Adjoint Logic with a 2-Category of Modes

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Abstract

We generalize the adjoint logics of Benton and Wadler (1996) and Reed (2009) to allow multiple different adjunctions between the same categories. This provides insight into the structural proof theory of cohesive homotopy type theory, which integrates the synthetic homotopy theory of homotopy type theory with the synthetic topology of Lawvere's axiomatic cohesion. Reed's calculus is parametrized by a preorder of modes, where each mode determines a category, and there is an adjunction between categories that are related by the preorder. Here, we consider a logic parametrized by a 2-category of modes, where each mode represents a category, each mode morphism represents an adjunction, and each mode 2-morphism represents a pair of conjugate natural transformations. Using this, we give mode theories that describe adjoint triples of the sort used in cohesive homotopy type theory. We give a sequent calculus for this logic, show that identity and cut are admissible, show that this syntax is sound and complete for pseudofunctors from the mode 2-category to the 2-category of adjunctions, and investigate some constructions in the example mode theories.

1. Introduction

An adjunction $F \dashv U$ between categories $\mathscr{C}$ and $\mathscr{D}$ consists of a pair of functors $F : \mathscr{C} \to \mathscr{D}$ and $U : \mathscr{D} \to \mathscr{C}$ such that maps $FC \longrightarrow_{\mathscr{D}} D$ correspond naturally to

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maps $C \longrightarrow_{\mathscr{C}} UD$. A prototypical adjunction, which provides a mnemonic for the notation, is where U takes the underlying set of some algebraic structure such as a group, and F is the free structure on a set—the adjunction property says that a structure-preserving map from FC to D corresponds to a map of sets from C to UD (because the action on the structure is determined by being a homomorphism). Adjunctions are important to the proof theories and λ-calculi of modal logics, because the composite FU is a comonad on $\mathscr{D}$, while UF is a monad on $\mathscr{C}$. Benton (1995); Benton and Wadler (1996) describe an adjoint λ-calculus for mixing linear logic and structural/cartesian logic, with functors U from linear to cartesian and F from cartesian to linear; the $!A$ modality of linear logic arises as the comonad FU, while the monad of Moggi's metalanguage (Moggi, 1991) arises as UF. Reed (Reed, 2009) describes a generalization of this idea to situations involving more than one category: the logic is parametrized by a preorder of *modes*, where every mode p determines a category, and there is an adjunction $F \dashv U$ between categories p and q (with $F : q \to p$) exactly when $q \geq p$. For example, the intuitionistic modal logics of Pfenning and Davies Pfenning and Davies (2001) can be encoded as follows: the necessitation modality $\Box$ is the comonad FU for an adjunction between "truth" and "validity" categories, the lax modality $\bigcirc$ is the monad UF of an adjunction between "truth" and "lax truth" categories, while the possibility modality $\diamond$ requires a more complicated encoding involving four adjunctions between four categories. While specific adjunctions such as $(- \times A) \dashv (A \to -)$ arise in many logics, adjoint logic provides a formalism for abstract/uninterpreted adjunctions.

In Reed's logic, modes are specified by a preorder, which allows at most one adjunction between any two categories (more precisely, there can be two isomorphic adjunctions if both $p \geq q$ and $q \geq p$). However, it is sometimes useful to consider multiple different adjunctions between the same two categories. A motivating example is axiomatic cohesion (Lawvere, 2007), which Lawvere proposed as a general categorical interface that describes *cohesive spaces* where points may be "stuck together" in some way. Models of axiomatic cohesion include topological spaces, manifolds with differentiable or smooth structures, and algebraic geometry (Schreiber, 2013; Johnstone, 2011). The interface consists of two categories $\mathscr{C}$ and $\mathscr{S}$, and a quadruple of adjoint functors $\Pi_0, \Gamma : \mathscr{C} \to \mathscr{S}$ and $\Delta, \nabla : \mathscr{S} \to \mathscr{C}$ where $\Pi_0 \dashv \Delta \dashv \Gamma \dashv \nabla$. The idea is that $\mathscr{S}$ is some category of "sets" that provides a notion of "point", and $\mathscr{C}$ is some category of cohesive spaces built out of these sets, where points may be stuck together in some way (e.g. via topology). Γ takes the underlying set of points of a cohesive space, forgetting the cohesive structure. This forgetful functor's right adjoint $\Gamma \dashv \nabla$ equips a set with codiscrete cohesion, where all points are stuck together; the adjunction says that a map *into* a codiscrete space is the same as a map of sets. The forgetful functor's left adjoint $\Delta \dashv \Gamma$ equips a set with *discrete cohesion*, where no points are stuck together; the adjunction says that a map *from* a discrete space is the same as a map of sets. The further left adjoint $\Pi_0 \dashv \Delta$, gives the set of connected components—i.e. each element of $\Pi_0 C$ is an equivalence class of points of C that are stuck together. Π_0 is important because it translates some of the cohesive information about a space into a setting where we no longer need to care about the cohesion. These functors must satisfy some additional laws, such as Δ and ∇ being fully faithful (maps between discrete or codiscrete cohesive spaces should be the same as maps of sets).

A variation on axiomatic cohesion called *cohesive homotopy type theory* (Schreiber, 2013; Schreiber and Shulman, 2012; Shulman, 2015b) is currently being explored in the setting of homotopy type theory and univalent foundations (Voevodsky, 2006; Univalent Foundations Program, 2013). Homotopy type theory uses Martin-Löf's intensional type theory as a logic of *homotopy spaces*: the identity type provides an ∞-groupoid structure on each type, and spaces such as the spheres can be defined by their universal properties using higher inductive types (Lumsdaine and Shulman, 2015; Shulman, 2011; Lumsdaine, 2011). Theorems from homotopy theory can be proved *synthetically* in this logic (Licata and Shulman, 2013; Licata and Brunerie, 2013; Licata and Finster, 2014; Licata and Brunerie, 2015; Hou, 2014; Cavallo, 2014), and these proofs can be interpreted in a variety of models (Shulman, 2015a; Kapulkin et al., 2012; Bezem et al., 2013). However, an important but subtle distinction is that there is no *topology* in synthetic homotopy theory: the "homotopical circle" is defined as a higher inductive type, essentially "the free ∞-groupoid on a point and a loop," which a priori has nothing to do with the "topological circle," $\{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$, where $\mathbb{R}^2$ has the usual topology. This is both a blessing and a curse: on the one hand, proofs are not encumbered by topological details; but on the other, internally to homotopy type theory, we cannot use synthetic theorems to prove facts about topological spaces.

Cohesive homotopy type theory combines the synthetic homotopy theory of homotopy type theory with the synthetic topology of axiomatic cohesion, using an adjoint quadruple of $(\infty,1)$-functors $\int \dashv \Delta \dashv \Gamma \dashv \nabla$. In this higher categorical generalization, $\mathscr{S}$ is an $(\infty,1)$-category of homotopy spaces (e.g. ∞-groupoids), and $\mathscr{C}$ is an $(\infty,1)$-category of cohesive homotopy spaces, which are additionally equipped with a topological or other cohesive structure at each level. The rules of type theory are now interpreted in $\mathscr{C}$, so that each type has an ∞-groupoid structure (given by the identity type) *as well as* a separate cohesive structure on its objects, morphisms, morphisms between morphisms, etc. For example, types have both morphisms, given by the identity type, and topological paths, given by maps that are continuous in the sense of the cohesion. As in the 1-categorical case, Γ forgets the cohesive structure, yielding the underlying homotopy space, while Δ and ∇ equip a homotopy space with the discrete and codiscrete cohesion. In the ∞-categorical case, Δ's left adjoint $\int A$ (pronounced "shape of A") generalizes from the connected components to the *fundamental homotopy space* functor, which makes a homotopy space from the topological/cohesive paths, paths between paths, etc. of A. This captures the process by which homotopy spaces arise from cohesive spaces; for example, one can prove (using additional axioms) that the shape of the topological circle is the homotopy circle Shulman (2015b). This allows synthetic homotopy theory to be used in proofs about topological spaces, and opens up possibilities for applications to other areas of mathematics and theoretical physics.

