THE BORSUK-ULAM THEORM IN REAL-COHESIVE HOMOTOPY TYPE THEORY

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Abstract. Borsuk-Ulam!

Writing Notes

Writing assignments:
- Amelia—section 5
- Chandrika—section 4
- Daniel—sections 2 and 3

Formalizing the cohomology proofs will be determined later.

1. Introduction

2. Overview of real-cohesive homotopy type theory

OUTLINE:
- HoTT as foundations
- Interpreting AlgTop theorems in HoTT is obstructed by discontinuous functions
- Relating continuous and discontinuous with flat and sharp, which are borrowed from cohesive topoi
- Formalizing flat and sharp in HoTT + axioms needed, e.g. Rflat
- Connecting sets used in AlgTop with HITs used in HoTT via shape

Homotopy type theory (HoTT) is an expression of a style of mathematics that expands the notion of "identity" to include logical identity, homotopy equivalence, and path connectedness. Experts call this style *Univalence foundations*. And as foundations, there is an ambitious program to encode all of mathematics in homotopy type theory. There is a growing community working to realize these ambitions and this paper belongs to this group.

Our present goal is to bring the classical theory of algebraic topology into the fray, and in particular the Borsuk-Ulam theorem. However, the HoTT approach to algebraic topology comes with an immediate challenge: the presence of so many fixed point theorems where, in the course of a proof, the fixed point must be specified precisely, not only up to homotopy. What is the problem with this? It is that homotopy type theory only works up to homotopy. Compare, for instance, the topological circle

$$\mathbb{S}^1 := \left\{(x,y) \in \mathbb{R}^2 | x^2 + y^2 = 1\right\}$$

with the homotopy type theoretical circle defined by a pair of constructors `base` and `loop`: `base` = `base`. One has infinitely many points that can be described exactly and the other has a single point. Brouwer's Fixed Point Theorem illustrates this problem nicely. We provide its statement and proof here for reference.

Theorem 2.1. *Let $\mathbb{D}^2$ denote the topological disk $\left\{(x,y) \in \mathbb{R}^2 | x^2 + y^2 \leq 1\right\}$. Any continuous map $f\colon \mathbb{D}^2 \to \mathbb{D}^2$ has a fixed point.*

Proof. Suppose that f is continuous but does not have a fixed point, hence $f(x) \neq x$ for all $x \in \mathbb{D}^2$. For each $x \in \mathbb{D}^2$, dray a ray from $f(x)$ to x. This ray intersects the circle in a point we denote by $s(x)$. This defines a continuous function $s\colon \mathbb{D}^2 \to \mathbb{S}^1$ with the property that $s(x) = x$ for all x on the boundary of $\mathbb{D}^2$. That implies that the identity on $\mathbb{S}^1$ factors as the inclusion $\mathbb{S}^1 \hookrightarrow \mathbb{D}^2$ followed by s. Appying the fundamental group function π_1 to this factorization gives that the identity on $\pi_1\left(\mathbb{S}^1\right) = \mathbb{Z}$ factors through $\pi_1\left(\mathbb{D}^2\right) = 1$ which is absurd. $\square$

Note how this proof relied on our precise specification of the point $s(x)$ on the circle. This point cannot be specified precisely in HoTT. Even if we did work with the only homotopical point on the circle, that is with $\mathtt{S}^1$, there is no way to relate $\mathbb{S}^1$ to $\mathtt{S}^1$ inside of type theory. Semantically speaking, this involves comparing a topological space with ∞-groupoids. This is done using the fundamental ∞-groupoid construction. No such construction exists in HoTT. This is the problem that real-cohesive homotopy theory solves. It does so by proposing to combine two already existing, but previously unrelated, type semantics: topological and ∞-groupoidal. With this proposal, there are three puzzles to be solved.

(a) We need to define a model for a *topological ∞-groupoid*.
(b) What rules or axioms can we equip HoTT with so that we can compare, for example, $\mathbb{S}^1$ to $\mathtt{S}^1$.
(c) Topology AMELIA: Type theory? is incompatible with the law of the excluded middle, which is required to prove these classic fixed point theorems. How can we resolve this?

Shulman's original paper on real-cohesive HoTT [**?**] discusses the solution to these puzzles in detail. Presently, we are content to simply say that the Lawvere's theory of cohesion offers a solution. Of course, we need to adapt cohesion to homotopy type theory and we leave the description of this to Shulman, but we do provide a high-level description of the role that cohesion plays.

A *category of cohesive space* is a pair of categories equipped with a string of adjunctions

$$\begin{tikzcd}
\text{Spaces} \arrow[d, shift right=6, "f_!"'] \arrow[d, leftarrow, shift right=2, "f^*"'] \arrow[d, shift left=2, "f_*"'] \arrow[d, leftarrow, shift left=6, "f^!"'] \\
\text{Sets}
\end{tikzcd}$$

with $f_! \dashv f^* \dashv f_* \dashv f^!$ and such that $f_!$ preserves finite products. As Lawvere puts it, the objects of Sets should be thought of as *abstract sets* which

> ... may by conceived of as a bag of dots which are devoid of properties apart from mutual distinctness [**?**].

On the other hand, the objects of Spaces should be thought of as abstract sets together with a sort of *cohesion* between the "dots". For our purposes, we think of cohesion as a topology though, in reality, this definition above axiomatizes the various forms that cohesion may take, each functor playing a different role. The $f_!$ functor tells us which points are "stuck" together through the cohesion by returning a set of connected components. The f^* functor endows a set with the discrete topology. The f_* functor forgets the topology of a space. The $f^!$ functor endows a set with the codiscrete topology on a set. From this string of adjoints, we get another adjoint string $\int \dashv \flat \dashv \sharp$ on Spaces comprised of the **shape operation** $\int \coloneqq f^* f_!$, the **flat operation** $\flat \coloneqq f^* f_*$, and the **sharp operation** $\sharp \coloneqq f^! f_*$.

