THE BORSUK-ULAM THEORM IN REAL-COHESIVE HOMOTOPY TYPE THEORY

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ABSTRACT. Borsuk-Ulam!

WRITING NOTES

Writing assignments:
- Amelia—section 5
- Chandrika—section 4
- Daniel—sections 2 and 3

Formalizing the cohomology proofs will be determined later.

1. INTRODUCTION

2. OVERVIEW OF REAL-COHESIVE HOMOTOPY TYPE THEORY

OUTLINE:
- HoTT as foundations
- Intepreting AlgTop theorems in HoTT is obsructed by discontinuous functions
- Relating continuous and discontinuous with flat and sharp, which are borrowed from cohesive topoi
- Formalizing flat and sharp in HoTT + axioms needed, e.g. Rflat
- Connecting sets used in AlgTop with HITs used in HoTT via shape

Homotopy type theory (HoTT) is an expression of a style of mathematics that expands the notion of "identity" to include logical identity, homotopy equivalence, and path connectedness. Experts call this style *Univalence foundations*. And as foundations, there is an ambitious program to encode all of mathematics in homotopy type theory. There is a growing community working to realize these ambitions and this paper belongs to this group.

Our present goal is to bring the classical theory of algebraic topology into the fray, and in particular the Borsuk-Ulam theorem. However, the HoTT approach to algebraic topology comes with on immediate challenge: the presence of so many fixed point theorems where, in the course of a proof, the fixed point must be specified precisely, not only up to homotopy. What is the problem with this? It is that homotopy type theory only works up to homotopy. Compare, for instance, the topological circle

$$\mathbb{S}^1 := \left\{(x, y) \in \mathbb{R}^2 | x^2 + y^2 = 1\right\}$$

with the homotopy type theoretical circle defined by a pair of constructors `base` and `loop`: `base` = `base`. One has infinitely many points that can be described exactly and the other has a single point. Brouwer's Fixed Point Theorem illustrates this problem nicely. We provide its statement and proof here for reference.

Theorem 2.1. *Let $\mathbb{D}^2$ denote the topological disk $\left\{(x,y) \in \mathbb{R}^2 | x^2+y^2 \leq 1\right\}$. Any continuous map $f\colon \mathbb{D}^2 \to \mathbb{D}^2$ has a fixed point.*

Proof. Suppose that f is continuous but does not have a fixed point, hence $f(x) \neq x$ for all $x \in \mathbb{D}^2$. For each $x \in \mathbb{D}^2$, dray a ray from $f(x)$ to x. This ray intersects the circle in a point we denote by $s(x)$. This defines a continuous function $s\colon \mathbb{D}^2 \to \mathbb{S}^1$ with the property that $s(x) = x$ for all x on the boundary of $\mathbb{D}^2$. That implies that the identity on $\mathbb{S}^1$ factors as the inclusion $\mathbb{S}^1 \hookrightarrow \mathbb{D}^2$ followed by s. Appying the fundamental group function π_1 to this factorization gives that the identity on $\pi_1\left(\mathbb{S}^1\right) = \mathbb{Z}$ factors through $\pi_1\left(\mathbb{D}^2\right) = 1$ which is absurd. □

Note how this proof relied on our precise specification of the point $s(x)$ on the circle. This point cannot be specified precisely in HoTT. Even if we did work with the only homotopical point on the circle, that is with $\mathtt{S}^1$, there is no way to relate $\mathbb{S}^1$ to $\mathtt{S}^1$ inside of type theory. Semantically speaking, this involves comparing a topological space with ∞-groupoids. This is done using the fundamental ∞-groupiod construction. No such construction exists in HoTT. This is the problem that real-cohesive homotopy theory solves. It does so by proposing to combine two already existing, but previously unrelated, type semantics: topological and ∞-groupoidal. With this proposal, there are three puzzles to be solved.

(a) We need to define a model for a *topological ∞-groupoid*.
(b) What rules or axioms can we equip HoTT with so that we can compare, for example, $\mathbb{S}^1$ to $\mathtt{S}^1$.
(c) Topology is incompatible with the law of the excluded middle, which is required to prove these classic fixed point theorems. How can we resolve this?

