an excisive couple of pairs in E, and it follows from property 5.6.20 that Φ_V, $\Phi_{V'}$, $\Phi_{V\cap V'}$, and $\Phi_{V\cup V'}$ map the exact Mayer-Vietoris sequence of $(E_V,\dot E_V)$ and $(E_{V'},\dot E_{V'})$ into the tensor product of the exact Mayer-Vietoris sequence of V and V' by $H_*(F,\dot{F};\ R)$. Since $H_*(F,\dot{F};\ R)$ is free over R, its tensor product with any exact sequence is exact. Therefore, if Φ_V, $\Phi_{V'}$, and $\Phi_{V\cap V'}$ are isomorphisms, it follows from the five lemma that $\Phi_{V\cup V'}$ is also an isomorphism. By induction, Φ_U is an isomorphism for any U which is a finite union of sufficiently small open sets. Let $\mathfrak{U}$ be the collection of these sets. Since any compact subset of B lies in some element of $\mathfrak{U}$, $H_*(B;R) \approx \lim_{\to} \{H_*(U;R)\}_{U\in\mathfrak{U}}$. Also, any compact subset of E lies in E_U for some $U \in \mathfrak{U}$, so $H_*(E,\dot{E};\ R) \approx \lim_{\to} \{H_*(E_U,\dot{E}_U;\ R)\}$. Because the tensor product commutes with direct limits and Φ corresponds to $\lim_{\to} \{\Phi_U\}_{U\in\mathfrak{U}}$ under these isomorphisms, Φ is also an isomorphism. ■

The above argument proves directly that Φ is an isomorphism for any coefficient module G. A similar argument does not appear possible for Φ^*, because it is not true that $H^*(B;R)$ is isomorphic to the inverse limit $\lim_{\leftarrow} \{H^*(U;R)\}_{U\in\mathfrak{U}}$. It should be noted that in theorem 9 we have said nothing about commutativity of Φ^* with cup products, because it is not true, in general, that Φ^* preserves cup products.

We now specialize to the case of sphere bundles. Because

$$H^r(E^{q+1},S^q;\ R) \approx \begin{cases} 0 & r \neq q+1 \\ R & r = q+1 \end{cases}$$

if ξ is a q-sphere bundle, a cohomology extension of the fiber in ξ is an element $U \in H^{q+1}(E_\xi,\dot{E}_\xi;\ R)$ such that for any $b \in B$, the restriction of U to $(p^{-1}(b),\ p^{-1}(b) \cap \dot{E})$ is a generator of $H^{q+1}(p^{-1}(b),\ p^{-1}(b) \cap \dot{E};\ R)$. Such a cohomology class is called an *orientation class* (*over* R) of the bundle. If orientations of the bundle exist, the bundle is called *orientable*. An *oriented sphere bundle* is a pair (ξ, U_ξ) consisting of a sphere bundle ξ and an orientation class of U_ξ of ξ.

If U is an orientation class of ξ over $\mathbf{Z}$ and if 1 is the unit element of R, then $\mu(U \otimes 1)$ is an orientation class of ξ over R. Therefore a sphere bundle orientable over $\mathbf{Z}$ is orientable over any R.

If (ξ,U_ξ) is an oriented sphere bundle over B and $f\colon B' \to B$, then $(f^*\xi,\bar{f}^*U_\xi)$ is an oriented sphere bundle over B' [where $\bar{f}\colon (E_{f^*\xi},\dot{E}_{f^*\xi}) \to (E_\xi,\dot{E}_\xi)$ is associated to f].

From theorem 9 we get the following *Thom isomorphism theorem.*

10 THEOREM *Let (ξ,U_ξ) be an oriented q-sphere bundle over B. There are natural isomorphisms for any R module G*

$$\Phi_\xi\colon H_n(E_\xi,\dot{E}_\xi;\ G) \xrightarrow{\approx} H_{n-q-1}(B;G) \qquad \Phi_\xi(z) = p_*(U_\xi \frown z)$$

$$\Phi_\xi^*\colon H^r(B;G) \xrightarrow{\approx} H^{r+q+1}(E_\xi,\dot{E}_\xi;\ G) \qquad \Phi_\xi^*(v) = p^*v \smile U_\xi$$

PROOF Let m and m^* be dual generators of $H_{q+1}(E^{q+1},S^q;\ R)$ and $H^{q+1}(E^{q+1},S^q;\ R)$, respectively, and define a cohomology extension θ by $\theta(m^*) = U_\xi$. Then Φ_ξ is the composite

$$H_n(E_\xi,\dot{E}_\xi;\ G) \xrightarrow{\Phi} H_{n-q-1}(B;G) \otimes H_{q+1}(E^{q+1},S^q;R) \approx H_{n-q-1}(B;G)$$

where the second map sends $z \otimes m$ to z. By theorem 9, Φ is an isomorphism, and so Φ_ξ is an isomorphism. A similar argument shows that Φ_ξ^* is an isomorphism. These isomorphisms are natural for induced bundles because of naturality properties of the cup and cap products. ∎

This result implies the exactness of the following *Thom-Gysin sequences* of a sphere bundle.

