PROOF We use the spectral sequence of theorem 9.2.17. For a finitely generated bigraded module E^r we define the Euler characteristic $\chi(E^r) = \sum_{s,t}(-1)^{s+t} \dim E^r_{s,t}$. Because we are considering a field as coefficients, it follows from the Künneth formula that

$$E^2_{s,t} \approx H_s(B;H_t(F)) \approx H_s(B) \otimes H_t(F)$$

Therefore $\chi(E^2) = \chi(B)\chi(F)$. Because $E^{r+1} \approx H(E^r)$, it follows (as in theorem 4.3.14) that

$$\chi(E^2) = \chi(E^3) = \cdots = \chi(E^r)$$

Because $E^2_{s,t} = 0$ if s and t are large enough, the same is true of $E^r_{s,t}$ for any r. Therefore $E^\infty = E^r$ for large enough r, and so $\chi(E^\infty) = \chi(B)\chi(F)$. By a standard property of dimension,

$$\dim [H_n(E)] = \sum_{s+t=n} \dim E^\infty_{s,t}$$

and so $\chi(E) = \chi(E^\infty) = \chi(B)\chi(F)$. ∎

We now compute the homomorphism induced by $i: F \subset E$ in terms of the spectral sequence. For $r \geq 2$, $E^{r+1}_{0,t}$ is a quotient of $E^r_{0,t}$ (because $E^r_{-r,t+r-1} = 0$ in a first-quadrant spectral sequence). Therefore there is an epimorphism $E^2_{0,t} \to E^\infty_{0,t}$. Because B is path connected, there is an isomorphism $H_t(F;G) \approx H_0(B;\ H_t(F;G))$. By using the spectral sequence of the fibration $F \to b_0$ and the functorial property of the spectral sequence, it follows that $i_*: H_t(F;G) \to H_t(E;G)$ is the composite

$$H_t(F;G) \approx H_0(B;\ H_t(F;G)) \approx E^2_{0,t} \to E^\infty_{0,t} = F_0H_t(E;G) \subset H_t(E;G)$$

This leads to the following *generalized Wang homology sequence.*

2 THEOREM *Let $p: E \to B$ be a fibration, with fiber F and simply connected base B which is a homology n-sphere (over R) for some $n \geq 2$ [that is, $H_q(B) = 0$ if $q \neq 0$ or n and $H_0(B) \approx R \approx H_n(B)$]. Then there is an exact sequence*

$$\cdots \to H_t(F;G) \xrightarrow{i_*} H_t(E;G) \to H_{t-n}(F;G) \to H_{t-1}(F;G) \xrightarrow{i_*} \cdots$$

PROOF Because $H_*(B)$ has no torsion, $E^2_{s,t} \approx H_s(B) \otimes H_t(F;G)$ in the spectral sequence of p. Therefore $E^2_{s,t} = 0$ unless $s = 0$ or n, and the only nonzero differential is $d^n: E^2_{n,t} \to E^2_{0,t+n-1}$. Hence there are exact sequences

$$0 \to E^\infty_{n,t} \to E^2_{n,t} \xrightarrow{d^n} E^2_{0,t+n-1} \to E^\infty_{0,t+n-1} \to 0$$

and

$$0 \to E^\infty_{0,t} \to H_t(E;G) \to E^\infty_{n,t-n} \to 0$$

These fit together into an exact sequence

$$\cdots \to H_t(E;G) \to E^2_{n,t-n} \xrightarrow{d^n} E^2_{0,t-1} \to H_{t-1}(E;G) \to \cdots$$

The result follows on observing that

$$E^2_{n,t-n} \approx H_n(B) \otimes H_{t-n}(F;G) \approx H_{t-n}(F;G)$$
$$E^2_{0,t-1} \approx H_0(B) \otimes H_{t-1}(F;G) \approx H_{t-1}(F;G)$$

and that on replacing $E^2_{0,t-1}$ by $H_{t-1}(F;G)$ in the exact sequence, the resulting map $H_{t-1}(F;G) \to H_{t-1}(E;G)$ is i_*. ∎

