$\boxed{\text{Encode - Decode}}$

have some type X defined as a colimit. want to understand $Id_X(x=y)$

$code : X \to X \to Type$ using univ. property of X and show (decode) that it is the identity type.

$$code(x,y) \underset{encode}{\overset{decode}{\rightleftarrows}} Id_X(x,y)$$

We find the encoding /decoding using Id-elim/J/ path induct.

"every equality with free end pts might as well be refl".

$$\begin{aligned} A &\longrightarrow \sum_{x:A}\sum_{y:A}(x=y) \\ a &\longmapsto (a,a,refl) \end{aligned}$$

this is a homotopy equivalence (shrink paths to starting point).

for any $a:A$, $\sum_{x:A}(a=x)$ is contractible by the same argument.

given $P : \prod_{x:A}\prod_{y:A}\prod_{p:x=y} Type$ (this is a type)

to define $f:P$, $f:\prod_{x:A}\prod_{y:A}\prod_{p:x=y} P(x,y,p)$

<u>encode</u>: $\prod_{x:X}\prod_{y:X}\prod_{p:x=y}$ ~~code~~ $code(x,y)$

enough to define $encode(refl) : code(x,x)$

encode(x,x,refl) by case analysis (since X is a colimit)

ex. ~~encode in~~ $X = A + B$

$\mathrm{code} : A+B \to A+B \to \mathrm{Type}$

$\mathrm{code}(\mathrm{inl}(a), \mathrm{inl}(a')) = (a =_A a')$

$\mathrm{code}(\mathrm{inl}(a), \mathrm{inr}(b)) = \emptyset$

$\mathrm{code}(\mathrm{inr}(b), \mathrm{inl}(a)) = \emptyset$

$\mathrm{code}(\mathrm{inr}(b), \mathrm{inr}(b')) = (b =_B b')$

encode(x,x,refl) by case analysis:

$\mathrm{encode}(\mathrm{inl}(a), \mathrm{inl}(a), \mathrm{refl}) = \mathrm{refl}_a : \mathrm{code}(\mathrm{inl}(a), \mathrm{inl}(a))$

$\mathrm{encode}(\mathrm{inr}(b), \mathrm{inr}(b), \mathrm{refl}) = \mathrm{refl}_b : \mathrm{code}(\mathrm{inr}(b), \mathrm{inr}(b))$

<u>decode</u> : $\prod_{x:X} \prod_{y:X} \prod_{p:\mathrm{code}(x,y)} (x=y)$

$\mathrm{decode}(\mathrm{inl}(a), \mathrm{inl}(a'), p) = \mathrm{ap}_{\mathrm{inl}}(p) : \mathrm{inl}(a) = \mathrm{inl}(a')$

where ap_f : paths in domain of f $\to$ paths in target of f

push path forward. $f : \cancel{X} \to \cancel{Y}$ some fcn.

$\mathrm{decode}(\mathrm{inl}(a), \mathrm{inr}(b), p) = \checkmark$ (there are no p's.)

$\underline{\quad\quad}_{b}$ " $\underline{\quad\quad}_{a}$ $= \checkmark$ $\underline{\quad\quad}$ " $\underline{\quad\quad}$

$\mathrm{decode}(\mathrm{inr}(b), \mathrm{inr}(b')) = \mathrm{ap}_{\mathrm{inr}}(p)$

want $\prod_{x:X} \prod_{y:X} \prod_{p:x=y}$ $\mathrm{decode}(\mathrm{encode}(p)) = p$
homotopy.

suffices: assume p is refl $y \equiv x$. for refl $\mathrm{decode}(\mathrm{encode}(\mathrm{refl}))$
$= \mathrm{refl}$ on the nose.

$$\begin{tikzcd} A \arrow[r, hook, "f"] \arrow[d, "g"'] & B \arrow[d, "i"] \\ C \arrow[r, "j"'] & X \end{tikzcd}$$

f injective, A, B, C sets. 1) is X a set?

2) is j injective?

3) more generally, what are Ids of a pushout of an embedding.

$$\begin{tikzcd} S^1 \times I \arrow[r, hook] \arrow[d, "\begin{array}{c}\text{orient}\\\text{double}\\\text{cover}\end{array}"'] & \mathbb{D}^2 \arrow[d] \\ \text{Möbius} \arrow[r] & X \end{tikzcd}$$

$X \overset{?}{=} \mathbb{RP}^2$ yes topologically. but as a groupoid, is this a set?

