We show the corners are propositions:

$$\mathrm{code}(\mathrm{inl}\, f(a), \mathrm{inl}\, f(a')) = \sum_{\bar{a}, \bar{a}' : A} (f(a) = f(\bar{a})) \times (f(a') = f(\bar{a}')) \times (g(\bar{a}) =_C g(\bar{a}'))$$

$$\simeq \sum_{\bar{a}, \bar{a}' : A} (a = \bar{a}) \times (a' = \bar{a}') \times (g(\bar{a}) =_C g(\bar{a}')) \quad \text{by } f \text{ inj.}$$

$$\simeq (g(a) =_C g(a')) \quad \text{this is a prop. b/c } C \text{ is a set.}$$

$$\mathrm{code}(\mathrm{inl}(f(a)), \mathrm{inr}\, g(a')) = \sum_{\bar{a} : A} (f(a) = f(\bar{a})) \times (g(a') = g(\bar{a}))$$

$$\simeq (g(a') = g(a)) \quad \text{this is a prop.}$$

$$\mathrm{code}(\mathrm{inr}(g(a)), \mathrm{inl}\, f(a')) = \sum_{\bar{a} : A} (g(a) = g(\bar{a})) \times (f(a') = f(\bar{a}))$$

$$\simeq (g(a) = g(a')) \quad \text{this is a prop}$$

$$\mathrm{code}(\mathrm{inr}(g(a)), \mathrm{inr}(g(a'))) \simeq (g(a) = g(a')) \quad \text{this is a prop.}$$

then the square must commute. □

code(inl(b), inl(b')) is wrong. refl_b is missing if b is not in the image of f.

To correct/redefine code(inl b, inl b'), we should take it as a pushout over $b = b' \ \sum_{a:A} f(a) = b$

$$\begin{tikzcd}
\sum_{a:A} (b = f(a)) \times (b' = f(a)) \arrow[r, "\gamma"] \arrow[d, "\lambda"] & (b = b') \arrow[d, "\mathrm{inl}_Y"] \\
\sum (b =_B f(a)) \times (b' =_B f(a')) \times (g(a) =_C g(a')) \arrow[r, "\mathrm{inr}_Y"'] \arrow[ur, phantom, "\ulcorner"] & \mathrm{code}(\mathrm{inl}\, b, \mathrm{inl}\, b')
\end{tikzcd}$$

Y is a proposition since only way for $(b = b')$ and $\sum b = f(a) \ldots g\ldots$ to be populated is if upper left is ~~$\emptyset$~~. not empty.

This ~~calculate~~ new defn should push through previous calculations because ap_{code} ~~is~~ only references b in $\mathrm{Im} f$, for which the definition is unchanged.

$\boxed{\text{Define Encode}}$ ~~for x : inl(b)~~)

$\mathrm{encode}(\mathrm{inl}\, b, \mathrm{inl}\, b, \mathrm{refl}) := \cancel{\mathrm{refl}}\ \mathrm{refl}_b)$

$\mathrm{encode}(\mathrm{inr}\, c, \mathrm{inr}\, c, \mathrm{refl}) := \cancel{\mathrm{refl}}\ \mathrm{refl}_c$

$\mathrm{encode}(\mathrm{inl}(f(a)), \mathrm{inr}\, g(a), \mathrm{glue}(a)) = (\mathrm{refl}_{f(a)}, \mathrm{refl}_{g(a)})$

might not be needed because $\mathrm{code}(\mathrm{inl}\, b, \mathrm{inl}\, b')$ is a proposition

now generate by composition and homotopy.

$\boxed{\text{Define Decode}}$

$$\mathrm{decode} : \mathrm{code}(\mathrm{inr}(c), \mathrm{inr}(c')) \to \mathrm{Id}(\mathrm{inr}(c), \mathrm{inr}(c'))$$
$$p \longmapsto ap_{\mathrm{inr}}(p)$$

$$\mathrm{code}(\mathrm{inl}(b), \mathrm{inr}(c)) \longrightarrow \mathrm{Id}(\mathrm{inl}(b), \mathrm{inr}(c))$$
$$(p, q) \longmapsto ap_{\mathrm{inr}}(q^{-1}) \cdot \mathrm{glue}(a) \cdot ap_{\mathrm{inl}}(p)$$

by symmetry $\mathrm{code}(\mathrm{inr}(c), \mathrm{inl}(b)) \to \mathrm{Id}(\mathrm{inr}(c), \mathrm{inl}(b))$

$\mathrm{code}(\mathrm{inl}(b), \mathrm{inl}(b')) \to \mathrm{Id}(\mathrm{inl}\, b, \mathrm{inl}\, b')$

$$\alpha : (b = b') \longrightarrow \mathrm{Id}(\mathrm{inl}\, b, \mathrm{inl}\, b')$$
$$p \longmapsto \mathrm{ap}_{\mathrm{inl}}(p)$$

$$\beta : \sum_{a,a'} (b = f(a)) \times (b' = f(a')) \times (g(a) = g(a')) \longrightarrow \mathrm{Id}(\mathrm{inl}\, b, \mathrm{inl}\, b')$$
$$(p, q, r) \longmapsto \mathrm{ap}_{\mathrm{inl}}(q) \cdot \mathrm{glue}_{a'}^{-1} \cdot \mathrm{ap}_{\mathrm{inr}}(r) \cdot \mathrm{glue}_a \cdot \mathrm{ap}_{\mathrm{inl}}(p)$$

check that this is well defined:

$\lambda(a, p, q) = (a, a, p, q, \mathrm{refl})$

$\gamma(a, p, q) = p \cdot q^{-1}$

$\cancel{\beta(\lambda(a, p, q)) = \beta(}$

$$\beta(\lambda(a, \mathrm{refl}, \mathrm{refl})) = \beta(a, a, \mathrm{refl}, \mathrm{refl}, \mathrm{refl})$$
$$= \mathrm{ap}_{\mathrm{inl}}(\mathrm{refl}) \cdot \mathrm{glue}(a)^{-1} \cdot \mathrm{ap}_{\mathrm{inr}}(\mathrm{refl}) \cdot \mathrm{glue}(a) \cdot \mathrm{ap}_{\mathrm{inl}}(\mathrm{refl})$$
$$= \mathrm{refl} \cdot \mathrm{glue}(a)^{-1} \cdot \cancel{\mathrm{ap}_{\mathrm{inr}}}(\mathrm{refl}) \cdot \mathrm{glue}\, a^{-1} \cdot \mathrm{refl}$$
$$= \mathrm{refl}$$

$$\alpha(\gamma(a, \mathrm{refl}, \mathrm{refl})) = \alpha(\mathrm{refl} \cdot \mathrm{refl}^{-1})$$
$$= \alpha(\mathrm{refl})$$
$$= ap_{inl}(\mathrm{refl})$$
$$= \mathrm{refl} \quad \checkmark$$

To finish defining decode, need to define

$ap_{decode}(\text{glue} \ldots)$ much like two page turns ago

for the defn. of code.

After this, must check encode/decode are inverses.