This paper begins an investigation into the structural proof theory of cohesive homotopy type theory, as a special case of generalizing Reed's adjoint logic to allow multiple adjunctions between the same categories. As one might expect, the first step is to generalize the mode preorder to a mode category, so that we can have multiple different morphisms $\alpha, \beta : p \geq q$. This allows the logic to talk about different but unrelated adjunctions between two categories. However, in order to describe an adjoint triple such as $\Delta \dashv \Gamma \dashv \nabla$, we need to know that the same functor Γ is both a left and right adjoint. To describe such a situation, we generalize to a 2-category of modes, and arrange the syntax of the logic to capture the following semantics. Each mode p determines a category (also denoted by p). Each morphism $\alpha : p \geq q$ determines adjoint functors $F_\alpha : p \to q$ and $U_\alpha : q \to p$ where $F_\alpha \dashv U_\alpha$. Each 2-cell $\alpha \Rightarrow \beta$ determines a morphism of adjunctions between $F_\alpha \dashv U_\alpha$ and $F_\beta \dashv U_\beta$, which consists of natural transformations $F_\beta \to F_\alpha$ and $U_\alpha \to U_\beta$ that are conjugate under the adjunction structure (Mac Lane, 1998, §IV.7). For example, an adjoint triple is specified by the mode 2-category with

- objects c and s
- 1-cells $\mathsf{d} : s \geq c$ and $\mathsf{n} : c \geq s$
- 2-cells $1_c \Rightarrow \mathsf{n} \circ \mathsf{d}$ and $\mathsf{d} \circ \mathsf{n} \Rightarrow 1_s$ satisfying some equations

The 1-cells generate $F_\mathsf{d} \dashv U_\mathsf{d}$ and $F_\mathsf{n} \dashv U_\mathsf{n}$, while the 2-cells are sufficient to prove that U_d is naturally isomorphic to F_n, so we can define $\Delta := F_\mathsf{n}$, $\nabla := U_\mathsf{n}$, and $\Gamma := U_\mathsf{d} \cong F_\mathsf{n}$ and have the desired adjoint triple. Indeed, you may recognize this 2-category as the "walking adjunction" with $\mathsf{d} \dashv \mathsf{n}$—that is, we give an adjoint triple by saying that the mode morphism generating the adjunction $\Delta \dashv \Gamma$ is itself left adjoint to the mode morphism generating the adjunction $\Gamma \dashv \nabla$.

The main judgement of the logic is a "mixed-category" entailment $A\,[\alpha] \vdash C$ where A has mode q and C has mode p and $\alpha : q \geq p$. Semantically, this judgement means a morphism from A to C "along" the adjunction determined by α—i.e. a map $F_\alpha\, A \to C$ or $A \to U_\alpha\, C$.[2] However, taking the mixed-mode judgement as

[2] We could instead use a structure that includes a basic notion of "morphisms along α,' such as

primitive makes for a nicer sequent calculus: U and F can be specified independently from each other, by left and right rules, in such a way that identity and cut (composition) are admissible, and the subformula property holds. While we do not consider focusing (Andreoli, 1992), we conjecture that the connectives can be given the same focusing behavior as in Reed (2009): F is positive and U is negative (which, because limits are negative and colimits are positive, matches what left and right adjoints should preserve).

The resulting logic has a good definition-to-theorem ratio: from simple sequent calculus rules for F and U, we can prove a variety of general facts that are true for any mode 2-category (F_α and U_α are functors; $F_\alpha U_\alpha$ is a comonad and $U_\alpha F_\alpha$ is a monad; F_α preserves colimits and U_α preserves limits), as well as facts specific to a particular theory (e.g. for the adjoint triple above, Γ preserves both colimits and limits, because it can be written either as U_{d} and F_{n}; the comonad $\flat := \Delta\Gamma$ and monad $\sharp := \nabla\Gamma$ are themselves adjoint). Moreover, we can use different mode 2-categories to add additional structure; for example, moving from the walking adjunction to the walking reflection (taking $\Delta\nabla = 1$) additionally gives that Δ and ∇ are full and faithful and that $\flat$ and $\sharp$ are idempotent, which are some of the additional conditions for axiomatic cohesion.

We make a few simplifying restrictions for this paper. First, we consider only single-hypothesis, single-conclusion sequents, deferring an investigation of products and exponentials to future work.

Second, on the semantic side, we consider only 1-categorical semantics of the derivations of the logic, rather than the ∞-groupoid semantics that we are ultimately interested in. More precisely, for any 2-category $\mathscr{M}$ of modes, we can interpret the logic using a pseudofunctor $S : \mathscr{M} \to \mathbf{Adj}$, where $\mathbf{Adj}$ is the 2-category of categories, adjunctions, and morphisms of adjunctions (conjugate pairs of natural transformations). By S, each mode determines a 1-category, and derivations in the logic are interpreted as morphisms in these categories. The action of S on 1- and 2-cells is used to interpret F and U. We show that the syntax forms such a pseudofunctor, and conjecture that the syntax is initial in some category or 2-category of pseudofunctors, but have not yet tried to make this precise.

Third, we consider only a logic of simple-types, rather than a dependent type theory. Consequently, we do not have an identity type available for proving equalities of proof terms. However, we need an equational theory to make many of the statements we would like to make (e.g. "UF is a monad" requires proving some

a Grothendieck bifibration over the 2-category of modes, or a pseudofunctor to the bicategory of profunctors that are both representable and corepresentable; these are equivalent to the structures used here.

equational laws), and the definitional equalities arising from admissibility of cut and identity are not sufficient. Thus, in addition to the sequent calculus itself, we give an equality judgement on sequent calculus derivations. This judgement is interpreted by actual equality of morphisms in the semantics above, but we intend some of these rules to be propositional equalities in an eventual adjoint type theory.

In Section 2, we define the rules of the logic, prove admissibility of identity and cut, and define an equational theory on derivations. In Section 3, we summarize some of the constructions that are possible in the logic for any mode specification $\mathscr{M}$, including showing that the syntax determines a pseudofunctor $\mathscr{M} \to \mathbf{Adj}$. In Section 4, we describe the 1-categorical semantics of the logic. Finally, in Sections 5 and 6, we examine some specific mode specifications for adjoint triples, and discuss the relationship to the rules for spatial type theory used in (Shulman, 2015b). All of the syntactic metatheory of the logic and the examples have been formalized in Agda (Norell, 2007), using a syntactic representation of the sequent calculus and the equational theory as inductive families.[3]

A preliminary version of this work was published in the proceedings of Logical Foundations of Computer Science, 2016. Relative to the conference version, this journal article contains additional examples and details throughout, a full proof of soundness (Section 4), and specialized sequent calculi for a specific mode theory (Section 6), which clarify the relationship with Shulman (2015b).

