HoTT = type theory + homotopy interpretation

formal logic
computer science

"=" of type theory as homotop

types ~~of~~ as-spaces

Q1. Can we do homotopy theory in type theory?

2011: Can we define a sphere?

2013: Book with $\pi_3(S^2)$

2017: • Localization and completion @ primes ①

• Formalizing synthetic homology ②

Q2. How does HoTT compare to other foundational systems and why do we care?

→ Building models of HoTT in higher categories.

$\{\text{type theories}\} \underset{\text{syntax}}{\overset{\text{semantics}}{\rightleftarrows}} \{\text{higher categories}\}$

HoTT ⇢ elementary ∞-toposes ④

want this correspondance

2010: model in sSet

2011: conjecture that there is a model in ∞-topos

what is {type theories}

what is that?

New type theories

• HoTT ⟷ types-as-(∞, 0) categories

• ??? ⟷ types-as-(∞, 1) categories

Directed type theory ⑤

Cubical Type Theory

Id-type of X = pathspace of X
"="

$[0,1] \to X$ paths.

$[0,1]^n \to X$ cubes

→ Build type theory implementing I and its powers

CCHM: CTT 6

(Differential) Cohesive HoTT

$S^1 = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2=1\}$

S^1:

base: S^1

loop: base. base

topology vs. homotopy theory

Q3. How to bring $\underbrace{\text{topology}}_{\text{point set topology}}$ into HoTT

- Modalities
- Brouwer's fixed pt. Thm in Cohesive HoTT
  What else? 7

Modalities in HoTT Egbert

propositions-as-types. modal operator is an operator on types.

monadic   comonadic

we study idempotent modalities

Voevodsky notion of a type X being contractible.

there is a term of type

$$X \text{ is contractible} \Longleftrightarrow \mathrm{isContr}(X) :\equiv \underbrace{\Sigma_{x:X} \Pi_{y:X}}_{\text{the paths vary continuously.}} \underbrace{x = y}_{\text{paths from } x \text{ to } y}$$

there is a term of type

$$X \text{ is a mere proposition} \Longleftrightarrow \mathrm{isProp}(X) :\equiv \Pi_{(x,y)} x = y$$

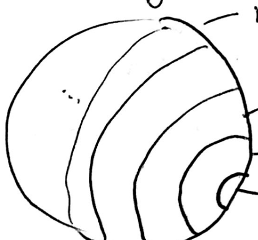
examples of propositions:

- Any contractible type
- The empty type
- The types isContr(X) and isProp(X) for any type X.

A type X is a set if there is a term of mere proposition

$$\Pi_{(x,y:X)} \mathrm{isProp}(x = y)$$

schematic diagram of a universe



n-types (homotopy groups of degree above n are trivial)

sets 0-types

propositions (-1)-types

contractible types

each "layer" in the previous picture is a sub-universe

see HoTT Book Ch. 7.

Defn a modal operator is a function $\bigcirc : \mathcal{U} \to \mathcal{U}$ from a universe to itself.

a modal unit we mean a family of functions $\eta^{\bigcirc} : \prod_{\{A:\mathcal{U}\}} A \to \bigcirc A$

a reflective subuniverse is a family

$$\mathrm{isModal} : \mathcal{U} \to \mathrm{Prop}$$

a reflective subuniverse is Σ-closed ~~to~~ if whenever isModal(X) and isModal(PX) for all $x : X$ we have $\mathrm{isModal}(\Sigma_{(x:X)} P(x))$

let Q be a mere proposition. The closed modality determined by Q consists of

- the subuniverse of types A satisfying $Q \to \mathrm{isContr}\,A$
- the modal operator $A \mapsto Q * A$ ie., the pushout (homotopy pushout)

$$\begin{tikzcd}
Q \times A \arrow[r] \arrow[d] & A \arrow[d] \\
Q \arrow[r] & Q * A
\end{tikzcd}$$

- the modal unit is the map $A \to Q * A$ in the pushout square

• It is Σ-closed since the dependent

If $U, V \subseteq \mathbb{R}$ and $\underbrace{U \cup V = \mathbb{R}}$ and $U \cap V = \emptyset$

means there is a homeo from $U \sqcup V$ to $\mathbb{R}$.

then $U = \emptyset$ or $V = \emptyset$

$\flat$ - discrete
$\sharp$ - codiscrete

~~P~~ $U, V : \mathbb{R} \to \mathrm{Prop}$.

because $U \cap V = \emptyset$, $U \cup V = U \sqcup V \text{"="} U + V$" why?

$\Rightarrow$ have a fcn $f : U + V \to 1 + 1$

$$f(\mathrm{Inl}(U)) = \mathrm{inl}(*)$$
$$f(\mathrm{Inr}(V)) = \mathrm{inr}(*)$$

If A is discrete $(A \to B) = (A \to \flat B)$ = all set maps from A to B

If A, B are discrete then $A + B$ is discrete:

proof

$$\begin{tikzcd} \flat(A+B) \arrow[r] & A+B \end{tikzcd}$$

$$\begin{tikzcd} & & A \arrow[dl] \arrow[dll, bend right] \\ \flat(A+B) \arrow[r] & A+B & \\ & & B \arrow[ul] \arrow[ull, bend left] \end{tikzcd}$$

$$\begin{tikzcd} A \arrow[r, dashed] \arrow[dr] & \flat(A+B) \arrow[d] \\ & A+B \end{tikzcd}$$

and similarly for B.

$\therefore$ $1+1$ is discrete (since 1 and 1 are discrete)

So

$$\begin{tikzcd} \mathbb{R} \arrow[r, "f"] \arrow[dr, "\eta"'] & 1+1 \\ & \int \mathbb{R} = 1 \arrow[u, "\bar{f}"'] \end{tikzcd}$$

f must factor through b/c $1+1$ is discrete. why?

~~$\mathrm{Im}(f) = 1$~~

either $\bar{f}(*) = \mathrm{inl}(x)$ or $\mathrm{inr}(*)$

if $\bar{f}(*) = \mathrm{inl}(*)$ then $f(*) = \mathrm{inl}(*) \Rightarrow V = \emptyset$