

Borsuk - Ulam

Statement: Given a continuous map $f: S^2 \rightarrow \mathbb{R}^2$, \exists a pt. $x \in S^2$ s.t. $f(x) = f(-x)$

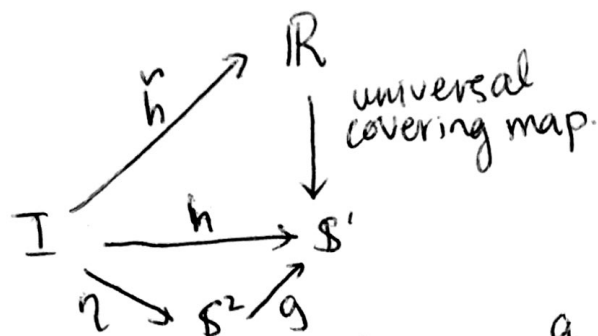
topology proof: If not, we get a map $g: S^2 \rightarrow S^1$ given by

$$g(x) = \frac{f(x) - f(-x)}{|f(x) - f(-x)|}$$

Define a loop $\eta: I \rightarrow S^2$ $\eta(s) = (\cos(2\pi s), \sin(2\pi s), 0)$

write $h: I \rightarrow S^1$ for $h = g \circ \eta$

we have $g(-x) = -g(x)$ so $h(s + 1/2) = -h(s)$



start lifting at 0
WLOG

This gives $\tilde{h}(s + 1/2) = \tilde{h}(s) + \frac{q}{2}$ with $q \in \mathbb{Z}$ odd.
 $s \in [0, 1/2]$

This q may depend on s continuously

but $q: I \rightarrow \mathbb{Z}$ odd must be constant (b/c of continuity)

$$\tilde{h}(1) = \tilde{h}(1/2) + \frac{q}{2} = \tilde{h}(0) + q$$

so h represents q times a generator for $\pi_1(S^1)$.

Since q is odd, $h: I \rightarrow S^1$ is not path null homotopic

but $\cancel{h: I \rightarrow S^1} \Rightarrow h = g \circ \eta$ and η is path null homotopic.

$\Rightarrow h$ is null homotopic $\Rightarrow \Leftarrow \square$

more general statement: for any continuous map
 $g: S^n \rightarrow \mathbb{R}^n$, \exists a pt. x s.t. $g(x) = g(-x)$.

prf. let $f(x) = g(x) - g(-x)$.

This is an odd map. we need to show $f(x) = 0$
for some x .

If not, replace $f(x)$ by $\frac{f(x)}{|f(x)|} = h(x)$

So $h: S^n \rightarrow S^{n-1}$ this is still odd.

Restrict h to the equator $S^{n-1} \subseteq S^n$. The restriction has odd degree by some Prop. This restriction is null homotopic via the restriction of h to one of the hemispheres bounded by S^{n-1} equator. this cannot be of odd degree \square

An Ω spectrum is $E: \mathbb{N} \rightarrow \text{Type}$
 $e: \prod_{n: \mathbb{N}} E(n)$

$$\prod_{n: \mathbb{N}} E(n) \cong_* \underbrace{\Omega E(n+1)}_{(e_n = e_n)}$$

X pointed type $(x: X)$ $\tilde{H}^n(X; E) = \tilde{H}^n(X_+, E)$
 this has the axioms of cohomology.

$$\tilde{H}(X; G) = \tilde{H}(X; kG)$$

\uparrow Eilenberg MacLane space

Cohomology in HoTT

Assume $S^n \xrightarrow{h} S^{n-1}$ $n > 2$ ~~is odd~~

$$\begin{array}{ccc} S^n & \xrightarrow{h} & S^{n-1} \\ \downarrow & \swarrow \text{odd} & \downarrow \\ \mathbb{R}P^n & \xrightarrow{h'} & \mathbb{R}P^{n-1} \end{array}$$

quotient by $\mathbb{Z}/2\mathbb{Z}$ action

$$h'_*: \mathbb{Z}/2\mathbb{Z} \rightarrow \mathbb{Z}/2\mathbb{Z}$$

h'_* is an isomorphism because of oddness of h