Shulman's original paper on real-cohesive HoTT [**?**] discusses the solution to these puzzles in detail. Presently, we are content to simply say that the Lawvere's theory of cohesion offers a solution. Of course, we need to adapt cohesion to homotopy type theory and we leave the description of this to Shulman, but we do provide a high-level description of the role that cohesion plays.

A *category of cohesive space* is a pair of categories equipped with a string of adjunctions

$$\begin{tikzcd} \text{Spaces} \arrow[d, shift right=4.5ex, "f_!"'] \arrow[d, shift left=4.5ex, "f_*"'] \\ \text{Sets} \arrow[u, shift right=1.5ex, "f^*"] \arrow[u, shift left=7.5ex, "f^!"] \end{tikzcd}$$

with $f_! \dashv f^* \dashv f_* \dashv f^!$ and such that $f_!$ preserves finite products. As Lawvere puts it, the objects of Sets should be thought of as *abstract sets* which

> ... may by conceived of as a bag of dots which are devoid of properties apart from mutual distinctness [**?**].

On the other hand, the objects of Spaces should be thought of as abstract sets together with a sort of *cohesion* between the "dots". For our purposes, we think of cohesion as a topology though, in reality, this definition above axiomatizes the various forms that cohesion may take, each functor playing a different role. The $f_!$ functor tells us which points are "stuck" together through the cohesion by returning a set of connected components. The f^* functor endows a set with the discrete topology. The f_* functor forgets the topology of a space. The $f^!$ functor endows a set with the codiscrete topology on a set. From this string of adjoints, we get another adjoint string $\int \dashv \flat \dashv \sharp$ on Spaces comprised of the **shape operation** $\int :\equiv f^* f_!$, the **flat operation** $\flat :\equiv f^* f_*$, and the **sharp operation** $\sharp :\equiv f^! f_*$.

To see how the axiomatic cohesion addresses the above puzzles, we will speak in the language of sets and categories instead of type theory. In other words, we restrict out attention to the semantics of the relevant type theory.

To solve the first puzzle, constructing a topological ∞-groupoid, we ask first that Spaces and Sets are toposes. A cohesive topos is also a *local and locally connected topos* which can be constructed using sheaves on a site that satisfy certain properties. By expanding this construction to the $(\infty, 1)$-category, we can obtain cohesive $(\infty, 1)$-toposes using ∞-sheaves on a site as shown by Schreiber [**?**]. The objects of a resulting cohesive $(\infty, 1)$-topos are precisely the topological groupoids we seek.

The second puzzle involves comparing a space with its homotopy type. Again modifying axiomatic cohesion, we replace the categories Sets and Spaces with the $(\infty, 1)$-categories of Spaces and $\infty -$ Groupoids. We also replace the functors with ∞-functors. The validity of this rests on work by Schreiber [**?**]. In this setup, applying $\int$ to a space returns the fundamental ∞-groupoid, an excellent proxy for the homotopy type.

Axiomatic cohesion also provides a solution to the final puzzle: the failure of the *continuous* excluded middle. Given that we are working with topological objects, we require that excluded middle holds continuously, but in general it does not. Given a space X and subspace U, there is no continuous inverse to the inclusion $U + (X \setminus U) \to X$ because, even though the underlying sets are the same, the topologies are different. If we can introduce discontinuous functions $X \to U + (X \setminus U)$, then we can find a discontinuous inverse to the inclusion, therefore obtaining a modified, "discontinuous" law of the excluded middle. Hence, the existence of a law of the excluded middle in our context hinges on the introduction of such discontinuous functions. To this end, recall that $\flat$ retopolozes discretely and $\sharp$ retopologizes codiscretely. If $\hom(X, Y)$ is the space of continous functions from X to Y, then both $\hom(\flat X, Y)$ and $\hom(X, \sharp Y)$ contain the discontinuous, by which we mean not necessarily continuous, functions from X to Y.

Moving towards syntax means introducing into homotopy type theory the constructors that mirror the semantics of $\int$, $\flat$, and $\sharp$. Upon adding these constructors, we obtain *cohesive homotopy type theory.* The "real" part of name comes from an additional axiom included so that we can capture the topology syntactically using continuous paths from the reals. This axiom states

> A crisp type $\mathtt{A}$ is discrete if and only if the function that returns a constant path $\mathtt{A} \to (\mathbb{R} \to \mathtt{A})$ is an equivalence.