To see how the axiomatic cohesion addresses the above puzzles, we will speak in the language of sets and categories instead of type theory. In other words, we restrict out attention to the semantics of the relevant type theory.

To solve the first puzzle, constructing a topological ∞-groupoid, we ask first that Spaces and Sets are toposes. A cohesive topos is also a *local and locally connected topos* which can be constructed using sheaves on a site that satisfy certain properties. By expanding this construction to the $(\infty, 1)$-category, we can obtain cohesive $(\infty, 1)$-toposes using ∞-sheaves on a site as shown by Schreiber [**?**]. The objects of a resulting cohesive $(\infty, 1)$-topos are precisely the topological groupoids we seek.

The second puzzle involves comparing a space with its homotopy type. Again modifying axiomatic cohesion, we replace the categories Sets and Spaces with the $(\infty, 1)$-categories of Spaces and $\infty -$ Groupoids. We also replace the functors with ∞-functors. The validity of this rests on work by Schreiber [**?**]. In this setup, applying $\int$ to a space returns the fundamental ∞-groupoid, an excellent proxy for the homotopy type.

Axiomatic cohesion also provides a solution to the final puzzle: the failure of the *continuous* excluded middle. Given that we are working with topological objects, we require that excluded middle holds continuously, but in general it does not. Given a space X and subspace U, there is no continuous inverse to the inclusion $U + (X \setminus U) \to X$ because, even though the underlying sets are the same, the topologies are different. If we can introduce discontinuous functions $X \to U + (X \setminus U)$, then we can find a discontinuous inverse to the inclusion, therefore obtaining a modified, "discontinuous" law of the excluded middle. Hence, the existence of a law of the excluded middle in our context hinges on the introduction of such discontinuous functions. To this end, recall that $\flat$ retopolozes discretely and $\sharp$ retopologizes codiscretely. If $\hom(X, Y)$ is the space of continuous functions from X to Y, then both $\hom(\flat X, Y)$ and $\hom(X, \sharp Y)$ contain the discontinuous, by which we mean not necessarily continuous, functions from X to Y.

Moving towards syntax means introducing into homotopy type theory the constructors that mirror the semantics of $\int$, $\flat$, and $\sharp$. Upon adding these constructors, we obtain *cohesive homotopy type theory.* The "real" part of name comes from an additional axiom included so that we can capture the topology syntactically using continuous paths from the reals. This axiom states

> A crisp type $\mathtt{A}$ is discrete if and only if the function that returns a constant path $\mathtt{A} \to (\mathbb{R} \to \mathtt{A})$ is an equivalence.

Calling $\mathtt{A}$ a crisp variable means that we perform constructions on it without regarding the topology, such as defining maps $\flat\mathtt{A} \to \mathtt{Y}$ or $\mathtt{X} \to \sharp\mathtt{A}$. When including this axiom along with the syntactic versions of shape, flat, and sharp, we get *real-cohesive homotopy type theory.*

3. Translating Borsuk-Ulam to homotopy type theory

OUTLINE:

- **Subsection 1.** Give statements for BU-classic, BU-odd, BU-retract) a la wikipedia. The proof strategy: show BU-retract implies BU-odd which is equivalent to BU-classic, then prove BU-retract. Give the proof for BU-retract.
- **Subsection 2.** Translate the classical statement into propositions as types. We want to model classical proof as much as possible. Notes on intuitionistic logic: contrapositive holds one way, i.e. $p \to q$ entails $-q \to -p$, and the inverse isn't true. The rule $p \to -q$ is equivalent to $q \to -p$ in intuitionistic logic. Due to this, our proof strategy is BU-retract implies not not BU-odd which is equivalent to not not BU-classic. But not not BU-classic is sharp BU-classic. Prove BU-retract.
- To close out the section, list the ingredients we need to prove BU-retract.

3.1. **The classical Borsuk-Ulam Theorem.** In classical algebraic topology, the Borsuk-Ulam theorem is given as the three equivalent statements listed below. The first of these statements is taken as the standard presentation.

Statement 1. *If $f : \mathbb{S}^n \to \mathbb{R}^n$ is a continuous function, then there exists a point $x \in \mathbb{S}^n$ such that $f(x) = f(-x)$, where $-x$ is the antipodal point of x.*

Statement 2. *Any odd continuous function $g : \mathbb{S}^n \to \mathbb{R}^n$ must have a zero.*

Statement 3. *There is no odd continuous function $h : \mathbb{S}^n \to \mathbb{S}^{n-1}$.*

Of the many approaches to proving Borsuk-Ulam, we share one that parallels our approach to proving the HoTT version. That is, show that each of the three statements are truly equivalent, then prove Statement 3. Though the classical version of this approach is well-documented, we sketch it below.

To see that Statement 2 implies Statement 1 note that given a continuous function f, one can construct a continuous function $g(x) = f(x) - f(-x)$ that is, by definition, odd. To see the opposite implication, start with a continuous odd function g which by assumption has some $x \in \mathbb{S}^n$ such that $g(x) = g(-x)$. Then, since g is odd, $g(-x) = -g(x)$ holds for this x, which implies that $g(x) = -g(x)$, hence $g(x) = 0$.

Statement 2 implies 3. Indeed, any odd continuous $h\colon \mathbb{S}^n \to \mathbb{S}^{n-1} \subseteq \mathbb{R}^n$ must have a zero, but $0 \notin \mathbb{S}^{n-1}$, so no such h exists. Conversely, suppose that an odd continuous $g\colon \mathbb{S}^n \to \mathbb{R}^n$ has no zero, then the map $h(x) = g(x)/|g(x)|$ defines an odd continuous function $h\colon \mathbb{S}^n \to \mathbb{S}^{n-1}$.

