PROOF LOGIC

There are three related statements for the Borsuk-Ulam theorem

Statement C**:** $(f\colon \mathbb{S}^n \to \mathbb{R}^n) \Rightarrow (\exists x \in \mathbb{S}^n . f(x) = f(-x))$. Symbolize this as $C_0 \Rightarrow C_1$.

Statement O**:** ($g\colon \mathbb{S}^n \to \mathbb{R}^n$ odd and cont. $\Rightarrow \exists x \in \mathbb{S}^n . g(x) = 0$. Symbolize this as $O_0 \Rightarrow O_1$

Statement R**:** $\exists h\colon \mathbb{S}^n \to \mathbb{S}^{n-1}$ odd and cont.

The Borsuk-Ulam theorem are the the three classically equivalent statement

$$C \Leftrightarrow O \Leftrightarrow -R.$$

The logic underlying classical strategy to prove this is as follows.

(1) $C \models O$

(2) $O \models C$

(3) $O \models -R$ where we instead prove

$$R \models -O$$
$$R \models -(O_0 \Rightarrow O_1)$$
$$R \models (O_0 \wedge -O_1) \qquad (R \models O_0 \text{ holds})$$
$$R \models -O_1 \qquad (\text{prove this})$$

Note these are two way equivalences in classical logic.

(4) $-R \models O$

$$-R \models O$$
$$-R \models (O_0 \Rightarrow O_1)$$
$$-R \models (-O_1 \Rightarrow -O_0)$$
$$-(-O_1 \Rightarrow -O_0) \models R$$
$$(-O_1 \wedge O_0) \models R \qquad \text{prove this}$$