Let $p: E \to B$ be an orientable fibration with path-connected base and let $B' \subset B$ and $E' = p^{-1}(B')$. We now show how the homomorphism induced by $p: (E,E') \to (B,B')$ is determined from the spectral sequence. For $r \geq 2$, $E^{r+1}_{s,0}$ is a submodule of $E^r_{s,0}$ (because $E^r_{s+r,-r+1} = 0$). Therefore there is a monomorphism $E^\infty_{s,0} \to E^2_{s,0}$. The augmentation homomorphism $H_0(F;G) \to G$ induces a homomorphism $H_s(B,B';\ H_0(F;G)) \to H_s(B,B';\ G)$. By using the spectral sequence of the fibration $B \subset B$ and the functorial property of the spectral sequence, it follows that $p_*: H_s(E,E';\ G) \to H_s(B,B';\ G)$ is the composite

$$H_s(E,E';\ G) = F_sH_s(E,E';\ G) \to E^\infty_{s,0} \to E^2_{s,0} \approx H_s(B,B';\ H_0(F;G)) \to H_s(B,B';\ G)$$

This leads to the following *generalized Gysin homology sequence.*

3 THEOREM *Let $p: E \to B$ be an orientable fibration with path-connected base space and with fiber F a homology n-sphere (over R), where $n \geq 1$. If $B' \subset B$ and $E' = p^{-1}(B')$, there is an exact sequence*

$$\cdots \to H_s(E,E';\ G) \xrightarrow{p_*} H_s(B,B';\ G) \to H_{s-n-1}(B,B';\ G) \to H_{s-1}(E,E';\ G) \xrightarrow{p_*} \cdots$$

PROOF Because, in the spectral sequence of p,

$$E^2_{s,t} \approx H_s(B,B';\ H_t(F;G)) = 0 \qquad t \neq 0 \text{ or } n$$

the only nonzero differential is $d^{n+1}: E^2_{s,0} \to E^2_{s-n-1,n}$. Hence there are exact sequences

$$0 \to E^\infty_{s,0} \to E^2_{s,0} \xrightarrow{d^{n+1}} E^2_{s-n-1,n} \to E^\infty_{s-n-1,n} \to 0$$

and
$$0 \to E^\infty_{s-n,n} \to H_s(E,E';\ G) \to E^\infty_{s,0} \to 0$$

These fit together into an exact sequence

$$\cdots \to H_s(E,E';\ G) \to E^2_{s,0} \xrightarrow{d^{n+1}} E^2_{s-n-1,n} \to H_{s-1}(E,E';\ G) \to \cdots$$

The result follows on observing that

$$E^2_{s,0} \approx H_s(B,B';\ H_0(F;G)) \approx H_s(B,B';\ G)$$
$$E^2_{s-n-1,n} \approx H_{s-n-1}(B,B';\ H_n(F;G)) \approx H_{s-n-1}(B,B';\ G)$$

and that on replacing $E^2_{s,0}$ by $H_s(B,B';\ G)$ in the exact sequence, the resulting map $H_s(E,E';\ G) \to H_s(B,B';\ G)$ is p_*. ∎

4 LEMMA *Let $p: E \to B$ be an orientable fibration with path-connected base space and with path-connected fiber F. Assume that $H_q(B,B') = 0$ for $q < n$ and $H_q(F) = 0$ for $0 < q < m$ (all coefficients R). Then the homomor-*

phism $p_*: H_q(E,E') \to H_q(B,B')$ *is an isomorphism for* $q \le n + m - 1$ *and an epimorphism for* $q = n + m$.

PROOF For the spectral sequence we have

$$E^2_{s,t} \approx H_s(B,B';\, H_t(F)) \approx H_s(B,B') \otimes H_t(F) \oplus H_{s-1}(B,B') * H_t(F)$$

By the hypotheses, $E^2_{s,t} = 0$ if $s < n$ or $0 < t < m$. Therefore, if $q \le n + m - 1$, then $E^2_{s,q-s} = 0$, except possibly for the term $E^2_{q,0}$. It follows that $E^r_{s,q-s} = 0$, except for the term $E^r_{q,0}$, and $E^r_{q,0} \approx E^2_{q,0}$. Therefore $E^\infty_{q,0} \approx E^2_{q,0}$ and $E^\infty_{s,q-s} = 0$ if $s \ne q$. Hence

$$H_q(E,E') \approx H_q(B,B';\, H_0(F)) \approx H_q(B,B')$$

and the isomorphism is induced by p_*.