Now having the equivalnce of the three statements, proving them proceeds in three cases: $n = 1$, $n = 2$, and $n \geq 3$.

For $n = 1$, we prove Statement 2. Given an odd continuous $g\colon \mathbb{S}^1 \to \mathbb{R}^1$ and a chosen x, then either $g(x) = 0$ in which case we are done or, without loss of generality, $g(x) > 0$ which means that $g(-x) < 0$. Apply the intermediate value theorem.

Finally, holding $n \geq 3$, let $h : \mathbb{S}^n \to \mathbb{S}^{n-1}$ be a continuous odd function. Since h is odd, we can identify antipodal points and arrive at the induced continuous function $h' : \mathbb{RP}^n \to \mathbb{RP}^{n-1}$. Note that the fundamental groups of $\mathbb{RP}^n$ and $\mathbb{RP}^{n-1}$ are both $\mathbb{Z}/2\mathbb{Z}$. Thus, if the homomorphism of fundamental groups induced by h' is nontrivial, it will also be an isomorphism. Since the fundamental group of $\mathbb{RP}^n$ is $\mathbb{Z}/2\mathbb{Z}$, it has a nontrivial element. Therefore there must exist a representative loop γ in $\mathbb{RP}^n$ that is essential. The loop γ can then be lifted to $\mathbb{S}^n$, resulting in a path p between antipodal points in $\mathbb{S}^n$ (since $\mathbb{S}^n$ is the universal cover of $\mathbb{RP}^n$). Applying the odd map h to this path p, we get a path $h(p)$ between two antipodal points in $\mathbb{S}^{n-1}$, which then projects to a loop $h'(\gamma)$ in $\mathbb{RP}^{n-1}$. If $h'(\gamma)$ was null homotopic, the null homotopy would lift to $\mathbb{S}^{n-1}$, contradicting that $h(p)$ is a path between antipodal points in $\mathbb{S}^{n-1}$.

Using the Hurewicz homomorphism and universal coefficient theorem, we get an induced graded ring homomorphism on cohomology with $\mathbb{Z}/2\mathbb{Z}$ coefficients, $h^* : H^*(\mathbb{RP}^n;\mathbb{Z}/2\mathbb{Z}) \leftarrow H^*(\mathbb{RP}^{n-1};\mathbb{Z}/2\mathbb{Z})$, where $H^*(\mathbb{RP}^n;\mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}[a]/\langle a^{n+1}\rangle$ and $H^*(\mathbb{RP}^{n-1};\mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}[b]/\langle b^n\rangle$ (AMELIA: Why was this again?). Since $h' : \mathbb{RP}^n \to \mathbb{RP}^{n-1}$ induces an isomorphism on the fundamental groups, and the Hurewicz homomorphism is an isomorphism at the $i = 1$ level, the induced homomorphism h^* must send b to a. This implies that b^n is sent to a^n, which is a contradiction since $b^n = 0$ and $a^n \neq 0$. Thus, no such map continuous odd map $h : \mathbb{S}^n \to \mathbb{S}^{n-1}$ can exist.

3.2. **Translating the classical Borsuk-Ulam Theorem into HoTT.** We start by restating the three Borsuk-Ulam statements above in a homotopy type theoretic language. In our restated versions, we include the $\flat$ and $\sharp$ modes from real-cohesive HoTT to allow us to prove Borsuk-Ulam discontinuously.

Statement 4. *For any* $f : \mathbb{S}^n \to \mathbb{R}^n$, $\sum_{x:\mathbb{S}^n} f(x) = f(-x)$ *is inhabited.*

Statement 5. *Any odd continuous function* $g : \mathbb{S}^n \to \mathbb{R}^n$, $\sum_{x:\mathbb{S}^n} g(x) = 0$ *is inhabited.*

Statement 6. *The type* $\prod_{h:\mathbb{S}^n\to\mathbb{S}^{n-1}} \sum_{x:\mathbb{S}^n} h(x) = h(-x)$ *is not inhabited.*

To show these are equivalent is more subtle than the classical case because we are using intuitionistic logic. Statements 4 and 5 are equivalent for the same reason as in the classical case. The different lies in showing the equivalence between Statements 5 and 6.

Let us denote Statement 5 by O and Statement 6 by $\neg R$. Therefore, R is the statement that the type $\prod_{h:\mathbb{S}^n\to\mathbb{S}^{n-1}} \sum_{x:\mathbb{S}^n} h(x) = h(-x)$ is inhabited. The first task in showing the equivalence between O and $\neg R$ is that $O \Rightarrow \neg R$, or equivalently that $R \Rightarrow \neg O$. To remind readers how this equivalence works in intuitionistic logic, we have that

$$(O \Rightarrow \neg R) = (\neg O \vee \neg R) = (\neg R \vee \neg O) = (R \Rightarrow \neg O).$$

With the logic set, $R \Rightarrow \neg O$ follows from the same arguement as in the classical case.

4. Topological and homotopical real projective spaces

OUTLINE:

- Define n-disks as both sets and types, the latter which is simply 1, since they're contractible. Show that $\int\mathbb{D} = \mathtt{D}$
- Define n-spheres as sets. Use pushouts to glue disks together. Explain why we need to glue with a collar—i.e. the “topology” (as encoded by continuous paths $\mathbb{R} \to \mathtt{X}$ of a type $\mathtt{X}$. Show, via Shulman, that $\int\mathbb{S}^n = \mathtt{S}^n$
- Define $\mathbb{RP}^n$ as sets using pushouts and collaring. Recall Bulcholtz and Egbert's definition of HIT $\mathtt{RP}^n$. CHANDRIKA: First draft done

- Prove that $\int \mathbb{RP}^n = \mathtt{RP}^n$, using an idea like "shape preserves pushouts".
- Show that the homology of $\mathbb{RP}^n$ is what we want it to be (see Prop 5.1, in Amelia's section).