Calling $\mathtt{A}$ a crisp variable means that we perform constructions on it without regarding the topology, such as defining maps $\flat\mathtt{A} \to \mathtt{Y}$ or $\mathtt{X} \to \sharp\mathtt{A}$. When including this axiom along with the syntactic versions of shape, flat, and sharp, we get *real-cohesive homotopy type theory.*

3. Translating Borsuk-Ulam to homotopy type theory

OUTLINE:

- **Subsection 1.** Give statements for BU-classic, BU-odd, BU-retract) a la wikipedia. The proof strategy: show BU-retract implies BU-odd which is equivalent to BU-classic, then prove BU-retract. Give the proof for BU-retract.
- **Subsection 2.** Translate the classical statement into propositions as types. We want to model classical proof. The failure of contrpositive rule in constructive logic—(not q implies not p) is (p implies not not q)—means our proof strategy is BU-retract implies not not BU-odd which is equivalent to not not BU-classic. But not not BU-classic is sharp BU-classic. Prove BU-retract.
- To close out the section, list the ingredients we need to prove BU-retract.

4. Topological and homotopical real projective spaces

OUTLINE:

- Define n-disks as both sets and types, the latter which is simply 1, since they're contractible. Show that $\int \mathbb{D} = \mathtt{D}$
- Define n-spheres as sets. Use pushouts to glue disks together. Explain why we need to glue with a collar—i.e. the "topology" (as encoded by continuous paths $\mathbb{R} \to \mathtt{X}$ of a type $\mathtt{X}$. Show, via Shulman, that $\int \mathbb{S}^n = \mathtt{S}^n$
- Define $\mathbb{RP}^n$ as sets using pushouts and collaring. Recall Bulcholtz and Egbert's definition of HIT $\mathtt{RP}^n$. Prove that $\int \mathbb{RP}^n = \mathtt{RP}^n$

4.1. Defining $\mathbb{RP}^n$.

We define $\mathbb{RP}^n$ using push outs, tautological bundles, spheres, and an inductive process, following the work of Rijike [].

The base case for our induction is $\mathbb{RP}^1$.

Definition 4.1. We define $\mathbb{RP}^1$ as the push out seen in the diagram below, where $\mathbb{I}$ is the open unit interval, and the maps are defined by $a_1(-1, x) = \frac{x}{4}$ and $a_1(1, x) = \frac{x}{4} + \frac{3}{4}$, and $b_1(\pm 1, x) := (0, a(\pm 1, x))$.

$$
\begin{tikzcd}
\mathbb{S}^0 \times \mathbb{I} \arrow[r, hook, "a_1"] \arrow[d, hook, "b_1"] & \mathbb{I} \arrow[d] \\
\{0\} \times \mathbb{I} \arrow[r] & \mathbb{RP}^1
\end{tikzcd}
$$

It is intuitively helpful to think of a_1 as an inclusion of a tubular neighbourhood of the boundary of $[0, 1] \subseteq \mathbb{R}$, with the boundary points deleted (so that it is indeed a subset of the open interval $\mathbb{I}$). We informally call this a *thickened boundary.* In fact, b_1 is also an inclusion of a thickened boundary. The image of b_1 is the deleted tubular neighbourhood of the boundary of $\{0\} \times [0, 1] \subseteq \{0\} \times \mathbb{R}$.

We also need the following $\mathbb{R}$-bundle as part of our base case.

Definition 4.2. Let $\mathtt{U}_{\mathbb{R}}$ be the universe of types homeomorphic to $\mathbb{R}$. The *tautological bundle* $\mathrm{taut}^1 : \mathbb{RP}^1 \to \mathtt{U}_{\mathbb{R}}$ is defined by the constructors $\mathbb{I} \mapsto \mathbb{R}$, $\{0\} \times \mathbb{I} \mapsto \mathbb{R}$, and the equivalence $\lambda x.x$ for every point in $\{0\} \times \mathbb{I}$, and the equivalence $\lambda x. -x$ for every point in $\{1\} \times \mathbb{I}$.