If $q = n + m$, then $E^2_{s,n+m-s} = 0$ except for the terms $E^2_{n+m,0}$ and $E^2_{n,m}$. Since $E^2_{n+m-r,r-1} = 0$ for $r \ge 2$, it follows that

$$E^\infty_{n+m,0} \approx E^2_{n+m,0} \approx H_{n+m}(B,B';\, H_0(F)) \approx H_{n+m}(B,B')$$

Therefore $p_*(H_{n+m}(E,E')) = H_{n+m}(B,B')$. ∎

We use this to prove the following *homotopy excision theorem.*[1]

5 THEOREM *Let* A, B, *and* $A \cap B$ *be path-connected subspaces of a simple space* X *such that*

(*a*) *Either* $X = \operatorname{int} A \cup \operatorname{int} B$, *or* $X = A \cup B$ *where* A *and* B *are closed subsets of* X *such that* $A \cap B$ *is a strong deformation retract of some neighborhood in* A (*or in* B).

(*b*) $A \cap B$, A, B, *and* X *have isomorphic fundamental groups.*

(*c*) $(A, A \cap B)$ *is n-connected and* $(B, A \cap B)$ *is m-connected, where* $n, m \ge 1$.

Then the homomorphism

$$j_\#: \pi_q(A, A \cap B) \to \pi_q(X,B)$$

induced by the excision map $j: (A, A \cap B) \subset (X,B)$ *is an isomorphism for* $q \le n + m - 1$ *and an epimorphism for* $q = n + m$.

PROOF First we reduce consideration to the case $X = \operatorname{int} A \cup \operatorname{int} B$. If A and B are closed in X and $A \cap B$ is a strong deformation retract of some neighborhood U in B, let $A' = A \cup U$ and observe that A is a strong deformation retract of A'. Furthermore, $A' \cap B = U$, and the inclusion map $(A, A \cap B) \subset (A', A' \cap B)$ is a homotopy equivalence, so that $(A', A' \cap B)$ is n-connected. By the exactness of the homotopy sequence of the triple $(B, A' \cap B, A \cap B)$ and the fact that $(A' \cap B, A \cap B)$ is k-connected for all k, we see that $(B, A' \cap B)$ is m-connected. Note that

$$X = A \cup (B - A) \subset \operatorname{int} A' \cup \operatorname{int} B,$$

[1] A more general form of this theorem can be found in A. L. Blakers and W. S. Massey, The homotopy groups of a triad, II, *Annals of Mathematics,* vol. 55, pp. 192–201, 1952.

and so A' and B satisfy conditions (a), (b), and (c). Since there is a commutative triangle

$$\begin{tikzcd} \pi_q(A, A \cap B) \arrow[rr, "\approx"] \arrow[dr, "j_\#"'] & & \pi_q(A', A' \cap B) \arrow[dl, "j'_\#"] \\ & \pi_q(X,B) & \end{tikzcd}$$

we are reduced to proving that $j'_\#$ has the desired properties.

Similarly, if $A \cap B$ is a strong deformation retract of some neighborhood V in A, let $B' = V \cup B$ and observe that B is a strong deformation retract of B'. Then $A \cap B' = V$, and it follows, as in the case above, that $(A, A \cap B')$ is n-connected and $(B', A \cap B')$ is m-connected. Since $X = (A - B) \cup B$ is contained in int $A \cup$ int B', we see that A and B' satisfy conditions (a), (b), and (c). From the commutativity of the square

$$\begin{tikzcd} \pi_q(A, A \cap B) \arrow[r, "\approx"] \arrow[d, "j_\#"'] & \pi_q(A, A \cap B') \arrow[d, "j''_\#"] \\ \pi_q(X,B) \arrow[r, "\approx"] & \pi_q(X,B') \end{tikzcd}$$

we are reduced to proving that $j''_\#$ has the desired properties.