2. Sequent Calculus and Equational Theory

2.1. Sequent Calculus

The logic is parametrized by a strict 2-category $\mathscr{M}$ of modes. We write p, q for the 0-cells (modes), $\alpha, \beta, \gamma, \delta : p \geq q$ for the 1-cells from q to p, and $e : \alpha \Rightarrow \beta$ for the 2-cells. The notation $p \geq q$ for the 1-cells follows Reed (2009), but in our case $\mathscr{M}$ is a general category, so there can be more than one morphism $p \geq q$. We use the notation $p \geq q$ for an arrow from q to p (an arrow points "lesser to greater") to match the sequent calculus, where the p-mode is on the left and the q-mode on the right ("validity is greater than truth"). We write $\beta \circ \alpha$ for 1-cell composition in function composition order (i.e. if $\beta : r \geq q$ and $\alpha : q \geq p$ then $\beta \circ \alpha : r \geq p$), $e_1 \cdot e_2$ for vertical composition of 2-cells in diagrammatic order, and $e_1 \circ_2 e_2$ for horizontal composition of 2-cells in "congruence of $\circ$" order (if $e_1 : \alpha \Rightarrow \alpha'$ and $e_2 : \beta \Rightarrow \beta'$ then $e_1 \circ_2 e_2 : \alpha \circ \beta \Rightarrow \alpha' \circ \beta'$). The equations for 2-cells say that $\cdot$ is associative with unit 1_α for any α, that $\circ_2$ is associative with unit 1_1, and that the interchange law $(e_1 \cdot e_2) \circ_2 (e_3 \cdot e_4) = (e_1 \circ_2 e_3) \cdot (e_2 \circ_2 e_4)$ holds. For convenience, we think of

[3] See `github.com/dlicata335/hott-agda/tree/master/metatheory/adjointlogic`.

the mode category as being fixed at the outset, and the syntax and judgements of the logic as being indexed by the actual semantic objects/morphisms/2-morphisms of this category; therefore, equal morphisms in the mode 2-category automatically determine equal propositions, judgements, and derivations. An alternative would be to give a syntax and explicit equality judgement for the mode category, which would be helpful if we needed a mode theory where equality of morphisms or 2-morphisms were undecidable.

In the pseudofunctor semantics, each object p of the mode category $\mathscr{M}$ will determine a category (also denoted by p). Syntactically, the judgement $A\ \mathsf{type}_p$ will mean that A determines an object of the category p. A morphism $\alpha : q \geq p$ in $\mathscr{M}$ determines an adjunction $F_\alpha \dashv U_\alpha$, with $F_\alpha : q \to p$ and $U_\alpha : p \to q$; note that the right adjoints are covariant and the left adjoints contravariant. Syntactically, the action on objects is given by $F_\alpha\, A\ \mathsf{type}_p$ when $A\ \mathsf{type}_q$ and $U_\alpha\, A\ \mathsf{type}_q$ when $A\ \mathsf{type}_p$. The "pseudo" of the pseudofunctor means that, for example, the types $F_\beta\ (F_\alpha\, A)$ and $F_{\alpha\circ\beta}\, A$ will be isomorphic but not definitionally equal. Finally, a 2-cell $e : \alpha \Rightarrow \beta$ in $\mathscr{M}$ determines natural transformations $U_\alpha \to U_\beta$ and $F_\beta \to F_\alpha$ which are "conjugate" (Mac Lane, 1998, §IV.7); again, the right adjoints are covariant and the left adjoints are contravariant. Syntactically, these natural transformations will be definable using the sequent calculus rules.

In addition to the connectives $F_\alpha\, A$ and $U_\alpha\, A$, we allow an arbitrary collection of atomic propositions (denoted P), each of which has a designated mode; these represent arbitrary objects of the corresponding categories. To add additional structure to a category or to all categories, we can add rules for additional connectives; for example, a rule $A + B\ \mathsf{type}_p$ if $A\ \mathsf{type}_p$ and $B\ \mathsf{type}_p$ (parametric in p) says that any category p has a coproduct type constructor.

The sequent calculus judgement has the form $A\,[\alpha] \vdash C$ where $A\ \mathsf{type}_q$ and $C\ \mathsf{type}_p$ and $\alpha : q \geq p$. This judgement represents a map from an object of some category q to an object of another category p along the adjunction $F_\alpha \dashv U_\alpha$. Semantically, this mixed-category map can be interpreted equivalently as an arrow $F_\alpha\, A \longrightarrow_p C$ or $A \longrightarrow_q U_\alpha\, C$. In the rules, we write A_p to indicate an elided premise $A\ \mathsf{type}_p$.

The rules for atomic propositions and for U and F are as follows:

$$\frac{1 \Rightarrow \alpha}{P\,[\alpha] \vdash P}\ \mathsf{hyp}$$

$$\frac{A_r\,[\alpha \circ \beta] \vdash C_p}{F_{\alpha:r \geq q}\, A_r\,[\beta : q \geq p] \vdash C_p}\ \mathsf{FL} \qquad \frac{\gamma : r \geq q \quad \gamma \circ \alpha \Rightarrow \beta \quad C_r\,[\gamma] \vdash A_q}{C_r\,[\beta : r \geq p] \vdash F_{\alpha:q \geq p}\, A_q}\ \mathsf{FR}$$

$$\frac{\gamma : q \geq p \quad \alpha \circ \gamma \Rightarrow \beta \quad A_q\,[\gamma] \vdash C_p}{U_{\alpha:r \geq q}\, A_q\,[\beta : r \geq p] \vdash C_p}\ \mathsf{UL} \qquad \frac{C_r\,[\beta \circ \alpha] \vdash A_p}{C_r\,[\beta : r \geq q] \vdash U_{\alpha:q \geq p}\, A_p}\ \mathsf{UR}$$

The rules for other connectives do not change α; for example, see the sequent calculus rules for coproducts in Figure 1.

These rules are guided by the usual design goals for sequent calculi: the only rules are the left and right rules for each connective, the rules have the subformula property (the premises only involve subformulas of the conclusion), and the identity and cut rules are admissible. To achieve these goals, it is necessary to treat the natural transformations $F_\beta \to F_\alpha$ and $U_\alpha \to U_\beta$ induced by a mode 2-cell $\alpha \Rightarrow \beta$ as an additional admissible structural rule: composing with such a natural transformation transforms a derivation of $A\,[\alpha] \vdash C$ into a derivation of $A\,[\beta] \vdash C$. The admissible rules are discussed further in Section 2.2 below.

Consider the rules FL and UR. When β is 1, these rules pass from $F_\alpha\, A\,[1] \vdash C$ and $A\,[1] \vdash U_\alpha\, C$ to $A\,[\alpha] \vdash C$, which makes sense because the judgement $A\,[\alpha] \vdash C$ is intended to mean either/both of these. When β is not 1, these rules compose the mode morphism in the connective with the mode morphism in the sequent. Semantically, this corresponds to one direction of the composition isomorphism between $F_{\alpha\circ\beta}\, A$ and $F_\beta\, F_\alpha\, A$ and similarly for U (see the Example below). While we do not formally give a focused sequent calculus, we conjecture that these two rules are *invertible*: whenever you have $F_\alpha\, A$ on the left or $U_\alpha\, A$ on the right, you can immediately apply the rule, no matter what is on the other side of the sequent.