11 THEOREM *Let (ξ, U_ξ) be an oriented q-sphere bundle with base B and projection $\dot{p} = p \mid \dot{E}\colon \dot{E} \to B$. For any R module G there are natural exact sequences*

$$\cdots \to H_n(\dot{E}_\xi;G) \xrightarrow{\dot{p}_*} H_n(B;G) \xrightarrow{\Psi_\xi} H_{n-q-1}(B;G) \xrightarrow{\rho} H_{n-1}(\dot{E}_\xi;G) \to \cdots$$
$$\cdots \to H^r(B;G) \xrightarrow{\dot{p}^*} H^r(\dot{E}_\xi;G) \xrightarrow{\rho^*} H^{r-q}(B;G) \xrightarrow{\Psi_\xi^*} H^{r+1}(B;G) \to \cdots$$

in which Ψ_ξ and Ψ_ξ^* have properties

$$\Psi_\xi(v \frown z) = (-1)^{(q+1)\deg v}\, \Psi_\xi^*(v) \frown z$$
$$\Psi_\xi^*(v_1 \smile v_2) = v_1 \smile \Psi_\xi^*(v_2)$$

PROOF There is a commutative diagram (with any coefficient module)

$$\begin{array}{ccccccc}
\cdots \to H_n(\dot{E}) & \xrightarrow{i_*} & H_n(E) & \xrightarrow{j_*} & H_n(E,\dot{E}) & \xrightarrow{\partial} & H_{n-1}(\dot{E}) \to \cdots \\
& \dot{p}_* \searrow & \approx\downarrow p_* & & \approx\downarrow \Phi_\xi & & \\
& & H_n(B) & & H_{n-q-1}(B) & &
\end{array}$$

the top row of which is exact. Since p is a deformation retraction of E onto B, p_* is an isomorphism. By theorem 10, Φ_ξ is an isomorphism. The desired sequence is obtained by defining $\Psi_\xi = \Phi_\xi j_* p_*{}^{-1}$ and $\rho = \partial\Phi_\xi{}^{-1}$. Similarly, the cohomology sequence is defined by $\Psi_\xi^* = p^{*-1} j^* \Phi_\xi^*$ and $\rho^* = \Phi_\xi^{*-1}\delta$. We verify the formula for Ψ_ξ.

$$\begin{aligned}
\Psi_\xi(v \frown z) &= \Phi_\xi j_* p_*{}^{-1}(v \frown z) = \Phi_\xi j_*(p^*(v) \frown p_*{}^{-1}(z)) \\
&= \Phi_\xi(p^*(v) \frown j_* p_*{}^{-1}(z)) = p_*(U \frown [p^*(v) \frown j_* p_*{}^{-1}(z)]) \\
&= p_*(j^*[U \smile p^*(v)] \frown p_*{}^{-1}(z)) \\
&= (-1)^{(q+1)\deg v} p_*[j^* \Phi_\xi^*(v) \frown p_*{}^{-1}(z)] \\
&= (-1)^{(q+1)\deg v}\, \Psi_\xi^*(v) \frown z \quad ∎
\end{aligned}$$

Note that the isomorphisms Φ and Φ^* of the Thom isomorphism theorem depend on the choice of the orientation class U of the bundle. Therefore the homomorphisms ρ and Ψ and ρ^* and Ψ^* of the Thom-Gysin sequences also depend on the orientation class. In case B is path connected and U and U' are orientation classes of a sphere bundle over B, it follows from theorem 10 that there is an element $r \in R$ such that

$$U' = p^*(r \times 1) \smile U = r[p^*(1) \smile U]$$

If $b_0 \in B$, then

$$U' \mid (p^{-1}(b_0), p^{-1}(b_0) \cap \dot{E}) = r[U \mid (p^{-1}(b_0), p^{-1}(b_0) \cap \dot{E})]$$

Therefore we have the next result.