In either case we have shown that it suffices to prove the theorem under the hypothesis that $X = \text{int } A \cup \text{int } B$, and we make this assumption now. By corollary 8.3.8, there is a fibration $p: E \to X$ such that E is simply connected and $p_\#: \pi_q(E) \approx \pi_q(X)$ for $q > 1$. Let E_A and E_B be the parts of E over A and B, respectively, and note that $E_A \cap E_B$ is the part of E over $A \cap B$. From theorem 7.2.8 it follows that $(E_A, E_A \cap E_B)$ is n-connected and $(E_B, E_A \cap E_B)$ is m-connected. Using (b) and the exactness of the homotopy sequence of a fibration, it is easy to see that $E_A \cap E_B$, E_A, and E_B are all simply connected. Since it is obvious that $E \subset p^{-1}(\text{int } A) \cup p^{-1}(\text{int } B) \subset \text{int } E_A \cup \text{int } E_B$, we have reduced the theorem to the case where all the spaces in question are simply connected by virtue of the commutativity of the square

$$\begin{tikzcd} \pi_q(E_A,E_A \cap E_B) \arrow[r, "\approx"] \arrow[d, "\bar{j}_\#"'] & \pi_q(A, A \cap B) \arrow[d, "j_\#"] \\ \pi_q(E,E_B) \arrow[r, "\approx"] & \pi_q(X,B) \end{tikzcd}$$

Thus, assume $X = \text{int } A \cup \text{int } B$ and that $A \cap B$, A, B, and X are all simply connected. We replace the inclusion map $A \subset X$ by the homotopically equivalent mapping path fibration $p: P \to X$ as in theorem 2.8.9. Then P is the space of paths $\omega: (I,0) \to (X,A)$ in the compact-open topology, and $p(\omega) = \omega(1)$. The fiber F of p over a point $a_0 \in A \cap B$ is the space of paths in X which start in A and end at a_0. If $p': PX \to X$ is the path fibration of all paths in X which end at a_0 and $p'(\omega) = \omega(0)$, then $F = p'^{-1}(A)$. Since PX is contractible, there are isomorphisms

$$\pi_q(X,A) \xleftarrow[\approx]{p'_\#} \pi_q(PX,F) \xrightarrow[\approx]{\partial} \pi_{q-1}(F)$$

Because $X = \operatorname{int} A \cup \operatorname{int} B$, the excision map $j'\colon (B, A \cap B) \subset (X,A)$ induces isomorphisms in homology. It follows from the relative Hurewicz isomorphism theorem and the m-connectedness of $(B, A \cap B)$ that (X,A) is also m-connected. Therefore F is $(m-1)$-connected, and so $H_q(F) = 0$ for $0 < q < m$.

Let $E' = p^{-1}(B)$ and observe that since X is simply connected, the fibration $p\colon P \to X$ is orientable. Since $j_*\colon H_q(A, A \cap B) \approx H_q(X,B)$, it follows that $H_q(X,B) = 0$ for $q < n + 1$. By lemma 4, the homomorphism

$$p_*\colon H_q(P,E') \to H_q(X,B)$$

is an isomorphism for $q \le n + m$ and an epimorphism for $q = n + m + 1$. The map $j\colon (A, A \cap B) \subset (X,B)$ has a lifting $\bar{j}\colon (A, A \cap B) \to (P,E')$, where $\bar{j}(a)$ is the constant path at a for all $a \in A$. There is a commutative triangle

$$\begin{tikzcd} H_q(A, A \cap B) \arrow[r, "\bar{j}_*"] \arrow[dr, "j_*"', "\approx"] & H_q(P,E') \arrow[d, "p_*"] \\ & H_q(X,B) \end{tikzcd}$$

Therefore $\bar{j}_*$ is an isomorphism for $q \le n + m$. Since $\bar{j} \mid A\colon A \to P$ is a homotopy equivalence, it follows from the five lemma that the homomorphism

$$(\bar{j} \mid A \cap B)_*\colon H_q(A \cap B) \to H_q(E')$$

is an isomorphism for $q \le n + m - 1$.

Because $\pi_1(E') \approx \pi_1(F) \approx \pi_2(X,A)$, and the latter group is a quotient group of $\pi_2(X)$ since $\pi_1(A) \approx \pi_1(X)$, we see that E' has an abelian fundamental group. Since $A \cap B$ is simply connected, it follows from the absolute Hurewicz isomorphism theorem that E' is also simply connected. By the Whitehead theorem, the homomorphism

$$(\bar{j} \mid A \cap B)_\#\colon \pi_q(A \cap B) \to \pi_q(E')$$

is an isomorphism for $q \le n + m - 2$ and an epimorphism for $q = n + m - 1$. Since $\bar{j} \mid A\colon A \to E$ is a homotopy equivalence, it follows from the five lemma that the homomorphism

$$\bar{j}_\#\colon \pi_q(A, A \cap B) \to \pi_q(P,E')$$

is an isomorphism for $q \le n + m - 1$ and an epimorphism for $q = n + m$. The result follows from this and the commutativity of the triangle

$$\begin{tikzcd} \pi_q(A, A \cap B) \arrow[r, "\bar{j}_\#"] \arrow[dr, "j_\#"'] & \pi_q(P,E') \arrow[d, "p_\#", "\approx"'] \\ & \pi_q(X,B) \end{tikzcd} \quad \blacksquare$$