Next, consider FR. The rule is a bit complex because it involves three different aspects of the pseudofunctor structure. First, in the case where γ is the identity 1-cell and $\beta = \alpha$ and the 2-cell is the identity, the rule gives functoriality of F (see the Example below). In the case where $\gamma = \beta$ and the 2-cell is the identity, the rule gives the other direction of the composition isomorphism between $F_{\alpha\circ\beta}\, A$ and $F_\beta\, F_\alpha\, A$ (see Example 2). In the case where γ is 1 and the rightmost premise is the identity sequent $A\,[1] \vdash A$, the rule gives a natural transformation $F_\beta \to F_\alpha$ induced by $e : \alpha \Rightarrow \beta$ (see Example 3). This is necessary because composition with such a natural transformation cannot always be pushed inside an application of functoriality, because a morphism from $\gamma \circ \alpha$ might not be constructed from a

Sequent calculus rules:

$$\frac{C_q\,[\alpha] \vdash A_p}{C_q\,[\alpha] \vdash A_p + B_p}\ \mathsf{Inl} \qquad \frac{C_q\,[\alpha] \vdash B_p}{C_q\,[\alpha] \vdash A_p + B_p}\ \mathsf{Inr} \qquad \frac{A_q\,[\alpha] \vdash C_p \quad B_q\,[\alpha] \vdash C_p}{A_q + B_q\,[\alpha] \vdash C_p}\ \mathsf{Case}$$

Definitions of $-_*(-)$ and ident_- and $\mathsf{cut}\ -\ -$:

$$\begin{array}{rcl}
e_*(\mathsf{Inl}(D)) & := & \mathsf{Inl}(e_*(D)) \\
e_*(\mathsf{Inr}(D)) & := & \mathsf{Inr}(e_*(D)) \\
e_*(\mathsf{Case}(D_1, D_2)) & := & \mathsf{Case}(e_*(D_1), e_*(D_2)) \\
\\
\mathsf{ident}_{A+B} & := & \mathsf{Case}(\mathsf{Inl}(\mathsf{ident}_A), \mathsf{Inr}(\mathsf{ident}_B)) \\
\\
\mathsf{cut}\ (\mathsf{Inl}(D))\ (\mathsf{Case}(E_1, E_2)) & := & \mathsf{cut}\ D\ E_1 \\
\mathsf{cut}\ (\mathsf{Inr}(D))\ (\mathsf{Case}(E_1, E_2)) & := & \mathsf{cut}\ D\ E_2 \\
\mathsf{cut}\ D\ (\mathsf{Inl}(E)) & := & \mathsf{Inl}(\mathsf{cut}\ D\ E) \\
\mathsf{cut}\ D\ (\mathsf{Inr}(E)) & := & \mathsf{Inr}(\mathsf{cut}\ D\ E) \\
\mathsf{cut}\ (\mathsf{Case}(D_1, D_2))\ E & := & \mathsf{Case}(\mathsf{cut}\ D_1\ E, \mathsf{cut}\ D_2\ E) \text{if } E \text{ is not a right rule}
\end{array}$$

Equational theory:

$$\frac{D : A + B\,[\alpha] \vdash C}{D \approx \mathsf{Case}(\mathsf{cut}\ (\mathsf{Inl}(\mathsf{ident}_A))\ D, \mathsf{cut}\ (\mathsf{Inr}(\mathsf{ident}_B))\ D)}$$

$$\frac{}{\mathsf{Inl}(\mathsf{UL}^{\gamma}_{e}(D)) \approx \mathsf{UL}^{\gamma}_{e}(\mathsf{Inl}(D))} \qquad \frac{}{\mathsf{Inr}(\mathsf{UL}^{\gamma}_{e}(D)) \approx \mathsf{UL}^{\gamma}_{e}(\mathsf{Inr}(D))}$$

Figure 1: Sequent calculus and equations for coproducts

morphism from γ. In the general form of the rule, we combine these three ingredients: given $\alpha : q \geq p$ and $\beta : r \geq p$, to prove $F_\alpha\, A$ from C, choose a natural transformation that splits β as $\gamma \circ \alpha$ for some $\gamma : r \geq q$, and apply functoriality of α, which leaves proving $C\,[\gamma] \vdash A$. The UL rule is dual.

Because we are interested not only in provability, but also in the equational theory of proofs in this logic, one might think the next step would be to annotate the sequent judgement with a proof term, writing e.g. $x : A[\alpha] \vdash M : B$. However, the proof terms M would have exactly the same structure as the derivations of this typing judgement.[4] so we instead use the derivations themselves as the proof terms. This corresponds to an "intrinsic representation" in Agda. We sometimes write $D : A\,[\alpha] \vdash B$ to indicate "typing" in the metalanguage; i.e. this should be read "D is a derivation tree of the judgement $A\,[\alpha] \vdash B$."

We now give some examples to illustrate the rules; these examples and many more like them are in the companion Agda code.

Example: Functoriality. We expect F and U to be functors; the functorial action of F on terms is derived as follows. Given $\alpha : q \geq p$ and $D : A\,[1_q] \vdash B$, we have

$$\dfrac{\dfrac{\overline{1_q : q \geq q} \quad \overline{1 : 1_q \circ \alpha \Rightarrow \alpha \circ 1_p} \quad D : A\,[1_q] \vdash B}{A\,[\alpha \circ 1_p] \vdash F_\alpha\, B}\;\mathsf{FR}}{F_\alpha\, A\,[1_p] \vdash F_\alpha\, B}\;\mathsf{FL}$$

Because $\mathscr{M}$ is a strict 2-category, the identity 2-cell has type $1_q \circ \alpha \Rightarrow \alpha \circ 1_p$.

Example: Comonad. The composites FU and UF should be a comonad and a monad respectively; define $\Box_\alpha A := F_\alpha\,(U_\alpha\, A)$ and $\bigcirc_\alpha A := U_\alpha\,(F_\alpha\, A)$ for any $\alpha : q \geq p$. The comonad operations on $\Box_\alpha A$ are defined as follows:

$$\dfrac{\dfrac{\overline{1 : p \geq p} \quad \overline{1 : \alpha \circ 1_p \Rightarrow \alpha} \quad \dfrac{\overline{1 : 1 \Rightarrow 1}}{P\,[1] \vdash P}\;\mathsf{hyp}}{U_\alpha\, P\,[\alpha] \vdash A}\;\mathsf{UL}}{\Box_\alpha P\,[1] \vdash P}\;\mathsf{FL}$$

$$\dfrac{\dfrac{\overline{1 : q \geq q} \quad \overline{1 : \alpha \Rightarrow \alpha} \quad \dfrac{\dfrac{\overline{1 : q \geq q} \quad \overline{1 : \alpha \Rightarrow \alpha} \quad \dfrac{\dfrac{\overline{1 : p \geq p} \quad \overline{1 : \alpha \Rightarrow \alpha} \quad \dfrac{\overline{1 \Rightarrow 1}}{P\,[1] \vdash P}\;\mathsf{hyp}}{U_\alpha\, P\,[\alpha] \vdash P}\;\mathsf{UL}}{U_\alpha\, P\,[1] \vdash U_\alpha\, P}\;\mathsf{UR}}{U_\alpha\, P\,[\alpha] \vdash \Box_\alpha P}\;\mathsf{FR}}{U_\alpha\, P\,[1] \vdash U_\alpha\, \Box_\alpha P}\;\mathsf{UR}}{U_\alpha\, P\,[\alpha] \vdash \Box_\alpha \Box_\alpha P}\;\mathsf{FR}}{\Box_\alpha P\,[1] \vdash \Box_\alpha \Box_\alpha P}\;\mathsf{FL}$$

The same derivations work for an arbitrary type A if we replace the top-right derivations of $P\,[1] \vdash P$ by the admissible identity principle $A\,[1] \vdash A$, defined below (indeed, we can always substitute an arbitrary type for an atomic proposition).

