Semantics of HoTT objects are continuous ∞-groupoids X

they have an ∞-groupoid $X(\mathbb{R}^n)$ maps from $\mathbb{R}^n$ to X (continuous)

think ∞-groupoids in spaces:

$$\begin{tikzcd} \vdots \arrow[d, shift left] \arrow[d, shift right] \\ X_1 \arrow[d, shift left] \arrow[d, shift right] \\ X_0 \end{tikzcd}$$

X_1 topological space of morphisms

X_0 topological space of objects

We saw previously ~~$\forall x:\mathbb{R},\ x<0 \vee x\geq 0$~~

Also (similarly) ~~$\forall x:\mathbb{R},\ x\leq 0 \vee x\geq 0$~~

but $\forall x:\mathbb{R}\ \ x<\frac{\varepsilon}{2} \vee x>\frac{-\varepsilon}{2}$ any $\varepsilon>0$

- $\int$ shape modality
  - $(\int X\to Y)=(X\to Y)$ Y discrete
  - discrete types are closed under colimits because of b
  - $\int X$ is the infinity fundamental groupoid of X.

proof that $\int \mathbb{S}^1 = S^1$:

$I :=$ open interval

$$\begin{tikzcd} I\sqcup I \arrow[r] \arrow[d] & I \arrow[d] \\ I \arrow[r] & \mathbb{S}^1 \end{tikzcd}$$ topological circle

$\int$

$$\begin{tikzcd} 1+1 \arrow[r] \arrow[d] & 1 \arrow[d] \\ 1 \arrow[r] & S^1 \underset{eq}{\simeq} \int \mathbb{S}^1 \end{tikzcd}$$

since $\int I = 1$

Brouwer Fixed pt

Axiom Rb: A type X is discrete iff $X \to (\mathbb{R} \to X)$ is an equiv.

Lemma 1: $\int \mathbb{D}^2$ is contractible

prf
- $\int \mathbb{R}$ is contractible (uses Axiom Rb)
- $\int$ preserves $\times$ (product). needs proof.

$\Rightarrow$ $\int \mathbb{R} \times \mathbb{R}$ is contractible.

- $\mathbb{D}^2$ is a retract of $\mathbb{R}^2$ because we can write a retracting formula
- $\int$ is functorial, so it preserves retracts $\square$

Lemma 2: (uses univalence axiom UA and Rb)
$\mathbb{S}^1$ is not a retract of $\mathbb{R}^2$.

prf: suppose not. then $\int \mathbb{S}^1$ would be a retract of $\int \mathbb{D}^2$.
- $\int \mathbb{S}^1$ is not contractible.
- $\int \mathbb{D}^2$ is ~~not~~ contractible. $\square$

[Category theory.

homset

$\mathcal{C}$ cat $\quad \mathcal{C}(X,Y) \times \mathcal{C}(X,Z) \cong \mathcal{C}(X, Y \times Z)$ nat in X

(can compose with projections to get from target to domain)

$$\begin{tikzcd}
X \arrow[rr, bend left] \arrow[dr, dashed, "\exists!"] \arrow[ddr, bend right] & & Y \\
& Y \times Z \arrow[ur] \arrow[d] & \\
& Z &
\end{tikzcd}$$

$$\begin{tikzcd}
X \arrow[dr, dashed] \arrow[drr, bend left, "f"] \arrow[ddr, bend right, "g"'] & & \\
& A \times_C B \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner" very near start] & A \arrow[d, "p"] \\
& B \arrow[r, "q"'] & C
\end{tikzcd}$$

pullback

In these examples, $Y \times Z$ and $A \times_C B$ are <u>limits</u>

the universal property of limits

universal property of <u>colimits</u>

<u>ex.</u> coproduct

<u>Adjunction</u> $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$

$$\mathcal{C}(A, GB) \cong \mathcal{D}(FA, B)$$

right adjoint   left adjoint

$$\begin{tikzcd}
\mathrm{Sets} \arrow[r, bend left, "\text{discrete}"] \arrow[r, bend right, "\text{codiscrete}"'] & \mathrm{Top} \arrow[l, "u"']
\end{tikzcd}$$

right adjoints preserve limits
(left)   (colimits)

Yoneda $\Rightarrow G(B \times C) \cong GB \times GC$

B, C in $\mathcal{D}$

$$\mathcal{C}(A, G(B \times C)) \cong \mathcal{D}(FA, B \times C) \cong \mathcal{D}(FA, B) \times \mathcal{D}(FA, C)$$
$$\cong \mathcal{C}(A, GB) \times \mathcal{C}(A, GC) \cong \mathcal{C}(A, GB \times GC)$$

Lemma (using LEM, Rb, T)

P.76 of Michael Schulman's Brouwer fixed pt. paper.

Axiom T: $x>0$ is fibre-wise codiscrete #

$$\{x>0\} \hookrightarrow \mathbb{R}$$

If $\exists f: \mathbb{D}^2 \longrightarrow \mathbb{D}^2$ with no fixed pt. then S^1 is a retract of $\mathbb{D}^2$

prf. by assumption $f(z) \neq z \ \forall z$ in $\mathbb{D}^2$

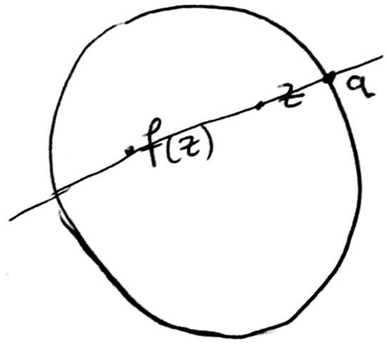
write $z=(x,y)$ and $f(z)=(u,v)$

$$(x-u)^2+(y-v)^2 \neq 0$$

$$\Rightarrow (x-u)^2+(y-v)^2 > 0$$

the line between $f(z)$ and z is $(tx+(1-t)u, ty+(1-t)v)$

$\mathbb{D}^2$



want to construct a.