4.1. **Disks, Spheres and their shapes.**

4.2. **Defining $\mathbb{RP}^n$.** We define $\mathbb{RP}^n$ using push outs, tautological bundles, spheres, and an inductive process, similar to the work of Rijike [].

The base case for our induction is $\mathbb{RP}^1$.

Definition 4.1. We define $\mathbb{RP}^1$ as the push out seen in the diagram below, where $\mathbb{I}$ is the open unit interval, and the maps are defined by $a_1(-1,x) = \frac{x}{4}$ and $a_1(1,x) = \frac{x}{4} + \frac{3}{4}$, and $b_1(\pm 1, x) := (0, a(\pm 1, x))$.

$$\begin{tikzcd}
\mathbb{S}^0 \times \mathbb{I} \arrow[r, hook, "a_1"] \arrow[d, hook, "b_1"] & \mathbb{I} \arrow[d] \\
\{0\} \times \mathbb{I} \arrow[r] & \mathbb{RP}^1
\end{tikzcd}$$

It is helpful to think of a_1 as an inclusion of a tubular neighbourhood of the boundary of $[0,1] \subseteq \mathbb{R}$, with the boundary points deleted (so that it is indeed a subset of the open interval $\mathbb{I}$). We call this a *thickened boundary*. In fact, b_1 is also an inclusion of a thickened boundary. The image of b_1 is the deleted tubular neighbourhood of the boundary of $\{0\} \times [0,1] \subseteq \{0\} \times \mathbb{R}$.

Definition 4.2. Let A, B be two cohesive types, with A an open subset of B. Let $\bar{A}$ be the closure of A in B. Suppose $\bar{A}$ has a non-empty boundary, $\partial \bar{A}$. Let N' be a tubular neighbourhood of $\partial \bar{A}$. Let $N := N' \setminus A^C$, where A^C is the complement of A. Even though A has no boundary, we call N a *thickened boundary of A, relative to B.*

We also need the following $\mathbb{R}$-bundle as part of our base case.

Definition 4.3. (Möbius strip) Let $\mathtt{U}_{\mathbb{R}}$ be the universe of types homeomorphic to $\mathbb{R}$. The *tautological bundle* $\mathcal{M}^1 : \mathbb{RP}^1 \to \mathtt{U}_{\mathbb{R}}$ is defined by the constructors $\mathbb{I} \mapsto \mathbb{R}$, $\{0\} \times \mathbb{I} \mapsto \mathbb{R}$, and the equivalence $\lambda x.x$ for every point in $\{0\} \times \mathbb{I}$, and the equivalence $\lambda x. - x$ for every point in $\{1\} \times \mathbb{I}$.

We note that the homeomorphism $\tau : \mathbb{R} \to \mathbb{I}$, $\tau x.\arctan(x) + \frac{1}{2}$, can be used to define a thickened boundary of $\mathcal{M}^1$. We define $\tau(\mathcal{M}^1)$ as a $\mathbb{I}$-bundle given by the composition $\tau \circ \mathcal{M}^1$. Note that there is a natural inclusion $\tau(\mathcal{M}^1) \hookrightarrow \mathcal{M}^1$ induced by the inclusion of $\mathbb{I}$ into $\mathbb{R}$. Any thickened boundary of $\tau(\mathcal{M}^1)$ relative to $\mathcal{M}^1$ has a pre-image by τ. With a slight abuse of notation, we call this a thickened boundary of $\mathcal{M}^1$. CHANDRIKA: Is this correct language? (Where $\mathcal{M}^1$ is used as the map defining a bundle and also as the bundle itself?)

From here, we proceed inductively with the definitions below. The definition of $\mathbb{RP}^n$ depends on the definition of $\mathcal{M}^{n-1}$, which depends on $\mathbb{RP}^{n-1}$, which depends on $\mathcal{M}^{n-2}$, and so on.

Definition 4.4. $\mathbb{RP}^n$ is defined (up to homeomorphism) as the following pushout.

$$\begin{tikzcd} \mathbb{S}^{n-1} \times \mathbb{I} \arrow[r, hook, "a_n"] \arrow[d, hook, "b_n"] & \mathbb{D}^n \arrow[d] \\ \mathcal{M}^{n-1} \arrow[r] & \mathbb{RP}^n \end{tikzcd}$$

where a_n and b_n are both inclusions of $\mathbb{S}^{n-1} \times \mathbb{I}$ as thickened boundaries of $\mathcal{M}^{n-1}$ and $\mathbb{D}^n$ (relative to $\mathbb{R}^n$). The choice of thickened boundary does not change the homeomorphism type of $\mathbb{RP}^n$.

CHANDRIKA: Are others worried about only defining it up to homeomorphism? My thinking was that in type theory, things are only really ever defined up to equivalence, because of univalence.

Definition 4.5. The tautological bundle $\mathcal{M}^n : \mathbb{RP}^n \to \mathtt{U}_{\mathbb{R}}$ is defined by the constructors $\mathbb{D}^n \mapsto \mathbb{R}$ and $\mathcal{M}^{n-1} \mapsto \mathbb{R}$, and the automorphism of $\mathbb{R}$, $\lambda x.x$ for every point of $\mathbb{S}^{n-1} \times \mathbb{I}$.