An advantage of using a cut-free sequent calculus is that we can observe some non-provabilities: for example, there is not in general a map $P\,[1_p] \vdash \Box_\alpha P$. By

[4]This is not the case in a dependently typed language that has a judgemental equality conversion rule in the derivations but not the terms.

inversion, a derivation must begin with FR, but to apply this rule, we need a $\gamma : p \geq q$ and a 2-cell $\gamma \circ \alpha \Rightarrow 1$, which may not exist.

The monad operations on $\bigcirc_\alpha A$ are dual.

Example: F preserves coproducts. As another example, we show a map from $F_\alpha\ (P+Q)$ to $F_\alpha\ P + F_\alpha\ Q$, which is part of an isomorphism:

$$\dfrac{\dfrac{\dfrac{\overline{1 : q \geq q} \quad \overline{1 : \alpha \Rightarrow \alpha} \quad \overline{P\,[1] \vdash P}\;\mathsf{hyp}\,1}{P\,[\alpha] \vdash F_\alpha\ P}\;\mathsf{FR}}{P\,[\alpha] \vdash F_\alpha\ P + F_\alpha\ Q}\;\mathsf{Inl} \qquad \dfrac{\dfrac{\overline{1 : q \geq q} \quad \overline{1 : \alpha \Rightarrow \alpha} \quad \overline{Q\,[1] \vdash Q}\;\mathsf{hyp}\,1}{Q\,[\alpha] \vdash F_\alpha\ Q}\;\mathsf{FR}}{Q\,[\alpha] \vdash F_\alpha\ P + F_\alpha\ Q}\;\mathsf{Inr}}{\dfrac{P+Q\,[\alpha] \vdash F_\alpha\ P + F_\alpha\ Q}{F_\alpha\ (P+Q)\,[1] \vdash F_\alpha\ P + F_\alpha\ Q}\;\mathsf{FL}}\;\mathsf{Case}$$

The key idea is that we can apply the left rule for F first, and then case-analyze the $P+Q$, before choosing between Inl and Inr on the right; this direction of map doesn't exist for $U_\alpha\ (P+Q)$ because the left rule for U cannot be applied until after applying UR.

Example: F/U on identity and composition. For identity and composition of 1-cells, an obvious question is the relationship between $F_1\ A$ and $U_1\ A$ and A, between $F_{\beta\circ\alpha}\ A$ and $F_\alpha\ (F_\beta\ A)$, and between $U_{\beta\circ\alpha}\ A$ and $U_\beta\ (U_\alpha\ A)$. We do not have definitional equalities of types, but the types in each group are isomorphic (in a sense that will be made precise below). For example, the maps for F are as follows:

$$\dfrac{\overline{P\,[1 \circ 1] \vdash P}\;\mathsf{hyp}\,1}{F_1\ P\,[1] \vdash P}\;\mathsf{FL} \qquad \dfrac{\overline{1 : p \geq p} \quad \overline{1 : 1 \circ 1 \Rightarrow 1} \quad \overline{P\,[1] \vdash P}\;\mathsf{hyp}\,1}{P\,[1] \vdash F_1\ P}\;\mathsf{FR}$$

$$\dfrac{\dfrac{\beta : r \geq p \quad \overline{1 : (\beta \circ \alpha) \circ 1 \Rightarrow \beta \circ \alpha} \quad \dfrac{\overline{1 : r \geq r} \quad \overline{1 : 1 \circ \alpha \Rightarrow \alpha} \quad \overline{P\,[1] \vdash P}\;\mathsf{hyp}\,1}{P\,[\beta] \vdash F_\beta\ P}}{P\,[(\beta \circ \alpha) \circ 1] \vdash F_\alpha\ (F_\beta\ P)}\;\mathsf{FR}}{F_{\beta\circ\alpha}\ P\,[1] \vdash F_\alpha\ (F_\beta\ P)}\;\mathsf{FL}$$

$$\dfrac{\dfrac{\dfrac{\overline{1 : r \geq r} \quad \overline{1 : 1 \circ (\beta \circ \alpha) \Rightarrow \beta \circ (\alpha \circ 1)} \quad \overline{P\,[1] \vdash P}\;\mathsf{hyp}\,1}{P\,[\beta \circ (\alpha \circ 1)] \vdash F_{\beta\circ\alpha}\ P}\;\mathsf{FR}}{F_\beta\ P\,[\alpha \circ 1] \vdash F_{\beta\circ\alpha}\ P}\;\mathsf{FL}}{F_\alpha\ (F_\beta\ P)\,[1] \vdash F_{\beta\circ\alpha}\ P}\;\mathsf{FL}$$

Example: F/U on 2-cells. For every 2-cell $e : \alpha \Rightarrow \beta$, we have derivations of $F_\beta\, P\,[1] \vdash F_\alpha\, P$ and $U_\alpha\, P\,[1] \vdash U_\beta\, P$.

$$\dfrac{\dfrac{\overline{1 : q \geq q} \quad e : 1 \circ \alpha \Rightarrow \beta \quad \overline{P\,[1] \vdash P}\ \mathsf{hyp}\,1}{P\,[\beta] \vdash F_\alpha\, P}\ \mathsf{FR}}{F_\beta\, P\,[1] \vdash F_\alpha\, P}\ \mathsf{FL} \qquad \dfrac{\dfrac{\overline{1 : p \geq p} \quad e : \alpha \circ 1 \Rightarrow \beta \quad \overline{P\,[1] \vdash P}\ \mathsf{hyp}\,1}{U_\alpha\, P\,[\beta] \vdash P}\ \mathsf{UL}}{U_\alpha\, P\,[1] \vdash U_\beta\, P}\ \mathsf{UR}$$

These derivations determine an adjunction morphism between $F_\beta \dashv U_\beta$ and $F_\alpha \dashv U_\alpha$. The other directions (e.g. $F_\alpha\, P\,[1] \vdash F_\beta\, P$) are not provable in general.

2.2. Admissible Rules

Adjunction morphisms. As discussed above, composition with the natural transformations $F_\beta \to F_\alpha$ and $U_\alpha \to U_\beta$ induced by a 2-cell $e : \alpha \Rightarrow \beta$ is an admissible rule, which we write as $e_*(D) : A\,[\beta] \vdash B$:

$$\dfrac{\alpha \Rightarrow \beta \quad A\,[\alpha] \vdash C}{A\,[\beta] \vdash C}\ -_*(-)$$

The definition of this operation pushes the natural transformation into the premises of a derivation until it reaches a rule that builds in a transformation (FR,UL,hyp):

$$\begin{array}{rcl}
e_*(\mathsf{hyp}\, e') & := & \mathsf{hyp}\,(e' \cdot e) \\
e_*(\mathsf{FR}^{\gamma}_{e'}(D)) & := & \mathsf{FR}^{\gamma}_{e' \cdot e}(D) \\
e_*(\mathsf{FL}(D)) & := & \mathsf{FL}((1 \circ_2 e)_*(D)) \\
e_*(\mathsf{UL}^{\gamma}_{e'}(D)) & := & \mathsf{UL}^{\gamma}_{e' \cdot e}(D) \\
e_*(\mathsf{UR}(D)) & := & \mathsf{UR}((e \circ_2 1)_*(D))
\end{array}$$

The hypothesis rule and FR and UL build in some movement along a 2-cell, so in those cases we compose the e with the 2-cells that are already present. For FL and UR (and for the rules for coproducts), the operation commutes with the rule.