We note that $\mathbb{R}$ is homeomorphic to $\mathbb{I}$, which is a subset of $[0, 1]$. Therefore while taut^1 is an $\mathbb{R}$-bundle, we may consider a deleted tubular neighbourhood of the boundary of the corresponding $[0, 1]$-bundle as a subset of taut^1. CHANDRIKA: Is this correct language? (Where taut^1 is used as the map defining a bundle and also as the bundle itself?) Better check with how Egbert writes it.

From here, we proceed inductively. $\mathbb{RP}^n$ is defined as the following pushout.

$$\begin{tikzcd} \mathbb{S}^{n-1}\times\mathbb{I} \arrow[r, hook, "a_n"] \arrow[d, hook, "b_n"] & \mathbb{D}^n \arrow[d] \\ \mathrm{taut}^{n-1} \arrow[r] & \mathbb{RP}^n \end{tikzcd}$$

where a_n and b_n are both inclusions of $\mathbb{S}^{n-1}\times\mathbb{I}$ as thickened boundary of taut^{n-1} and $\mathbb{D}^n$. The tautological bundle $\mathrm{taut}^n : \mathbb{RP}^n \to \mathtt{U}_{\mathbb{R}}$ is defined by the constructors $\mathbb{D}^n \mapsto \mathbb{R}$ and $f : \mathrm{taut}^{n-1} \to U_R$ where f is projection to the base space, followed by the map defining the bundle structure on taut^{n-1}.

4.2. Showing $\int \mathbb{RP}^n = \mathtt{RP}^n$.

5. Cohomology

OUTLINE:

- **Subsection 1.** Define cohomology for $\mathbb{Z}/2\mathbb{Z}$ coefficients and the EM-spaces for $\mathbb{R}\mathrm{P}^n$
- **Subsection 2.** Show that we get a commutative graded ring structure for cohomology of any type $\mathtt{X}$ with $\mathbb{Z}/2\mathbb{Z}$-coefficients. Follow Brunerie's thesis.
- **Subsection 3.** Compute $\mathbb{Z}/2\mathbb{Z}$-cohomology ring for $\mathbb{R}\mathrm{P}^n$ using Mayer-Vietoris. To do this we first need to compute cohomology for disks and spheres.

5.1. Cohomology and EM-spaces for $\mathbb{R}\mathbf{P}^n$.

We follow a similar construction for cohomology as found in [**?**], modifying their construction with $\mathbb{Z}$ coefficients to have coefficients in $\mathbb{Z}/2\mathbb{Z}$. In order to define cohomology, we must first define Eilenberg-MacLane spaces $K(\mathbb{Z}/2\mathbb{Z}, n)$. Eilenberg-MacLane spaces $K(G, n)$ were defined for arbitrary group G by Finster and Licata in [**?**].

We give a construction of $\mathbb{R}\mathrm{P}^n$ in Section 4, which shows that $\mathbb{R}\mathrm{P}^2$ will have the homotopy groups necessary to be the foundation of our Eilenberg-Maclane spaces.

Proposition 5.1.

$$\pi_k(\mathbb{R}P^2) = \begin{cases} 0 & \textit{for } n = 0 \\ \mathbb{Z}/2\mathbb{Z} & \textit{for } n = 1 \\ 0 & \textit{for } n > 1, \end{cases}$$

Proof. AMELIA: We discussed this on Oct 1 2019 and thought this would go in Chandrika's section ☐

Definition 5.2. For $n : \mathbb{N}$, the type **Eilenberg-MacLane space** $K(\mathbb{Z}/2\mathbb{Z}, n)$ is the n-truncated and $(n-1)$-connected pointed type defined by

$$K(\mathbb{Z}/2\mathbb{Z}, n) := \begin{cases} \mathbb{Z}/2\mathbb{Z} & \text{for } n = 0 \\ \left|\left|\Sigma^{n-1}\mathbb{R}\mathrm{P}^2\right|\right|_n & \text{for } n \geq 1, \end{cases}$$

where Σ^{n-1} indicates the reduced suspension $n-1$ times.

