

# THE BORSUK-ULAM THEOREM IN REAL-COHESIVE HOMOTOPY TYPE THEORY

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ABSTRACT. Borsuk-Ulam!

## WRITING NOTES

Writing assignments:

- Amelia—section 5
- Chandrika—section 4
- Daniel—sections 2 and 3

Formalizing the cohomology proofs will be determined later.

## 1. INTRODUCTION

## 2. OVERVIEW OF REAL-COHESIVE HOMOTOPY TYPE THEORY

OUTLINE:

- HoTT as foundations
- Interpreting AlgTop theorems in HoTT is obstructed by discontinuous functions
- Relating continuous and discontinuous with flat and sharp, which are borrowed from cohesive topoi
- Formalizing flat and sharp in HoTT + axioms needed, e.g. Rflat
- Connecting sets used in AlgTop with HITs used in HoTT via shape

Homotopy type theory (HoTT) is an expression of a style of mathematics that expands the notion of “identity” to include logical identity, homotopy equivalence, and path connectedness. Experts call this style *Univalence foundations*. And as foundations, there is an ambitious program to encode all of mathematics in homotopy type theory. There is a growing community working to realize these ambitions and this paper belongs to this group.

Our present goal is to bring the classical theory of algebraic topology into the fray, and in particular the Borsuk-Ulam theorem. However, the HoTT approach to algebraic topology comes with an immediate challenge. This is the presence of so many fixed point theorems where, in the course of a proof, the fixed point must be specified precisely, not only up to homotopy. What is the problem with this? Homotopy type theory only works up to homotopy. Compare, for instance, the topological circle

$$\mathbb{S}^1 := \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$$

with the homotopy type theoretical circle defined by a pair of constructors `base` and `loop`: `base = base`. One has infinitely many points that can be described exactly and the other has a single point. Brouwer’s Fixed Point Theorem serves as

a good illustration of this problem. We provide its statement and proof here for reference.

**Theorem 2.1.** *Let  $\mathbb{D}^2$  denote the topological disk  $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$ . Any continuous map  $f: \mathbb{D}^2 \rightarrow \mathbb{D}^2$  has a fixed point.*

*Proof.* Suppose that  $f$  is continuous but does not have a fixed point, hence  $f(x) \neq x$  for all  $x \in \mathbb{D}^2$ . For each  $x \in \mathbb{D}^2$ , draw a ray from  $f(x)$  to  $x$ . This ray intersects the circle in a point we denote by  $s(x)$ . This defines a continuous function  $s: \mathbb{D}^2 \rightarrow \mathbb{S}^1$  with the property that  $s(x) = x$  for all  $x$  on the boundary of  $\mathbb{D}^2$ . That implies that the identity on  $\mathbb{S}^1$  factors as the inclusion  $\mathbb{S}^1 \hookrightarrow \mathbb{D}^2$  followed by  $s$ . Applying the fundamental group function  $\pi_1$  to this factorization gives that the identity on  $\pi_1(\mathbb{S}^1) = \mathbb{Z}$  factors through  $\pi_1(\mathbb{D}^2) = 1$  which is absurd.  $\square$

Note how this proof relied on our precise specification of the point  $s(x)$  on the circle. This point cannot be specified precisely in HoTT. Even if we did work with the only homotopical point on the circle, that is with  $\mathbb{S}^1$ , there is no way to relate  $\mathbb{S}^1$  to  $\mathbb{S}^1$  inside of type theory. Semantically speaking, this involves comparing a topological space with  $\infty$ -groupoids. This is done using the fundamental  $\infty$ -groupoid construction. No such construction exists in HoTT. This is the problem that real-cohesive homotopy theory solves. It does so by proposing to combine two already existing, but previously unrelated, type semantics: topological and  $\infty$ -groupoidal. With this proposal, there are three puzzles to be solved.

- (a) We need to define a model for a *topological  $\infty$ -groupoid*.
- (b) What rules or axioms can we equip HoTT with so that we can compare, for example,  $\mathbb{S}^1$  to  $\mathbb{S}^1$ .
- (c) Topology is incompatible with the law of the excluded middle, which is required to prove these classic fixed point theorems. How can we resolve this?

For a detailed account of real-cohesive homotopy type theory, we point the reader to Shulman’s original paper [Shu18]. At present, we content ourselves with providing a high-level account, which we begin with the following puzzle. Given a

### 3. TRANSLATING BORSUK-ULAM TO HOMOTOPY TYPE THEORY

OUTLINE:

- **Subsection 1.** Give statements for BU-classic, BU-odd, BU-retract) a la wikipedia. The proof strategy: show BU-retract implies BU-odd which is equivalent to BU-classic, then prove BU-retract. Give the proof for BU-retract.
- **Subsection 2.** Translate the classical statement into propositions as types. We want to model classical proof. The failure of contrapositive rule in constructive logic—(not  $q$  implies not  $p$ ) is ( $p$  implies not not  $q$ )—means our proof strategy is BU-retract implies not not BU-odd which is equivalent to not not BU-classic. But not not BU-classic is sharp BU-classic. Prove BU-retract.
- To close out the section, list the ingredients we need to prove BU-retract.

### 4. TOPOLOGICAL AND HOMOTOPICAL REAL PROJECTIVE SPACES

OUTLINE:

- Define  $n$ -disks as both sets and types, the latter which is simply 1, since they're contractible. Show that  $\int \mathbb{D} = \mathbb{D}$
- Define  $n$ -spheres as sets. Use pushouts to glue disks together. Explain why we need to glue with a collar—i.e. the “topology” (as encoded by continuous paths  $\mathbb{R} \rightarrow \mathbf{X}$  of a type  $\mathbf{X}$ ). Show, via Shulman, that  $\int \mathbb{S}^n = \mathbb{S}^n$
- Define  $\mathbb{R}P^n$  as sets using pushouts and collaring. Recall Bulcholtz and Egbert's definition of HIT  $\mathbb{R}P^n$ . Prove that  $\int RRP^n = \mathbb{R}P^n$

## 5. COHOMOLOGY

OUTLINE:

- Subsection 1. Define cohomology for  $\mathbb{Z}/2\mathbb{Z}$  coefficients and the EM-spaces for  $\mathbb{R}P^n$
- **Subsection 2.** Show that we get a commutative graded ring structure for cohomology of any type  $\mathbf{X}$  with  $\mathbb{Z}/2\mathbb{Z}$ -coefficients. Follow Brunerie's thesis.
- **Subsection 3.** Compute  $\mathbb{Z}/2\mathbb{Z}$ -cohomology ring for  $\mathbb{R}P^n$  using Mayer-Vietoris. This needs us to first compute cohomology for disks and spheres.

## 6. THE BORSUK-ULAM THEOREM

OUTLINE:

- The proof is done by this point. Just put it all together and reconnect the dots for the reader.

## REFERENCES

- [BR17] Ulrik Buchholtz and Egbert Rijke. The real projective spaces in homotopy type theory. To appear in LICS'17. arXiv:1704.05770, 2017.
- [Bru16] Guillaume Brunerie. *On the homotopy groups of spheres in homotopy type theory*. PhD thesis, Université de Nice, 2016. arXiv:1606.05916.
- [Shu18] Michael Shulman. Brouwer's fixed-point theorem in real-cohesive homotopy type theory. *Mathematical Structures in Computer Science*, 28(6):856–941, 2018.