Identity. The identity rule is admissible:

$$\dfrac{}{A_p\,[1] \vdash A_p}\ \mathsf{ident}$$

The general strategy is "apply the invertible rule and then the focus rule and then the inductive hypothesis." For example, for $F_\alpha\, A$, the following reduces the problem to identity on A:

$$\dfrac{\dfrac{1_q : q \geq q \quad 1 : 1 \circ \alpha \Rightarrow \alpha \quad A\,[1] \vdash A}{A\,[\alpha] \vdash F_\alpha\, A}\ \mathsf{FR}}{F_{\alpha : q \geq p}\, A\,[1] \vdash F_\alpha\, A}\ \mathsf{FL}$$

As a function from types to derivations, we have

$$\begin{array}{rcl}
\mathsf{ident}_P & := & \mathsf{hyp}\,1 \\
\mathsf{ident}_{U_\alpha A} & := & \mathsf{UR}(\mathsf{UL}_1^1(\mathsf{ident}_A)) \\
\mathsf{ident}_{F_\alpha A} & := & \mathsf{FL}(\mathsf{FR}_1^1(\mathsf{ident}_A))
\end{array}$$

Cut. The following cut rule is admissible:

$$\frac{A_r\,[\beta] \vdash B_q \quad B_q\,[\alpha] \vdash C_p}{A_r\,[\beta \circ \alpha] \vdash C_p}\ \mathsf{cut}$$

For example, consider the principal cut for F:

$$\frac{\dfrac{e : \gamma \circ \alpha_1 \Rightarrow \beta \quad D : A\,[\gamma] \vdash B}{A\,[\beta] \vdash F_{\alpha_1}\,B}\ \mathsf{FR} \quad \dfrac{E : B\,[\alpha_1 \circ \alpha] \vdash C}{F_{\alpha_1}\,B\,[\alpha] \vdash C}\ \mathsf{FL}}{A\,[\beta \circ \alpha] \vdash C}\ \mathsf{cut}$$

In this case the cut reduces to

$$\frac{e \circ_2 1 : (\gamma \circ \alpha_1) \circ \alpha \Rightarrow \beta \circ \alpha \quad \dfrac{D : A\,[\gamma] \vdash B \quad E : B\,[\alpha_1 \circ \alpha] \vdash C}{A\,[\gamma \circ \alpha_1 \circ \alpha] \vdash C}\ \mathsf{cut}}{A\,[\beta \circ \alpha] \vdash C}\ -_*(-)$$

As a transformation on derivations, we have

$$\begin{array}{rcll}
\mathsf{cut}\ (\mathsf{hyp}\,e)\ (\mathsf{hyp}\,e') & := & \mathsf{hyp}\,(e \circ_2 e') & \\
\mathsf{cut}\ (\mathsf{FR}_e^\gamma(D))\ (\mathsf{FL}(E)) & := & (e \circ_2 1)_*(\mathsf{cut}\,D\,E) & \\
\mathsf{cut}\ (\mathsf{UR}(D))\ (\mathsf{UL}_e^\gamma(E)) & := & (1 \circ_2 e)_*(\mathsf{cut}\,D\,E) & \\
\mathsf{cut}\,D\ (\mathsf{FR}_e^\gamma(E)) & := & \mathsf{FR}_{1 \circ_2 e}^{\beta \circ \gamma}(\mathsf{cut}\,D\,E) & \\
\mathsf{cut}\,D\ (\mathsf{UR}(E)) & := & \mathsf{UR}(\mathsf{cut}\,D\,E) & \\
\mathsf{cut}\ (\mathsf{FL}(D))\ E & := & \mathsf{FL}(\mathsf{cut}\,D\,E) & \text{if } E \text{ is not a right rule} \\
\mathsf{cut}\ (\mathsf{UL}_e^\gamma(D))\ E & := & \mathsf{UL}_{e \circ_2 1}^{\gamma \circ \alpha}(\mathsf{cut}\,D\,E) & \text{if } E \text{ is not a right rule}
\end{array}$$

The first case is for atomic propositions. The next two cases are the principal cuts, when a right rule meets a left rule; these correspond to β-reduction in natural deduction. The next two cases are right-commutative cuts, which push any D inside a right rule for E. The final two cases are left commutative cuts, which push any E inside a left rule for D. The left-commutative and right-commutative cuts overlap when D is a left rule and E is a right rule; we resolve this arbitrarily by saying that right-commutative cuts take precedence. Using the equational theory below, we will be able to prove the unrestricted left-commutative rules.

Example: Adjunction. As an example using identity and cut, we give one of the maps from the bijection-on-hom-sets adjunction for F and U: given $\alpha : q \geq p$ we can transform $D : F_\alpha\, A\,[1] \vdash B$ into $A\,[1] \vdash U_\alpha\, B$:

$$\dfrac{\dfrac{\dfrac{}{1 : q \geq q} \quad \dfrac{}{1 : \alpha \Rightarrow \alpha} \quad \dfrac{}{A\,[1] \vdash A}\,\mathsf{ident}}{A\,[\alpha] \vdash F_\alpha\, A}\,\mathsf{FR} \qquad D : F_\alpha\, A\,[1] \vdash B}{\dfrac{A\,[\alpha] \vdash B}{A\,[1] \vdash U_\alpha\, B}\,\mathsf{UR}}\,\mathsf{cut}$$

2.3. Equations

When we construct proofs using the admissible rules $e_*(D)$ and ident_A and $\mathsf{cut}\; D\; E$, there is a natural notion of definitional equality induced by the above definitions of these operations—the cut- and identity-free proofs are normal forms, and a proof using cut or identity is equal to its normal form. However, to prove the desired equations in the examples below, we will need some additional "propositional" equations, which, because we are using derivations as proof terms, we represent by a judgement $D \approx D'$ on two derivations $D, D' : A\,[\alpha] \vdash C$. This judgement is the least congruence closed under the following rules. First, we have uniqueness/η rules. The rule for F says that any map from $F_\alpha\, A$ is equal to a derivation that begins with an application of the left rule and then cuts the original derivation with the right rule; the rule for U is dual.

$$\dfrac{D : F_\alpha\, A\,[\beta] \vdash C}{D \approx \mathsf{FL}(\mathsf{cut}\;(\mathsf{FR}^1_1(\mathsf{ident}_A))\; D)}\,\mathsf{F}\eta \qquad \dfrac{D : C\,[\beta] \vdash U_\alpha\, A}{D \approx \mathsf{UR}(\mathsf{cut}\; D\;(\mathsf{UL}^1_1(\mathsf{ident}_A)))}\,\mathsf{U}\eta$$

Second, we have rules arising from the 2-cell structure. For example, suppose we construct a derivation by $\mathsf{FR}^\gamma_e(D)$ for some $\gamma : r \geq q$ and $e : \gamma \circ \alpha \Rightarrow \beta$, but there is another morphism $\gamma' : r \geq q$ such that there is a 2-cell between γ and γ'. The following says that we can equally well pick γ' and suitably transformed e and D, using composition and $e_{2*}(-)$ to make the types match up.

$$\dfrac{e : \gamma \circ \alpha \Rightarrow \beta \quad D : C\,[\gamma'] \vdash A \quad e_2 : \gamma' \Rightarrow \gamma}{\mathsf{FR}^\gamma_e(e_{2*}(D')) \approx \mathsf{FR}^{\gamma'}_{((e_2 \circ_2 1)\cdot e)}(D')} \qquad \dfrac{e : \gamma \circ \alpha \Rightarrow \beta \quad D : C\,[\gamma'] \vdash A \quad e_2 : \gamma' \Rightarrow \gamma}{\mathsf{UL}^\gamma_e(e_{2*}(D')) \approx \mathsf{UL}^{\gamma'}_{((1 \circ_2 e_2)\cdot e)}(D')}$$

Semantically, these rules will be justified by some of the pseudofunctor laws.