[*wrong correction appears later

$$\text{code}(\text{inl}(b), \text{inl}(b')) = \sum_{a, a' : A} (b =_B f(a)) \times (b' =_B f(a')) \times (g(a) = g(a'))$$

$$\text{code}(\text{inr}(c), \text{inr}(c')) = (c =_C c')$$ b/c f ~~g~~ is injective

$$\text{code}(\text{inl}(b), \text{inr}(c)) = \sum_{a : A} (b = f(a)) \times (c = g(a))$$

$$\text{code}(\text{inr}(c), \text{inl}(b)) = \sum_{a : A} (c = g(a)) \times (b = f(a))$$

$$\text{ap}_{\text{code}}(\text{inr}(c), \text{glue}(a)) = \prod_{c : C} \prod_{a : A} \text{code}(\text{inr}(c), \text{inl}(f(a))) = \text{code}(\text{inr}(c), \text{inr}(g(a)))$$

this is b/c f is an injection.

$$= \prod_{c : C} \prod_{a : A} \left(\sum_{a' : A} (c = g(a)) \times (f(a) = f(a')) \right.$$

$$\cancel{\sum_{a' : A} (c = g(a)) \times} \overset{\shortparallel}{(c = g(a))}$$

$\overset{\phi}{\to}$ given $\gamma : (c =_C g(a'))$ $\quad p : (f(a) =_B f(a'))$

$a =_A a'$ in A b/c f is an injection, so $g(a) =_C g(a')$

so we get a path $\gamma : (c =_C g(a))$ (path composition)

$\overset{\psi}{\leftarrow}$ given $\gamma : (c =_C g(a))$ let $a' = a$ in A.

then we get $p : f(a) =_B f(a')$ by choosing

$p = \mathrm{refl}_{f(a)}$.

~~$\phi(a', \gamma, p) = g \circ ap$~~

$$\phi(a', \gamma, p) = \gamma \circ \overline{ap_g(ap_f^{-1}(p))}$$

$$\psi(\gamma) = (a, \gamma, \mathrm{refl}_{f(a)})$$

Alternatively: $\displaystyle\sum_{a':A} (c =_C g(a')) \times (f(a) =_B f(a'))$

$\simeq \sum (c = g(a')) \times (a = a')$ since f is injective.

$$\simeq \sum_{(a', r) : \boxed{\sum_{a':A} (a = a')}} (c = g(a'))$$

contractible.

$\simeq (c = g(a))$

by symmetry can define

$ap_{code}(\mathrm{glue}(a), \mathrm{inr}(c))$

Now define

$$ap_{code}(\mathrm{glue}(a), \mathrm{inl}(b)) = \prod_{b:B} \prod_{a:A} \mathrm{code}(\mathrm{inl}(f(a)), \mathrm{inl}(b))$$

$$\| \qquad \mathrm{code}(\mathrm{inr}(g(a)), \mathrm{inl}(b))$$

$$= \prod_{a_1 : A} \prod_{b : B} \left(\sum_{\substack{a_2^* : A \\ a_3 : A}} (b =_B f(a_2)) \times (\cancel{b}^{f(a_1)} =_B f(a_3)) \times (g(a_2) = g(a_3)) \right.$$

IS need to prove this equivalence

$$\left. \sum_{a_4 : A} \cancel{(c = g(a_4))} (g(a_1) = g(a_4)) \times (b = f(a_4)) \right)$$

proof of equivalence:

$$\sum_{\substack{a_2 : A \\ a_3 : A}} (b =_B f(a_2)) \times (f(a_1) =_B f(a_3)) \times (g(a_2) = g(a_3))$$

$$\simeq \sum_{\substack{a_2 : A \\ a_3 : A}} (b =_B f(a_2)) \times (a_1 =_A a_3) \times (g(a_2) = g(a_3))$$

since f is inject.

$$\simeq \sum_{\substack{a_2 : A \\ \cancel{a_3 : A}}} (b =_B f(a_2)) \times (g(a_2) = g(a_3))$$

$$r : \sum_{a_3 : A} (a_1 =_A a_3)$$

$$\simeq \sum_{a_2 : A} (b =_B f(a_2)) \times (g(a_2) = g(a_1))$$

$$\simeq \sum_{a_4 : A} (b = f(a_4)) \times (g(a_1) = g(a_4)) \qquad \square$$

Now define $ap_{code}(glue(a), glue(a))$

~~$ap_{code}(glue(a), glue(a)) = code($~~

we have shown each of these equalities.

~~ap_{code}~~

$$\begin{tikzcd}
code(inl(f(a)), inl f(a')) \arrow[r, equal, "X"] \arrow[d, equal, "W"'] & code(inl f(a), inr g(a')) \arrow[d, equal, "Y"] \\
code(inr(g(a)), inl f(a')) \arrow[r, equal, "Z"'] & code(inr g(a), inr(g(a')))
\end{tikzcd}$$

$ap_{code}(glue(a), glue(a))$ shows that this square of equalities commute.

X:
$$ap_{code}(inl(b), glue(a)) : \prod_{a:A} \prod_{b:B} code(inl(b), inl f(a)) \simeq code(inl(b), inr g(a))$$

Y:
$$ap_{code}(glue(a), inr(c)) : \prod_{a:A} \prod_{c:C} code(inl f(a), inr(c)) \simeq code(inr g(a), inr(c))$$

W:
$$ap_{code}(glue(a), inl(b)) : \prod_{a:A} \prod_{b:B} code(inl f(a), inl(b)) \simeq code(inr(g(a)), inl(b))$$

Z:
$$ap_{code}(inr(c), glue(a)) : \prod_{a:A} \prod_{c:C} code(inr(c), inl f(a)) \simeq code(inr(c), inr g(a))$$