Proposition 5.3. *This definition of $K(\mathbb{Z}/2\mathbb{Z}, n)$ does indeed define an Eilenberg-Maclane Space.*

Proof. AMELIA: This is Prop 5.1 and the fact (that I don't have a HoTT reference for) that EM spaces are unique. □

AMELIA: Finster and Licata constructed $K(G, 1)$ in a particular way. My concern is that I am not sure how that impacts which of their results transfer over. I think that their results in section 4 and 5 mean that as long as the $K(G, 1)$ has the right groups, using suspension and truncation works. In which case, it doesn't matter that we constructed $K(G, 1)$ differently than they did, their results should all port over. I'd really like a second opinion on this. AMELIA: On October 17 2019 we talked about this question of what results port over. we said: According to Mike, in classic topology, EM spaces are unique up to unique isomorphism, however in HoTT they are equivalent up to homotopy but the homotopy may not be unique. Our though was that our K(G,1) should be homotopy equivalent to theirs, so by univalence they're equal as types. So, their results for their construction of K(G,1) should apply to ours.

As noted in [?], Eilenberg-Maclane spaces have the following delooping property. This property allows us to construct a spectrum for ordinary cohomology with $\mathbb{Z}/2\mathbb{Z}$ coefficients.

Proposition 5.4.

$$K(\mathbb{Z}/2\mathbb{Z}, n) \simeq \Omega K(\mathbb{Z}/2\mathbb{Z}, n+1)$$

AMELIA: **** G showed equivalence of K_n and ΩK_{n+1}. I thought this was in Finster and Licata. But now I can only find a mention of it as a fact, not something proven. May need to go back and prove this.

Given this construction of the EM spaces, we define cohomology in the following way.

Definition 5.5. For a type $\mathtt{X}$ and $n : N$, the n**-th cohomology group of** $\mathtt{X}$ is the type

$$H^n(\mathtt{X}; \mathbb{Z}/2\mathbb{Z}) := ||\mathtt{X} \to K(\mathbb{Z}/2\mathbb{Z}, n)||_0 .$$

AMELIA: B also defines reduced cohomology as well. I'm not sure we need that. G proved a reduced MV sequence, so we may need it if we also use MV. Leaving it out for now.

5.2. **Commutative Graded Ring Structure.** Our strategy for describing the graded ring structure is to show that $||K(R, n) \wedge K(R, m)||_{n+m} = K(R \otimes R, n+m)$, then lift the ring multiplication of $R \otimes R \to R$ to a cup product on cohomology with R-coefficients.

AMELIA: basically copy pasted the following from "Notes", with some edits. I'll keep working through this.

Proposition 5.6. $||K(R, n) \wedge K(R, m)||_{n+m} = K(R \otimes R, n + m)$

AMELIA: is = the right symbol here? probably yes bc univalence?

Proof. Recall that $K(R \otimes R, n+m)$ is the unique up to homotopy space having the property that

$$\pi_k(K(R \otimes R, n+m)) \cong \begin{cases} 0, & k \neq n+m \\ R \otimes R, & k = n+m \end{cases}.$$

AMELIA: on 12/5, we decided to go with isomorphism symbols for groups instead of equalities. Univalence most likely means that we could write equals, but there doesn't seem to be a need.

Thus, by showing that $||K(R,n) \wedge K(R,m)||_{n+m}$ satisfies this property, we have the desired equivalence.

In order to establish the property for $k = n+m$, it is a known result (see [**?**, Prop 19.60]) that

$$\pi_{n+m}(||K(R,n) \wedge K(R,m)||_{n+m}) \cong R \otimes R.$$

In the case where $k > m+n$, $||K(R,n) \wedge K(R,m)||_{n+m}$ is truncated to be an $(n+m)$-type. So,

$$\pi_k(||K(R,n) \wedge K(R,m)||_{n+m}) \cong 0,$$

for $k > n+m$.

Finally, note that $||K(R,n) \wedge K(R,m)||_{n+m}$ is $(n+m-1)$-connected, which follows from $K(R,k)$ being $(k-1)$-connected, as seen in [**?**, Prop 4.3.1]. Thus,

$$\pi_k(||K(R,n) \wedge K(R,m)||_{n+m}) \cong 0$$

for $k \leq n+m-1$.

Thus we have shown the desired property and arrive at the homotopy equivalence

$$||K(R,n) \wedge K(R,m)||_{n+m} \simeq K(R \otimes R, n+m).$$

Invoking the univalence axiom, we conclude that

$$||K(R,n) \wedge K(R,m)||_{n+m} = K(R \otimes R, n+m).$$

□

Next, we construct the ring structure on cohomology. The strategy is to define the necessary structures and properties on EM-spaces (Def. 5.2) and then lift those to cohomology.