12 LEMMA *Two orientation classes U and U' of a sphere bundle over a path-connected base space B are equal if and only if for some $b_0 \in B$*

$$U \mid (p^{-1}(b_0), p^{-1}(b_0) \cap \dot{E}) = U' \mid (p^{-1}(b_0), p^{-1}(b_0) \cap \dot{E}) \quad \blacksquare$$

If B is not path connected, let $\{B_j\}$ be the set of path components of B and let $(E_j,\dot{E}_j)$ be the part of $(E,\dot{E})$ over B_j. Then

$$H^*(E,\dot{E};\, R) \approx \times_j\, H^*(E_j,\dot{E}_j;\, R)$$

and we also obtain the following result.

13 LEMMA *Two orientation classes U and U' of a sphere bundle with base space B are equal if and only if for all $b \in B$*

$$U \mid (p^{-1}(b), p^{-1}(b) \cap \dot{E}) = U' \mid (p^{-1}(b), p^{-1}(b) \cap \dot{E}) \quad \blacksquare$$

In case $R = \mathbf{Z}_2$, then $H^{q+1}(p^{-1}(b), p^{-1}(b) \cap \dot{E};\, \mathbf{Z}_2) \approx \mathbf{Z}_2$ for all $b \in B$. Therefore this module has a unique nonzero element, and we obtain the following consequence of lemma 13.

14 COROLLARY *Any two orientation classes over $\mathbf{Z}_2$ of a sphere bundle are equal.* $\blacksquare$

Thus, for $R = \mathbf{Z}_2$ the homomorphisms Φ, ρ, and Ψ and Φ^*, ρ^*, and Ψ^* are all unique.

The *characteristic class* Ω_ξ of an oriented q-sphere bundle (ξ, U_ξ) is defined to be the element

$$\Omega_\xi = \Psi_\xi^*(1) \in H^{q+1}(B;R)$$

This is functorial (that is, $\Omega_{f^*\xi} = f^*\Omega_\xi$). From the multiplicative properties of Ψ_ξ and Ψ_ξ^* in theorem 11 we obtain the following equations.

15 *For $z \in H_n(B;G)$*

$$\Psi_\xi(z) = \Omega_\xi \frown z$$

and for $v \in H^r(B;G)$

$$\Psi_\xi^*(v) = v \smile \Omega_\xi \quad \blacksquare$$

We now investigate the existence of orientation classes for a sphere bundle. Let (X,X') be a pair and let $\{A_j\}_{j \in J}$ be an indexed collection of subsets $A_j \subset X$. An indexed collection

$$\{u_j \in H^n(A_j, A_j \cap X';\, G)\}_{j \in J}$$

is said to be *compatible* if for all $j, j' \in J$

$$u_j \mid (A_j \cap A_{j'}, A_j \cap A_{j'} \cap X') = u_{j'} \mid (A_j \cap A_{j'}, A_j \cap A_{j'} \cap X')$$

The compatible collections $\{u_j\}$ constitute an R module $H^n(\{A_j\},X';\, G)$. Clearly, the restriction maps

$$H^n(X,X';\, G) \to H^n(A_j, A_j \cap X';\, G)$$

define a natural homomorphism $H^n(X,X'; G) \to H^n(\{A_j\},X'; G)$.

16 LEMMA *Let $(E,\dot{E})$ be a fiber-bundle pair with base B, projection $p: E \to B$, and fiber pair $(F,\dot{F})$. Assume that for some $n > 0$, $H_i(F,\dot{F}; R) = 0$ for $i < n$. Then*

(a) For all $A \subset B$ and all R modules G

$$H_i(p^{-1}(A), p^{-1}(A) \cap \dot{E}; G) = 0 = H^i(p^{-1}(A), p^{-1}(A) \cap \dot{E}; G) \qquad i < n$$

(b) If $\{V\}$ is any open covering of B, then in degree n the natural homomorphism is an isomorphism

$$H^n(E,\dot{E}; G) \approx H^n(\{p^{-1}V\},\dot{E}; G)$$

PROOF By the universal-coefficient formula, it suffices to prove (*a*) for $G = R$. If $A \subset B$ is such that $(p^{-1}(A), p^{-1}(A) \cap \dot{E})$ is homeomorphic to $A \times (F,\dot{F})$, then by the Künneth formula,

$$H_i(p^{-1}(A), p^{-1}(A) \cap \dot{E}; R) \approx H_i(A \times (F,\dot{F}); R) = 0 \qquad i < n$$

From this it follows (as in the proof of theorem 9) by induction on the number of coordinate neighborhoods of the bundle needed to cover A (using the Mayer-Vietoris sequence and the five lemma) that (*a*) holds for all compact $A \subset B$. By taking direct limits, (*a*) holds for any A.