4.3. **Showing $\int\mathbb{RP}^n = \mathtt{RP}^n$.** Our first observation is that shape is a left adjoint operator, and so it preserves colimits. Therefore we have the following commutative diagram.

$$\begin{tikzcd} \int(\mathbb{S}^{n-1} \times \mathbb{I}) \arrow[r] \arrow[d] & \int(\mathbb{D}^n) \arrow[d] \\ \int(\mathcal{M}^{n-1}) \arrow[r] & \int(\mathbb{RP}^n) \end{tikzcd}$$

Based on work of Shulman [**?**], we have $\int(\mathbb{S}^{n-1} \times \mathbb{I}) = S^{n-1}$ and $\int(\mathbb{D}^n)$ is the unit type. Next, we find $\int(\mathcal{M}^{n-1})$ so that we may understand $\int\mathbb{RP}^n$. We begin by showing that $\mathcal{M}^n$ is a push out for all n. Note that $\mathcal{M}^n = \sum_{x\in\mathbb{RP}^n} f(x)$ where f is the map taking each point in $\mathbb{RP}^n$ to the fibre over it (see Definition 4.5). Consider the following commutative diagram, where the bottom square in the cube is the push out defining $\mathbb{RP}^n$, and f is the map defining the bundle $\mathcal{M}^n$ (in the top, front, right position in the cube). The maps s and q are induced by the push out. The top square consists of bundles over each of the types in the bottom square with fibres are types homeomorphic to $\mathbb{R}$. The map λ above s is defined by $\lambda(x, y).(s(x), y)$. The remaining maps in on the top square are similarly defined by each of the maps they lie over.

$$\begin{tikzcd}
 & \sum_{x\in\mathbb{S}^n\times\mathbb{I}} f\circ s\circ b_n(x) \arrow[rr] \arrow[dl] \arrow[dd, "p_4"] & & \sum_{x\in\mathbb{D}^n} f\circ q(x) \arrow[dl] \arrow[dd, "p_2"] \\
\sum_{x\in\mathcal{M}^{n-1}} f\circ s(x) \arrow[rr, "\lambda"] \arrow[dd, "p_3"] & & \sum_{x\in\mathbb{RP}^n} f(x) \arrow[dd, "p_1"] & \\
 & \mathbb{S}^n\times\mathbb{I} \arrow[rr, "a_n"] \arrow[dl, "b_n"] & & \mathbb{D}^n \arrow[dl, "q"] \\
\mathcal{M}^{n-1} \arrow[rr, "s"] & & \mathbb{RP}^n \arrow[dr, "f"] & \\
 & & & \mathtt{U}_{\mathbb{R}}
\end{tikzcd}$$

Because we are in a Van Kampen category, it suffices to show that the vertical faces of the cube are pullbacks in order to prove that the top face of the cube is a pushout. CHANDRIKA: is the language in this paragraph correct?

4.3.1. *Showing that the front face of is a pullback.* We would like to show the following.

$$\sum_{x\in\mathcal{M}^{n-1}} f\circ s(x) = \sum_{a\in\sum_{c\in\mathbb{RP}^n} f(c)} \quad \sum_{b\in\mathcal{M}^{n-1}} (s(b)=p_1(a))$$

In order to do this we define a homotopy equivalence between the left hand side and the right hand side. Let

$$g : \sum_{x\in\mathcal{M}^{n-1}} f\circ s(x) \longrightarrow \sum_{a\in\sum_{c\in\mathbb{RP}^n} f(c)} \quad \sum_{b\in\mathcal{M}^{n-1}} (s(b)=p_1(a))$$

be defined by $g(x,y) := (\lambda(x,y), p_3(x,y), \mathrm{refl}_{s\circ p_3(x,y)})$. Note that by the definition of λ, and the fact that p_3 is projection to the first coordinate, the right hand side simplifies to $((s(x),y),x,\mathrm{refl}_{s(x)})$. This is in the codomain because the diagram commutes. Define

$$h : \sum_{a\in\sum_{c\in\mathbb{RP}^n} f(c)} \quad \sum_{b\in\mathcal{M}^{n-1}} (s(b)=p_1(a)) \longrightarrow \sum_{x\in\mathcal{M}^{n-1}} f\circ s(x)$$

by $h((c,r),b,k) := (b,r)$. The right hand side is a type in the codomain because $r : f(c)$, and k is a witness that $(s(b)=c)$, so $r : f\circ s(b)$. Now we check that $g\circ h$ and $h\circ g$ are homotopic to identity. Indeed,

$$\begin{aligned}
g\circ h((c,r),b,k) &= g(b,r) \\
&= ((s(b),r),b,\mathrm{refl}_{s(b)}) \\
&= ((c,r),b,\mathrm{ref;}_{s(b)}),
\end{aligned}$$

since $s(b)=c$ (as witnessed by k). It remains to show that $\mathrm{refl}_{s(b)}=k$, but this must be true because $\mathbb{RP}^n$ is a set.

4.4. **The $\mathbb{Z}$-homology of $\mathbb{RP}^n$.**

5. Cohomology

OUTLINE:

- **Subsection 1.** Define cohomology for $\mathbb{Z}/2\mathbb{Z}$ coefficients and the EM-spaces for $\mathbb{RP}^n$ AMELIA: Done except for some questions/references/anxiety and maybe a proof of prop 5.4
- **Subsection 2.** Show that we get a commutative graded ring structure for cohomology of any type `X` with $\mathbb{Z}/2\mathbb{Z}$-coefficients. Follow Brunerie's thesis.
- **Subsection 3.** Compute $\mathbb{Z}/2\mathbb{Z}$-cohomology ring for $\mathbb{RP}^n$ using Mayer-Vietoris. To do this we first need to compute cohomology for disks and spheres.

