THE BORSUK-ULAM THEORM IN REAL-COHESIVE HOMOTOPY TYPE THEORY

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Abstract. Borsuk-Ulam!

Writing Notes

Writing assignments:
- Amelia—section 5
- Chandrika—section 4
- Daniel—sections 2 and 3

Formalizing the cohomology proofs will be determined later.

1. Introduction

2. Overview of real-cohesive homotopy type theory

OUTLINE:
- HoTT as foundations
- Intepreting AlgTop theorems in HoTT is obsructed by discontinuous functions
- Relating continuous and discontinuous with flat and sharp, which are borrowed from cohesive topoi
- Formalizing flat and sharp in HoTT + axioms needed, e.g. Rflat
- Connecting sets used in AlgTop with HITs used in HoTT via shape

3. Translating Borsuk-Ulam to homotopy type theory

OUTLINE:
- **Subsection 1.** Give statements for BU-classic, BU-odd, BU-retract) a la wikipedia. The proof strategy: show BU-retract implies BU-odd which is equivalent to BU-classic, then prove BU-retract. Give the proof for BU-retract.
- **Subsection 2.** Translate the classical statement into propositions as types. We want to model classical proof. The failure of contrpositive rule in constructive logic—(not q implies not p) is (p implies not not q)—means our proof strategy is BU-retract implies not not BU-odd which is equivalent to not not BU-classic. But not not BU-classic is sharp BU-classic. Prove BU-retract.
- To close out the section, list the ingredients we need to prove BU-retract.

4. Topological and homotopical real projective spaces

OUTLINE:

- Define n-disks as both sets and types, the latter which is simply 1, since they're contractible. Show that $\int \mathbb{D} = \mathtt{D}$
- Define n-spheres as sets. Use pushouts to glue disks together. Explain why we need to glue with a collar—i.e. the "topology" (as encoded by continuous paths $\mathbb{R} \to \mathtt{X}$ of a type $\mathtt{X}$. Show, via Shulman, that $\int \mathbb{S}^n = \mathtt{S}^n$
- Define $\mathbb{R}\mathrm{P}^n$ as sets using pushouts and collaring. Recall Bulcholtz and Egbert's definition of HIT $\mathtt{RP}^n$. Prove that $\int RRP^n = \mathtt{RP}^n$

5. Cohomology

OUTLINE:

- Subsection 1. Define cohomology for $\mathbb{Z}/2\mathbb{Z}$ coefficients and the EM-spaces for $\mathbb{R}\mathrm{P}^n$
- **Subsection 2.** Show that we get a commutative graded ring structure for cohomology of any type $\mathtt{X}$ with $\mathbb{Z}/2\mathbb{Z}$-coefficients. Follow Brunerie's thesis.
- **Subsection 3.** Compute $\mathbb{Z}/2\mathbb{Z}$-cohomology ring for $\mathbb{R}\mathrm{P}^n$ using Mayer-Vietoris. This needs us to first compute cohomology for disks and spheres.

6. The Borsuk-Ulam theorem

OUTLINE:

- The proof is done by this point. Just put it all together and reconnect the dots for the reader.

References

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[Bru16] Guillaume Brunerie. *On the homotopy groups of spheres in homotopy type theory*. PhD thesis, Université de Nice, 2016. arXiv:1606.05916.

[Shu17] Michael Shulman. Brouwer's fixed-point theorem in real-cohesive homotopy type theory. To appear in MSCS. arXiv:1509.07584, 2017.