The addition and subtraction operations of type $K(\mathbb{Z}/2\mathbb{Z}, n) \times K(\mathbb{Z}/2\mathbb{Z}, n) \to K(\mathbb{Z}/2\mathbb{Z}, n)$ follow directly from [**?**]. We restate them below.

AMELIA: I've worked this far as of 12/10

Proposition 5.7. [**?**, Prop. 5.1.4] *The maps* $+ : K(\mathbb{Z}/2\mathbb{Z}, n) \times K(\mathbb{Z}/2\mathbb{Z}, n) \to K(\mathbb{Z}/2\mathbb{Z}, n)$ *and* $- : K(\mathbb{Z}/2\mathbb{Z}, n) \times K(\mathbb{Z}/2\mathbb{Z}, n) \to K(\mathbb{Z}/2\mathbb{Z}, n)$ *are*

Let's do multiplication. From the multiplication $\mathbb{Z}/2\mathbb{Z} \otimes \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$, we get a multiplication $K(\mathbb{Z}/2\mathbb{Z} \otimes \mathbb{Z}/2\mathbb{Z}, n) \to K(\mathbb{Z}/2\mathbb{Z}, n)$ by applying the functor

$$K(-,n) \colon \mathbf{Ring} \to \mathbf{Ho}(\mathbf{Top}_*)$$

But

$$||K(R,n) \wedge K(R,m)||_{n+m} = K(R \otimes R, n+m).$$

for any ring R as we showed above, and so we can treat our multiplication as

$$\widehat{\mu} \colon ||K(\mathbb{Z}/2\mathbb{Z}, n) \wedge K(\mathbb{Z}/2\mathbb{Z}, m)||_{n+m} \to K(\mathbb{Z}/2\mathbb{Z}, n+m).$$

However, we *really* want the domain of multiplication to be $K(\mathbb{Z}/2\mathbb{Z}, n) \times K(\mathbb{Z}/2\mathbb{Z}, m)$. To get this, we precompose $\widehat{\mu}$ by several canonical arrows resulting in the composite

$$K(\mathbb{Z}/2\mathbb{Z}, n) \times K(\mathbb{Z}/2\mathbb{Z}, m) \xrightarrow{\texttt{proj}} K(\mathbb{Z}/2\mathbb{Z}, n) \wedge K(\mathbb{Z}/2\mathbb{Z}, m) \xrightarrow{|-|_{n+m}} ||K(\mathbb{Z}/2\mathbb{Z}, n) \wedge K(\mathbb{Z}/2\mathbb{Z}, m)||_{n+m} \xrightarrow{\widehat{\mu}} K(\mathbb{Z}/2\mathbb{Z},$$

that we call *multiplication* μ. This induces the cup product

$$\smile \colon H^n(X) \times H^m(X) \to H^{n+m}(X)$$

on cohomology. Recall that we define the n-th cohomology with $\mathbb{Z}/2\mathbb{Z}$-coefficients to be $H^n(X) :\equiv ||X \to_* K_n||_0$. This is the standard definition in homotopy type theory, we do not use singular cochains, because taking the set up cochains is not a continuous process. Define $\smile (|\alpha|, |\beta|))$, for $\alpha \colon X \to_* K_n$ and $\beta] from X \to K_m$, to be the truncation of the pairing $\langle \alpha, \beta \rangle$ followed by μ:

$$\smile (|\alpha|, |\beta|)) : ||X \to K_n \times K_m \to K_{n+m}||_0.$$

Thus $\smile (|\alpha|, |\beta|))$ type-checks. It remains to show the usual ring properties hold for $H^n(X)$. Distributivity follows directly from Guillaume's argument in Proposition 5.1.7 which uses only connectivity hence carries through to our context without issue. To show 'graded comutativity', it suffices to show standard comutativity because of the $\mathbb{Z}/2\mathbb{Z}$-coefficients.

5.3. Computing the Cohomology Ring of $\mathbb{RP}^n$.

6. The Borsuk-Ulam theorem

OUTLINE:

- The proof is done by this point. Just put it all together and reconnect the dots for the reader.