5.1. Cohomology and EM-spaces for $\mathbb{RP}^n$.

We follow a similar construction for cohomology as found in [**?**], modifying their construction with $\mathbb{Z}$ coefficients to have coefficients in $\mathbb{Z}/2\mathbb{Z}$. In order to define cohomology, we must first define Eilenberg-MacLane spaces $K(\mathbb{Z}/2\mathbb{Z}, n)$. Eilenberg-MacLane spaces $K(G, n)$ were defined for arbitrary group G by Finster and Licata in [**?**].

We give a construction of $\mathbb{RP}^n$ in Section 4, which shows that $\left|\left|\mathbb{RP}^2\right|\right|_1$ will have the homotopy groups necessary to be the foundation of our Eilenberg-MacLane spaces.

Proposition 5.1.

$$\pi_n(\left|\left|\mathbb{RP}^2\right|\right|_1) = \begin{cases} 0 & \textit{for } n = 0 \\ \mathbb{Z}/2\mathbb{Z} & \textit{for } n = 1 \\ 0 & \textit{for } n > 1, \end{cases}$$

Proof. AMELIA: We discussed this on Oct 1 2019 and thought this would go in Chandrika's section □

AMELIA: (3/15/22) We need to say how we're viewing $\mathbb{Z}/2\mathbb{Z}$ as a space below since we later refer to this as a functor landing in spaces

Definition 5.2. For $n : \mathbb{N}$, the type **Eilenberg-MacLane space** $K(\mathbb{Z}/2\mathbb{Z}, n)$ is the n-truncated and $(n-1)$-connected pointed type defined by

$$K(\mathbb{Z}/2\mathbb{Z}, n) := \begin{cases} \mathbb{Z}/2\mathbb{Z} & \text{for } n = 0 \\ \left|\left|\Sigma^{n-1}\mathbb{RP}^2\right|\right|_n & \text{for } n \geq 1, \end{cases}$$

where Σ^{n-1} indicates the reduced suspension $n-1$ times.

Proposition 5.3. *This definition of $K(\mathbb{Z}/2\mathbb{Z}, n)$ does indeed define an Eilenberg-MacLane Space.*

Proof. This follows from Prop 5.1 and AMELIA: the fact (that I don't have a HoTT reference for) that EM spaces are unique. □

AMELIA: Finster and Licata constructed $K(G, 1)$ in a particular way. My concern is that I am not sure how that impacts which of their results transfer over. I think that their results in section 4 and 5 mean that as long as the $K(G, 1)$ has the right groups, using suspension and truncation works. In which case, it doesn't matter that we constructed $K(G, 1)$ differently than they did, their results should all port over. I'd really like a second opinion on this. AMELIA: On October 17 2019 we talked about this question of what results port over. we said: According to Mike, in classic topology, EM spaces are unique up to unique isomorphism, however in HoTT they are equivalent up to homotopy but the homotopy may not be unique. Our thought was that our K(G,1) should be homotopy equivalent to theirs, so by univalence they're equal as types. So, their results for their construction of K(G,1) should apply to ours.

As noted in [**?**], Eilenberg-MacLane spaces have the following delooping property. This property allows us to construct a spectrum for ordinary cohomology with $\mathbb{Z}/2\mathbb{Z}$ coefficients.

Proposition 5.4.

$$K(\mathbb{Z}/2\mathbb{Z}, n) \simeq \Omega K(\mathbb{Z}/2\mathbb{Z}, n+1)$$

AMELIA: **** G showed equivalence of K_n and ΩK_{n+1}. I thought this was in Finster and Licata. But now I can only find a mention of it as a fact, not something proven. May need to go back and prove this.

Given this construction of the Eilenberg-MacLane spaces, we define cohomology in the following way.

Definition 5.5. For a type $\mathtt{X}$ and $n : \mathbb{N}$, the **n-th cohomology group of $\mathtt{X}$ with coefficients in** $\mathbb{Z}/2\mathbb{Z}$ is the type

$$H^n(\mathtt{X}; \mathbb{Z}/2\mathbb{Z}) := ||\mathtt{X} \to K(\mathbb{Z}/2\mathbb{Z}, n)||_0 .$$

AMELIA: B also defines reduced cohomology as well. I'm not sure we need that. G proved a reduced MV sequence, so we may need it if we also use MV. Leaving it out for now.

5.2. **Commutative Graded Ring Structure.** For a ring R, the multiplication operation on R can be viewed as a homomorphism $R \otimes R \to R$, where $R \otimes R$ is the tensor product of the underlying abelian groups. Our strategy for describing the graded ring structure is to show that $||K(R,n) \wedge K(R,m)||_{n+m} = K(R \otimes R, n+m)$, then lift the ring multiplication of $R \otimes R \to R$ to a cup product on cohomology with R-coefficients.

AMELIA: 3/8/22: Probably shouldn't go into more detail on this tensor in the finished write up, but for my own sanity: For abelian groups G and H, $G \otimes H$ is the quotient $G \times H / \sim$ where

- $(g_1, h) + (g_2, h) \sim (g_1 + g_2, h)$
- $(g, h_1) + (g, h_2) \sim (g, h_1 + h_2)$.

So if we define $\cdot : R \otimes R \to R$ by $\cdot(a, b) = ab$, then the distributive axiom of rings and bilinearity of tensor gives us a group homomorphism because:

$$\cdot((a_1, b) + (a_2, b)) = \cdot(a_1 + a_2, b) = (a_1 + a_2)b = a_1 b + a_2 b = \cdot(a_1, b) + \cdot(a_2, b)$$

Proposition 5.6. $||K(R,n) \wedge K(R,m)||_{n+m} = K(R \otimes R, n+m)$ *for* $n, m \geq 1$.

AMELIA: is $=$ the right symbol here? probably yes bc univalence?