It should be noted that the main argument above involved the case where A and B satisfy (*c*), satisfy (*b*) in the stronger form that all the spaces in question are simply connected, and satisfy the condition that $\{A,B\}$ is an excisive couple of subsets of X, which is a weak form of (*a*). It should also be observed that if A and B satisfy condition (a) of theorem 5, then if $(A, A \cap B)$ is n-connected [or $(B, A \cap B)$ is m-connected], it is easy to show that (X,B) is also n-connected [or (X,A) is m-connected]. Furthermore, if A and B satisfy a and c and $A \cap B$ is simply connected, then it follows that A and B are each simply connected and also that X is simply connected. Hence condition b is also satisfied, and theorem 5 is valid in this case.

6 COROLLARY *Let (X,A) be an n-connected relative CW complex, where $n \geq 2$, such that A is m-connected, where $m \geq 1$. Then the collapsing map $k: (X,A) \to (X/A,x_0)$ induces a homomorphism*

$$k_\#: \pi_q(X,A) \to \pi_q(X/A)$$

which is an isomorphism for $q \leq m + n$ and an epimorphism for $q = m + n + 1$.

PROOF Let CA be the unreduced cone over A and regard it as a space whose intersection with X is A. Since A is m-connected and CA is contractible, it follows that (CA,A) is $(m + 1)$-connected. We shall apply theorem 5, with A and B replaced by X and CA, respectively. Since $X \cap CA = A$ is a strong deformation retract of some neighborhood in CA, a of theorem 5 is satisfied. Since A is simply connected and c is also satisfied, it follows, as in the remarks above, that b is satisfied too. Hence the hypotheses of theorem 5 are satisfied, and it follows that $j: (X,A) \subset (X \cup CA, CA)$ induces a homomorphism

$$j_\#: \pi_q(X,A) \to \pi_q(X \cup CA, CA)$$

which is an isomorphism for $q \leq n + m$ and an epimorphism for $q = n + m + 1$. It follows from lemma 7.1.5 that the collapsing map $k': (X \cup CA, CA) \to (X \cup CA, CA)/CA$ is a homotopy equivalence. The result follows from the commutativity of the triangle

$$\begin{tikzcd} \pi_q(X,A) \arrow[r, "j_\#"] \arrow[dr, "k_\#"'] & \pi_q(X \cup CA, CA) \arrow[d, "\approx"', "k'_\#"] \\ & \pi_q(X/A) \end{tikzcd} \qquad \blacksquare$$

7 COROLLARY *Let $f: (X',A') \to (X,A)$ be a relative homeomorphism between relative CW complexes both of which are n-connected, with $n \geq 2$, and such that A' and A are m-connected, with $m \geq 1$. Then f induces an isomorphism*

$$f_\#: \pi_q(X',A') \approx \pi_q(X,A) \qquad q \leq n + m$$

PROOF Let $k': (X',A') \to (X'/A',x_0')$ and $k: (X,A) \to (X/A,x_0)$ be the collapsing maps. Then f induces a homeomorphism $f': X'/A' \to X/A$ such that $f' \circ k' = k \circ f$. Since f' induces isomorphisms of the homotopy groups in all dimensions, the result follows from corollary 6. $\blacksquare$

We use this last result to study the suspension map

$$S: \pi_q(S^n) \to \pi_{q+1}(S^{n+1})$$

in more detail. Since $S^{n+1} = S(S^n)$, there is a characteristic map $\mu'\colon S^n \to \Omega S^{n+1}$ for the path fibration $PS^{n+1} \to S^{n+1}$. From the commutativity of the triangle

$$\begin{tikzcd} \pi_q(S^n) \arrow[rr, "\mu'_\#"] \arrow[dr, "S"'] & & \pi_q(\Omega S^{n+1}) \\ & \pi_{q+1}(S^{n+1}) \arrow[ur, "\approx", "\partial"'] & \end{tikzcd}$$

it suffices to study the map $\mu'_\#$.