For (*b*), let $\{W\}$ be the collection of finite unions of elements of $\{V\}$. By (*a*) and the universal-coefficient formula for cohomology, there is a commutative diagram

$$\begin{array}{ccc} H^n(E,\dot{E}; G) & \approx & \operatorname{Hom}(H_n(E,\dot{E};R), G) \\ \downarrow & & \downarrow\approx \\ \lim_{\leftarrow}\{H^n(p^{-1}(W), p^{-1}(W) \cap \dot{E}; G)\} & \approx & \lim_{\leftarrow}\{\operatorname{Hom}(H_n(p^{-1}(W), p^{-1}(W) \cap \dot{E}; R),G)\} \end{array}$$

Hence we need only prove that a compatible collection $\{u_V\}_{V\in\{V\}}$ extends to a unique compatible collection $\{u_W\}_{W\in\{W\}}$. This follows by using Mayer-Vietoris sequences again and from the fact that $H^i(p^{-1}(W), p^{-1}(W) \cap \dot{E}; G) = 0$ for $i < n$. ∎

For sphere bundles we have the following immediate consequence.

17 COROLLARY *A sphere bundle ξ with base B is orientable if and only if there is a covering $\{V\}$ of B and a compatible family $\{u_V\}$, where u_V is an orientation class of $\xi \mid V$ for each $V \in \{V\}$.* ∎

Since a trivial sphere bundle is orientable, corollaries 17 and 14 imply the following result.

18 COROLLARY *Any sphere bundle has a unique orientation class over $\mathbf{Z}_2$.* ∎

By theorem 2.8.12, there is a contravariant functor from the fundamental groupoid of the base space B of a sphere bundle ξ to the homotopy category which assigns to $b \in B$ the fiber pair $(E_b,\dot{E}_b)$ over b and to a path class $[\omega]$ in B a homotopy class $h[\omega] \in [E_{\omega(0)},\dot{E}_{\omega(0)}; E_{\omega(1)},\dot{E}_{\omega(1)}]$. For fixed R there is then a covariant functor from the fundamental groupoid of B to the category of R modules which assigns to $b \in B$ the module $H^{q+1}(E_b,\dot{E}_b;\ R)$ and to a path class $[\omega]$ the homomorphism

$$h[\omega]^*\colon H^{q+1}(E_{\omega(1)},\dot{E}_{\omega(1)};\ R) \to H^{q+1}(E_{\omega(0)},\dot{E}_{\omega(0)};\ R)$$

19 THEOREM *A sphere bundle ξ is orientable over R if and only if for every closed path ω in B, $h[\omega]^* = 1$.*

PROOF If ξ is orientable with orientation class $U \in H^{q+1}(E,\dot{E};\ R)$, for any small path ω in B (and hence for any path)

$$h[\omega]^*(U \mid (E_{\omega(1)},\dot{E}_{\omega(1)})) = U \mid (E_{\omega(0)},\dot{E}_{\omega(0)})$$

Since $U \mid (E_b,\dot{E}_b)$ is a generator of $H^{q+1}(E_b,\dot{E}_b;\ R)$, this implies that $h[\omega]^* = 1$ for any closed path ω.

Conversely, if $h[\omega]^* = 1$ for every closed path ω in B, there exist generators $U_b \in H^{q+1}(E_b,\dot{E}_b;\ R)$ such that for any path class $[\omega]$ in B, $h[\omega]^*(U_{\omega(1)}) = U_{\omega(0)}$. If V is any subset of B such that $\xi \mid V$ is trivial, it is easy to see that there is an orientation class U_V of $\xi \mid V$ such that $U_V \mid (E_b,\dot{E}_b) = U_b$ for all $b \in V$. If $\{V\}$ is an open covering of B by sets such that $\xi \mid V$ is trivial for all V, then $\{U_V\}$ is a compatible family of orientations, and by corollary 17, ξ is orientable. ∎

20 COROLLARY *A sphere bundle with a simply connected base is orientable over any R.* ∎

8 THE COHOMOLOGY ALGEBRA

The cup product in cohomology makes the cohomology (over R) of a topological pair a graded R algebra. In the first part of this section we define the relevant algebraic concepts and compute this algebra over $\mathbf{Z}_2$ for a real projective space and over any R for complex and quaternionic projective space. This is applied to prove the Borsuk-Ulam theorem.