Proof. Recall that $K(R \otimes R, n+m)$ is the unique up to homotopy space having the property that

$$\pi_k(K(R \otimes R, n+m)) \cong \begin{cases} 0, & k \neq n+m \\ R \otimes R, & k = n+m \end{cases}.$$

AMELIA: on 12/5, we decided to go with isomorphism symbols for groups instead of equalities. Univalence most likely means that we could write equals, but there doesn't seem to be a need.

Thus, by showing that $||K(R,n) \wedge K(R,m)||_{n+m}$ satisfies this property, we have the desired equivalence.

In order to establish the property for $k = n+m$, it is a known result (see [**?**, Prop 19.60]) that for $n, m \geq 1$

$$\pi_{n+m}(||K(R,n) \wedge K(R,m)||_{n+m}) \cong R \otimes R.$$

In the case where $k > m+n$, $||K(R,n) \wedge K(R,m)||_{n+m}$ is truncated to be an $(n+m)$-type. So,

$$\pi_k(||K(R,n) \wedge K(R,m)||_{n+m}) \cong 0,$$

for $k > n+m$.

Finally, note that $||K(R,n) \wedge K(R,m)||_{n+m}$ is $(n+m-1)$-connected, which follows from $K(R,k)$ being $(k-1)$-connected, as seen in [**?**, Prop 4.3.1]. Thus,

$$\pi_k(||K(R,n) \wedge K(R,m)||_{n+m}) \cong 0$$

for $k \leq n+m-1$.

We have shown the desired property and arrive at the homotopy equivalence

$$||K(R,n) \wedge K(R,m)||_{n+m} \simeq K(R \otimes R, n+m).$$

Invoking the univalence axiom, we conclude that

$$||K(R,n) \wedge K(R,m)||_{n+m} = K(R \otimes R, n+m).$$

□

Next, we construct the ring structure on cohomology. The strategy is to define the necessary structures and properties on Eilenberg-MacLane spaces (Def. 5.2) and then lift those to cohomology.

The addition and additive inverse operations follow directly from [**?**], and satisfy the properties to make $K(\mathbb{Z}/2\mathbb{Z}, n)$ an abelian group. When $n = 0$, addition and additive inverses are defined as usual in $\mathbb{Z}/2\mathbb{Z}$. For $n > 0$, addition and additive inverses are defined via the equivalence $K(\mathbb{Z}/2\mathbb{Z}, n) \simeq \Omega K(\mathbb{Z}/2\mathbb{Z}, n+1)$ using the composition and inverse of loops respectively. We restate the operation properties below.

Proposition 5.7. [**?**, Prop. 5.1.4] *The maps* $+ : K(\mathbb{Z}/2\mathbb{Z}, n) \times K(\mathbb{Z}/2\mathbb{Z}, n) \to K(\mathbb{Z}/2\mathbb{Z}, n)$ *and* $- : K(\mathbb{Z}/2\mathbb{Z}, n) \to K(\mathbb{Z}/2\mathbb{Z}, n)$ *have the following properties for all* $n : \mathbb{N}$ *and* x*,* y*,* $z : K(\mathbb{Z}/2\mathbb{Z}, n)$*:*

$$x + 0 = x,$$
$$0 + x = x,$$
$$x + (-x) = 0,$$
$$(-x) + x = 0,$$
$$(x + y) + z = x + (x + z),$$

and

$$x + y = y + x.$$

We now define the ring operation multiplication. From the multiplication $\mathbb{Z}/2\mathbb{Z}\otimes\mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$, we induce a multiplication $K(\mathbb{Z}/2\mathbb{Z}\otimes\mathbb{Z}/2\mathbb{Z}, n) \to K(\mathbb{Z}/2\mathbb{Z}, n)$ by applying the functor

$$K(-,n)\colon \mathbf{Ring} \to \mathbf{Ho}(\mathbf{Top}_*).$$

AMELIA: Do we know that $K(-,n)$ is a functor in HoTT? It is a functor, but I'm not sure we ever proved it and I don't think it's explicitly stated in Finster-Licata.

As we showed in Proposition 5.6,

$$||K(R,n) \wedge K(R,m)||_{n+m} = K(R\otimes R, n+m)$$

for any ring R, and so we can treat our multiplication as

$$\widehat{\mu}\colon ||K(\mathbb{Z}/2\mathbb{Z},n) \wedge K(\mathbb{Z}/2\mathbb{Z},m)||_{n+m} \to K(\mathbb{Z}/2\mathbb{Z}, n+m).$$

However, we want the domain of multiplication to be $K(\mathbb{Z}/2\mathbb{Z},n)\times K(\mathbb{Z}/2\mathbb{Z},m)$, without the truncation. To get this, we precompose $\widehat{\mu}$ by several canonical arrows resulting in the composite

$$K(\mathbb{Z}/2\mathbb{Z},n)\times K(\mathbb{Z}/2\mathbb{Z},m) \xrightarrow{\mathtt{proj}} K(\mathbb{Z}/2\mathbb{Z},n)\wedge K(\mathbb{Z}/2\mathbb{Z},m) \xrightarrow{|-|_{n+m}} \cdots$$
$$\to ||K(\mathbb{Z}/2\mathbb{Z},n) \wedge K(\mathbb{Z}/2\mathbb{Z},m)||_{n+m} \xrightarrow{\widehat{\mu}} K(\mathbb{Z}/2\mathbb{Z}, n+m)$$

that we call *multiplication* μ. The proceeding definition only works when $n, m \geq 1$. In the special case where n or m is 0, we use the fact that $K(\mathbb{Z}/2\mathbb{Z},0) = \mathbb{Z}/2\mathbb{Z}$ and we define μ as one would expect: $\mu(x,0) = \mu(0,x) = 0$ and $\mu(x,1) = \mu(1,x) = x$.