Let X^{2n} be the space obtained from $S^n \times S^n$ by identifying (z,z_0) with (z_0,z) for all $z \in S^n$ (where z_0 is a base point of S^n). We regard S^n as imbedded in X^{2n} as the set of points corresponding to $S^n \times z_0$ in $S^n \times S^n$. Then X^{2n} is a CW complex consisting of S^n and a single $2n$-cell attached by a map $\alpha_n\colon S^{2n-1} \to S^n$.

8 LEMMA *There is a map $g\colon X^{2n} \to \Omega S^{n+1}$, where $n \geq 2$, which is a $(3n-1)$-equivalence such that $g \mid S^n = \mu'$.*

PROOF Let $\mu\colon S^n \times \Omega S^{n+1} \to \Omega S^{n+1}$ be the map defined by $\mu(z,\omega) = \omega * \mu'(z)$. By corollary 8.5.8, μ is homotopic to a clutching function for the fibration $PS^{n+1} \to S^{n+1}$. Let $f\colon S^n \times S^n \to \Omega S^{n+1}$ be defined by $f(z,z') = \mu'(z') * \mu'(z)$. There is a commutative diagram

$$\begin{tikzcd} H_{n+1}(C_-S^n,S^n) \otimes H_n(\Omega S^{n+1}) \arrow[r, "\approx"'] \arrow[d, "\partial \otimes 1", "\approx"'] & H_{2n+1}((C_-S^n,S^n) \times \Omega S^{n+1}) \arrow[dr, "\mu_* \partial", "\approx"'] & \\ H_n(S^n) \otimes H_n(\Omega S^{n+1}) \arrow[r, "\approx"'] & H_{2n}(S^n \times \Omega S^{n+1}) \arrow[r, "\mu_*"] & H_{2n}(\Omega S^{n+1}) \\ H_n(S^n) \otimes H_n(S^n) \arrow[u, "1 \otimes \mu'_*", "\approx"'] \arrow[rr, "\approx"'] & & H_{2n}(S^n \times S^n) \arrow[ul, "(1 \times \mu')_*"'] \arrow[u, "f_*"'] \end{tikzcd}$$

Therefore $f_*\colon H_{2n}(S^n \times S^n) \approx H_{2n}(\Omega S^{n+1})$. Since $f \mid S^n \vee S^n$ is homotopic to the map sending (z,z_0) to $\mu'(z)$ and (z_0,z) to $\mu'(z)$, f is homotopic to a map f' such that $f'(z,z_0) = \mu'(z) = f'(z_0,z)$. Then f' defines a map $g\colon X^{2n} \to \Omega S^{n+1}$ such that $g \circ k = f'$, where $k\colon S^n \times S^n \to X^{2n}$ is the quotient map. Then $g \mid S^n = \mu'$, and since $H_n(S^n) \approx H_n(X^{2n})$, $g_*\colon H_n(X^{2n}) \approx H_n(\Omega S^{n+1})$. Since $k_*\colon H_{2n}(S^n \times S^n) \approx H_{2n}(X^{2n})$, it follows that $g_*\colon H_{2n}(X^{2n}) \approx H_{2n}(\Omega S^{n+1})$. The only nontrivial homology groups of X^{2n} are in degrees, 0, n, and $2n$, and in degrees $< 3n$ the only nontrivial homology groups of ΩS^{n+1} are in degrees 0, n, and $2n$. Therefore $g_*\colon H_q(X^{2n}) \approx H_q(\Omega S^{n+1})$ for $q < 3n$. Since $n \geq 2$, X^{2n} and ΩS^{n+1} are both simply connected. By the Whitehead theorem, the homomorphism

$$g_\#\colon \pi_q(X^{2n}) \to \pi_q(\Omega S^{n+1})$$

is an isomorphism for $q < 3n - 1$ and an epimorphism for $q = 3n - 1$. ■

Let $\bar{\alpha}_n\colon (E^{2n},S^{2n-1}) \to (X^{2n},S^n)$ be the characteristic map for the $2n$-cell of X^{2n} corresponding to the attaching map $\alpha_n\colon S^{2n-1} \to S^n$. Then $\bar{\alpha}_n$ is a