Borsuk - Ulam

Statement: Given a continuous map $f: S^2 \longrightarrow \mathbb{R}^2$, $\exists$ a pt. $x \in S^2$ s.t. $f(x) = f(-x)$

topology proof: If not, we get a map $g: S^2 \to S^1$ given by

$$g(x) = \frac{f(x) - f(-x)}{|f(x) - f(-x)|}$$

Define a loop $\eta: I \to S^2$ $\eta(s) = (\cos(2\pi s), \sin(2\pi s), 0)$

write $h: I \longrightarrow S^1$ for $h = g \circ \eta$

we have $g(-x) = -g(x)$ so $h(s+\frac{1}{2}) = -h(s)$

$$\begin{tikzcd} & & \mathbb{R} \arrow[d, "\text{universal covering map.}"] \\ I \arrow[urr, "\tilde{h}"] \arrow[rr, "h"] \arrow[dr, "\eta"'] & & S^1 \\ & S^2 \arrow[ur, "g"'] & \end{tikzcd}$$

start lifting at 0 WLOG

This gives $\tilde{h}(s+\frac{1}{2}) = \tilde{h}(s) + \frac{q}{2}$ with $q \in \mathbb{Z}$ odd.

$s \in [0, \frac{1}{2}]$

This q may depend on s continuously

but $q: I \to \mathbb{Z}$ odd must be constant (b/c of continuity)

$\tilde{h}(1) = \tilde{h}(\frac{1}{2}) + \frac{q}{2} = \tilde{h}(0) + q$

so h represents q times a generator for $\pi_1(S^1)$.

Since q is odd, $h: I \longrightarrow S^1$ is not path nullhomotopic

but $\cancel{h: I \longrightarrow}$ $h = g \circ \eta$ and η is path null homotopic.

$\Rightarrow h$ is null homotopic $\Rightarrow\Leftarrow$ $\square$

<u>more general statement</u>: for any continuous map $g: S^n \to \mathbb{R}^n$, $\exists$ a pt. x s.t. $g(x) = g(-x)$.

<u>prf</u>. let $f(x) = g(x) - g(-x)$.

This is an odd map. we need to show $f(x) = 0$ for some x.

If not, replace $f(x)$ by $\frac{f(x)}{|f(x)|} = h(x)$

So $h : S^n \to S^{n-1}$ this is still odd.
Restrict h to the equator $S^{n-1} \subseteq S^n$. The restriction has odd degree by some Prop. This restriction is null homotopic via the restriction of h to one of the hemispheres bounded by S^{n-1} equator.
this cannot be of odd degree $\square$

An Ω spectrum is $E : \mathbb{N} \to \mathrm{Type}$
$e : \prod_{n:\mathbb{N}} E(n)$

$$\prod_{n:\mathbb{N}} E(n) \simeq_* \underbrace{\Omega E(n+1)}_{(e_n = e_n)}$$

X pointed type $(x_0 : X)$ $\tilde{H}^n(X;E) = \tilde{H}^n(X_+;E)$
this has the axioms of cohomology.

$\tilde{H}^n(X;G) = \tilde{H}^n(X;KG)$
Eilenberg Maclane space

$\boxed{\text{Cohomology in HoTT}}$

Assume $S^n \xrightarrow{h} S^{n-1}$ $n > 2$ ~~is odd.~~
odd

$$\begin{tikzcd} S^n \arrow[r, "h"] \arrow[d] & S^{n-1} \arrow[d] \\ \mathbb{R}P^n \arrow[r, "h'"] & \mathbb{R}P^{n-1} \end{tikzcd}$$

quotient by $\mathbb{Z}/2\mathbb{Z}$ action

$h'_* : \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ h'_* is an isomorphism because of oddness of h