The final rules say that left rules of negatives and right rules of positives commute. These are needed to prove the left-commutative cut equations in the case where E is a right rule, because we chose to give right-commutative cuts precedence definitionally (the full left-commutative equations seem necessary for proving some of the theorems we want to prove below). For U and F, we have

$$\frac{(1 \circ_2 e_1) \cdot e_2 = (e_3 \circ_2 1) \cdot e_4}{\mathsf{UL}_{e_2}(\mathsf{FR}_{e_1}(D)) \approx \mathsf{FR}_{e_4}(\mathsf{UL}_{e_3}(D))}$$

We elide the details of the typing of the 2-cells e_i, which will not be needed below; they are the most general thing that makes both sides of the conclusion type check.

2.3.1. Admissible rules

The following additional equality rules are admissible for logic containing the U/F rules described above and the coproduct rules in Figure 1. The rules in each line (except the first) are proved by mutual induction, and use the preceding lines:

1. For each D, $e_*(D)$ is functorial on the 2-cell identity and vertical composition:

   $$\frac{}{1_*(D) = D} \qquad \frac{}{(e_1 \cdot e_2)_*(D) = e_{2*}(e_{1*}(D))}$$

   It is important for the remaining proofs that these are definitional equalities, not $\approx$, so that we can use them "in context" before we know that cut is well-defined on $\approx$.
2. $e_*(-)$ is well-defined on $\approx$:

   $$\frac{D \approx D'}{e_*(D) \approx e_*(D')}$$

   and $e_*(-)$ commutes with cut:

   $$\frac{e : \alpha \Rightarrow \alpha' \quad e' : \beta \Rightarrow \beta' \quad D : A\,[\alpha] \vdash B \quad D' : B\,[\beta] \vdash C}{(e \circ_2 e')_*(\mathsf{cut}\, D\, D') \approx \mathsf{cut}\ (e_*(D))\ (e'_*(D'))}$$
3. Cut is associative:

   $$\frac{}{\mathsf{cut}\, D_1\ (\mathsf{cut}\, D_2\, D_3) \approx \mathsf{cut}\ (\mathsf{cut}\, D_1\, D_2)\ D_3}$$
4. Cut is well-defined on $\approx$, identities are units for cut, and the left-commutative cut rules hold always (they are true definitionally only when E is not a right rule).

   $$\frac{}{\mathsf{cut}\, D\ \mathsf{ident} \approx D} \quad \frac{}{\mathsf{cut}\ \mathsf{ident}\ D \approx D} \quad \frac{D \approx D'}{\mathsf{cut}\, D\, E \approx \mathsf{cut}\, D'\, E} \quad \frac{E \approx E'}{\mathsf{cut}\, D\, E \approx \mathsf{cut}\, D\, E'}$$

   $$\frac{}{\mathsf{cut}\ (\mathsf{FL}(D))\ E \approx \mathsf{FL}(\mathsf{cut}\, D\, E)} \quad \frac{}{\mathsf{cut}\ (\mathsf{UL}^{\gamma}_{e}(D))\ E \approx \mathsf{UL}^{\gamma \circ \alpha}_{e \circ_2 1}(\mathsf{cut}\, D\, E)}$$

3. Syntactic Constructions

In this section, we investigate some constructions on F and U that hold for any mode specification $\mathscr{M}$. Taken together, the first set of constructions shows that the syntax forms a pseudofunctor $\mathscr{M} \to \mathbf{Adj}$, where $\mathbf{Adj}$ is the 2-category of categories, adjunctions, and conjugate natural transformations. This is a standard way of proving logical completeness in categorical logic: we think of the pseudofunctors $\mathscr{M} \to \mathbf{Adj}$ as the class of models of the syntax, so this shows that if something is true in all models $\mathscr{M} \to \mathbf{Adj}$, then it is true in the syntax, because the syntax forms a model.

Since the objects of $\mathbf{Adj}$ are categories, the first step is to associate a category with each mode p. The rules for $\approx$ in the previous section imply that for each p, there is a category whose objects are $A\ \mathsf{type}_p$ and whose morphisms are derivations of $A\,[1_p] \vdash B$ quotiented by $\approx$, with identities given by ident and composition given by cut. We write $D \bullet E$ as an infix notation for $\mathsf{cut}\, D\, E$; this corresponds to writing composition in this category in diagrammatic order. Some standard category-theoretic terminology applied to categories of this form unpacks as follows:

1. For $A, B\ \mathsf{type}_p$, an isomorphism $A \cong B$ consists of a pair $D : A\,[1] \vdash B$ and $E : B\,[1] \vdash A$ such that $D \bullet E \approx \mathsf{ident}_A$ and $E \bullet D \approx \mathsf{ident}_B$.
2. For modes p and q, a functor from p to q consists of a function G_0 from types with mode p to types with mode q and a function G_1 from derivations $A\,[1] \vdash B$ to derivations $G_0 A\,[1] \vdash G_0 B$, such that $G_1(\mathsf{ident}_A) \approx \mathsf{ident}_{G_0 A}$ and $G_1(D \bullet E) \approx G_1(D) \bullet G_1(E)$.
3. For two functors $G, H : p \to q$, a natural transformation $t : G \to H$ consists of a family of derivations $D_A : G_0(A)\,[1] \vdash H_0(A)$ for each $A\ \mathsf{type}_p$, such that for any $D : A\,[1_p] \vdash B$, $\mathsf{cut}\, D_A\ (H_1(D)) \approx \mathsf{cut}\ (G_1(D))\ D_B$. A natural isomorphism consists of a natural transformation along with inverses demonstrating that each D_A is an isomorphism $G_0(A) \cong H_0(A)$.
4. For two functors $L, R : p \to q$, an adjunction $L \dashv R$ (using the natural-bijection-of-hom-sets definition) consists of functions $-^{\rhd} : (L_0(A)\,[1] \vdash B) \to (A\,[1] \vdash R_0(B))$ and $-^{\lhd} : (A\,[1] \vdash R_0(B)) \to (L_0(A)\,[1] \vdash B)$ which are mutually inverse and such that $-^{\rhd}$ is natural in A and B: for $D_1 : A'\,[1] \vdash A$ and $D_3 : B\,[1] \vdash B'$ and $D_2 : L_0(A)\,[1] \vdash B$, $(L_1(D_1) \bullet D_2 \bullet D_3)^{\rhd} \approx D_1 \bullet D_2^{\rhd} \bullet R_1(D_3)$ (it follows that $-^{\lhd}$ is natural as well).
5. A morphism of adjunctions (or "adjunction morphism") from $L^1 \dashv_1 R^1$ to $L^2 \dashv_2 R^2$ consists of two natural transformations $t^L : L^1 \to L^2$ and $t^R : R^2 \to R^1$ between the corresponding functors that are "conjugate" under the adjunction structure (Mac Lane, 1998, §IV.7). This means that for any $D : L^2(A)\,[1] \vdash B$ we have
$$(t^L \bullet D)^{\rhd_1} = D^{\rhd_2} \bullet t^R.$$

An adjunction isomorphism consists of an adjunction morphism together with inverses showing that t^L and t^R are each natural isomorphisms.[5]

6. Because we treat equality of morphisms as propositional/proof-irrelevant, two adjunction morphisms (t^L, t^R) and $(u^L, u^{R'})$ between the same two adjunctions are equal iff $t^L{}_A \approx u^L{}_A$ and $t^L{}_A \approx u^L{}_A$ for all A.