We will use this multiplication μ to induce the cup product

$$\smile\colon H^n(X)\times H^m(X) \to H^{n+m}(X)$$

on cohomology.

Recall that we define the n-th cohomology with $\mathbb{Z}/2\mathbb{Z}$-coefficients to be $H^n(\mathtt{X};\mathbb{Z}/2\mathbb{Z}) = ||\mathtt{X} \to K(\mathbb{Z}/2\mathbb{Z},n)||_0$. This is the standard definition in homotopy type theory, rather than singular cochains, because using singular cochains is not a continuous process.

Define $\smile(|\alpha|,|\beta|))$, for $\alpha\colon X \to_* K(\mathbb{Z}/2\mathbb{Z},n)$ and $\beta\colon X \to K(\mathbb{Z}/2\mathbb{Z},m)$, to be the truncation of the pairing $\langle\alpha,\beta\rangle$ followed by μ:

$$\smile(|\alpha|,|\beta|)) : ||X \to K(\mathbb{Z}/2\mathbb{Z},n)\times K(\mathbb{Z}/2\mathbb{Z},m) \to K(\mathbb{Z}/2\mathbb{Z},n+m)||_0.$$

Thus $\smile(|\alpha|,|\beta|))$ type-checks. It remains to show the usual ring properties hold for $H^n(X)$.

We prove the distributive property similarly to Proposition 5.1.8 in [**?**].

Proposition 5.8. *The multiplication map* $\mu : K(\mathbb{Z}/2\mathbb{Z},n)\times K(\mathbb{Z}/2\mathbb{Z},m) \to K(\mathbb{Z}/2\mathbb{Z},n+m)$ *distributes over addition.*

Proof. Let $a : K(\mathbb{Z}/2\mathbb{Z},n)$ and $b, c : K(\mathbb{Z}/2\mathbb{Z},m)$. We must show that $\mu(a, b+c) = \mu(a,b)+\mu(a,c)$.

First, consider the case where $m = 0$. Since $K(\mathbb{Z}/2\mathbb{Z},0) = \mathbb{Z}/2\mathbb{Z}$, b and c are each either 0 or 1. Note that since μ is induced by the multiplication map $\mathbb{Z}/2\mathbb{Z}\otimes\mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$, $\mu(x,0) = \mu(0,x) = 0$. So, if $b = 0$, it follows from the properties in Proposition 5.7 that

$$\mu(a,b+c) = \mu(a,0+c) = \mu(a,c) = \mu(a,0)+\mu(a,c) = \mu(a,b)+\mu(a,c).$$

Distributivity holds similarly in the case where $c = 0$. If $b = c = 1$, then

$$\mu(a,b)+\mu(a,c) = \mu(a,1)+\mu(a,1) = a+a,$$

which equals 0 because composing a loop with itself in $\mathbb{RP}^2$ results in a trivial loop. Thus when $m = 0$, μ distributes over addition.

Now consider $m > 0$. We prove distributivity by showing there is a filler of the following diagram. (We abbreviate $K(\mathbb{Z}/2\mathbb{Z}, i)$ by K_i to save space.)

$$\begin{tikzcd}
K_n \wedge (K_m \times K_m) \arrow[rr, "id\wedge +"] \arrow[d, "p"] & & K_n \wedge K_m \arrow[d, "\widehat{\mu}"] \\
(K_n \wedge K_m) \times (K_n \wedge K_m) \arrow[r, "\widehat{\mu}\times\widehat{\mu}"] & K_{n+m} \times K_{n+m} \arrow[r, "+"] & K_{n+m}
\end{tikzcd}$$

The map p on the left sends $\texttt{proj}(a, (b, c))$ to $(\texttt{proj}(a, b), \texttt{proj}(a, c))$.

AMELIA: 3/22/22: Leaning on G's thesis this much seems like lazy writing... but I also didn't want to just re-write a bunch of bits from his thesis. This is coming from the proof of prop 4.3.1 and prop 3.2.8(which has multiple steps)

As shown in Propositions 4.3.1 and 3.2.8 of [?], if A and B are two pointed types such that A is r-connected and B is s-connected, then map $i : A \vee B \to A \times B$ is $(r + s)$-connected. Since $K(\mathbb{Z}/2\mathbb{Z}, m)$ is $(m - 1)$-connected, $i : K(\mathbb{Z}/2\mathbb{Z}, m) \vee K(\mathbb{Z}/2\mathbb{Z}, m) \to K(\mathbb{Z}/2\mathbb{Z}, m) \times K(\mathbb{Z}/2\mathbb{Z}, m)$ is $(2m - 2)$-connected. Then the map

$$f : K(\mathbb{Z}/2\mathbb{Z}, n)\wedge(K(\mathbb{Z}/2\mathbb{Z}, m) \vee K(\mathbb{Z}/2\mathbb{Z}, m)) \to K(\mathbb{Z}/2\mathbb{Z}, m)\wedge(K(\mathbb{Z}/2\mathbb{Z}, m) \times K(\mathbb{Z}/2\mathbb{Z}, m))$$

given by

$$f(\texttt{proj}(a, \texttt{inl}(b))) := \texttt{proj}(a, (y, 0)) \text{ and } f(\texttt{proj}(a, \texttt{inr}(c))) := \texttt{proj}(a, (0, c))$$

is $(n + 2m - 2)$-connected.

AMELIA: need to finish this proof

□

To show 'graded commutativity', it suffices to show standard commutativity because of the $\mathbb{Z}/2\mathbb{Z}$-coefficients. AMELIA: (3/15/22) need to add a prop showing commutativity

5.3. **Computing the Cohomology Ring of $\mathbb{RP}^n$.**

6. The Borsuk-Ulam theorem

OUTLINE:

- The proof is done by this point. Just put it all together and reconnect the dots for the reader.