For the case of an H space, there is even more algebraic structure that can be introduced in the cohomology algebra. The cohomology of such a space is a Hopf algebra, and the second part of the section is devoted to its definition and some results about its structure. The section concludes with a proof of the Hopf theorem about the cohomology algebra of a compact connected H space.

A *graded R algebra* consists of a graded R module $A = \{A^q\}$ and a homomorphism of degree 0

$$\mu\colon A \otimes A \to A$$

called the *product* of the algebra (μ then maps $A^p \otimes A^q$ into A^{p+q} for all p and q). For $a, a' \in A$ we write $aa' = \mu(a \otimes a')$. The product is *associative* if $(aa')a'' = a(a'a'')$ for all $a, a', a'' \in A$ and is *commutative* if $aa' = (-1)^{\deg a \deg a'} a'a$ for all $a, a' \in A$.

1 EXAMPLE If (X,A) is a topological pair, then $H^*(X,A;\ R)$ is a graded R algebra whose product is the cup product (with respect to the multiplication pairing of R with itself to R). It follows from property 5.6.10 that this product is associative and from property 5.6.11 that it is commutative. If $A = \varnothing$, it follows from property 5.6.9 that 1 is a unit element of the algebra $H^*(X;R)$. $H^*(X,A;\ R)$ is called the *cohomology algebra* of (X,A) over R.

2 EXAMPLE The *polynomial algebra over R generated by x of degree* $n > 0$, denoted by $S_n(x)$, is defined by

$$[S_n(x)]^q = \begin{cases} 0 & q \not\equiv 0\ (n) \text{ or } q < 0 \\ \text{free } R \text{ module generated by } x_p & q = pn,\ p \geq 0 \end{cases}$$

with the product $(\alpha x_p)(\beta x_q) = (\alpha\beta)x_{p+q}$ for $\alpha,\ \beta \in R$. It is then clear that x_0 is a unit element and that $x_p = (x_1)^p$. If we denote x_1 by x, then $x_p = x^p$. Thus, disregarding the graded structure, $S_n(x)$ is simply the polynomial algebra over R in one indeterminate x. The *truncated polynomial algebra over R generated by x of degree n and height h*, denoted by $T_{n,h}(x)$, is defined to be the quotient of $S_n(x)$ by the graded ideal generated by x^h. If $h = 2$, this is called the *exterior algebra generated by x of degree n* and is denoted by $E_n(x)$.

If A and B are graded R algebras, their tensor product $A \otimes B$ is also a graded R algebra with product

$$(a \otimes b)(a' \otimes b') = (-1)^{\deg b \deg a'} aa' \otimes bb'$$

If A and B have associative or commutative products, so does $A \otimes B$.

3 EXAMPLE If R is a field and (X,A) and (Y,B) are topological pairs such that either $H_*(X,A;\ R)$ or $H_*(Y,B;\ R)$ is of finite type, it follows from theorem 5.5.11 that

$$H^*(X,A;\ R) \otimes H^*(Y,B;\ R) \approx H^*((X,A) \times (Y,B);\ R)$$

We compute the graded $\mathbf{Z}_2$ algebra $H^*(P^n;\mathbf{Z}_2)$ for real projective space P^n. Note that the double covering $p\colon S^n \to P^n$ is a 0-sphere bundle. We let $w_n \in H^1(P^n;\mathbf{Z}_2)$ be the characteristic class (over $\mathbf{Z}_2$) of this bundle.

4 THEOREM *For $n \geq 1$, $H^*(P^n;\mathbf{Z}_2)$ is a truncated polynomial algebra over $\mathbf{Z}_2$ generated by w_n of degree 1 and height $n + 1$.*

PROOF All coefficients in the proof will be $\mathbf{Z}_2$ and will be omitted. By corollary 5.7.18 and theorem 5.7.11, there is an exact Thom-Gysin sequence

$$\cdots \to H^q(S^n) \xrightarrow{p^*} H^q(P^n) \xrightarrow{\Psi^*} H^{q+1}(P^n) \xrightarrow{p^*} H^{q+1}(S^n) \to \cdots$$

starting on the left with $0 \to H^0(P^n) \xrightarrow{p^*} H^0(S^n)$ and terminating on the right with $H^n(S^n) \xrightarrow{p^*} H^n(P^n) \to 0$ [note that $H^q(P^n) = 0$ for $q > n$, because P^n is a polyhedron of dimension n]. Because $H^q(S^n) = 0$ for $0 < q < n$, it follows that

$$\Psi^*\colon H^q(P^n) \to H^{q+1}(P^n)$$