While these definitions are "external" (meta-theoretic), we are hopeful that it would be possible to internalize them in a dependent type theory based on adjoint logic. For example, although the above definition allows a functor to be given by arbitrary meta-theoretic functions, in all of the examples we consider, the action on objects is in fact given by a syntactic type with a "placeholder", and the action on morphisms is given by taking an assumed derivation D and applying rules to it. Similarly, all of the equalities are proved by chaining together the equality rules (including the admissible ones, such as associativity and identity of cut) from the previous section.

A pseudofunctor is a map between 2-categories that preserves identity and composition of 1-cells up to coherent isomorphism, rather than on the nose.[6] The rules describe a pseudofunctor because $F_1\,A \cong A \cong U_1\,A$ and $F_{\beta\circ\alpha}\,A \cong F_\alpha\,F_\beta\,A$ and $U_{\beta\circ\alpha}\,A \cong U_\beta\,U_\alpha\,A$, but these are not equalities of types.

THEOREM 3.1: SYNTAX DETERMINES A PSEUDOFUNCTOR. *The syntax of adjoint logic determines a pseudofunctor $\mathscr{M} \to \textbf{\textit{Adj}}$:*

1. *An object p of $\mathscr{M}$ is sent to the category whose objects are A type_p and morphisms are derivations of $(A\,[1_p] \vdash B)$ quotiented by $\approx$.*
2. *For each q, p, there is a functor from the category of morphisms $q \geq p$ to the category of adjoint functors between q and p.*
   - *Each $\alpha : q \geq p$ is sent to $F_\alpha \dashv U_\alpha$ in **Adj**—F_α and U_α are functors and they are adjoint.*
   - *Each 2-cell $e : \alpha \Rightarrow \beta$ is sent to a conjugate pair of transformations $(F(e), U(e)) : (F_\alpha \dashv U_\alpha) \to (F_\beta \dashv U_\beta)$, and this preserves 1 and $e_1 \cdot e_2$.*
3. *$F_1\,A \cong A$ and $U_1\,A \cong A$ naturally in A, and these are conjugate, so there is an adjunction isomorphism P^1 between $F_1 \dashv U_1$ and the identity adjunction.*
4. *$F_{\beta\circ\alpha}\,A \cong F_\alpha\,(F_\beta\,A)$ and $U_{\beta\circ\alpha}\,A \cong U_\beta\,(U_\alpha\,A)$ naturally in A, and these are conjugate, so there is an adjunction isomorphism $P^\circ(\alpha,\beta)$ between $F_{\beta\circ\alpha}$ $\dashv$*

[5]This definition is equivalent to "two inverse adjunction morphisms," similarly to how isomorphisms that are natural are the same as iso-natural transformations—we can recover the conjugation condition for one direction from the other.

[6]http://ncatlab.org/nlab/show/pseudofunctor

$U_{\beta\circ\alpha}$ and the composition of the adjunctions $F_\alpha \dashv U_\alpha$ and $F_\beta \dashv U_\beta$. Moreover, this family of adjunction isomorphisms is natural in α and β.

5. *Three coherence conditions between P^1 and P° are satisfied, which relate (1) P^1 and $P^\circ(\alpha,1)$, (2) P^1 and $P^\circ(1,\alpha)$, and (3) $P^\circ(\gamma,\beta\circ\alpha)$ composed with $P^\circ(\beta,\alpha)$ and $P^\circ(\gamma,\beta)$ composed with $P^\circ(\gamma\circ\beta,\alpha)$.*

Proof. We have given a flavor for some of the maps in the examples above; the complete construction is about 500 lines of Agda. There are many equations to verify—inverses, naturality, conjugation, and coherence—but they are all true for $\approx$. $\square$

Next, we summarize some expected constructions on $F_\alpha \dashv U_\alpha$. Of course, these standard facts are corollaries of the above; the point is that we can construct them directly using the rules of the logic.

LEMMA 3.2: SOME CONSTRUCTIONS ON ADJUNCTIONS. *Let $\alpha : q \geq p$. Then:*

1. *The composite functor $\Box_\alpha A := F_\alpha\, U_\alpha\, A$ is a comonad:*

   $\mathsf{counit} : \Box_\alpha A\,[1] \vdash A$ *naturally in A*

   $\mathsf{comult} : \Box_\alpha A\,[1] \vdash \Box_\alpha \Box_\alpha A$ *naturally in A*

   $\mathsf{comult} \bullet (\Box\,\mathsf{comult}) \approx \mathsf{comult} \bullet \mathsf{comult}$ *and* $\mathsf{comult} \bullet \mathsf{counit} \approx \mathsf{ident}$ *and* $\mathsf{comult} \bullet (\Box\,\mathsf{counit}) \approx \mathsf{ident}$.

2. *The composite functor $\bigcirc_\alpha A := U_\alpha\, F_\alpha\, A$ is a monad:*

   $\mathsf{unit} : A\,[1] \vdash \bigcirc_\alpha A$ *naturally in A*

   $\mathsf{mult} : \bigcirc_\alpha \bigcirc_\alpha A\,[1] \vdash \bigcirc_\alpha A$ *naturally in A*

   $(\bigcirc\,\mathsf{mult}) \bullet \mathsf{mult} \approx \mathsf{mult} \bullet \mathsf{mult}$ *and* $\mathsf{unit} \bullet \mathsf{mult} \approx \mathsf{ident}$ *and* $(\bigcirc\,\mathsf{unit}) \bullet \mathsf{mult} \approx \mathsf{ident}$.

3. *F preserves coproducts: $F_\alpha\,(A+B) \cong F_\alpha\, A + F_\alpha\, B$ naturally in A and B.*

Proof. We showed many of the maps above; the (co)monad laws, naturality conditions, and inverse laws are all true for $\approx$; the construction is about 150 lines of Agda. $\square$

4. Semantics

Next, we show that we can interpret the rules of adjoint logic in any pseudofunctor $S : \mathscr{M} \to \mathbf{Adj}$. This shows that the syntax is sound for these models. On the semantic side, we unpack the definition of a pseudofunctor as follows:

- We write $\mathscr{C}_p$ for $S(p)$. We write ";" for composition of morphisms in $\mathscr{C}_p$ in diagrammatic order.

- We write $\mathscr{F}_\alpha \dashv \mathscr{U}_\alpha$ for $S(\alpha)$, and $-^{\rhd\alpha}$ and $-^{\lhd\alpha}$ for the two maps of hom-sets of the adjunction. Naturality of the maps of hom-sets says that $(\mathscr{F}_\alpha\, m_1; m_2; m_3)^{\rhd\alpha} = m_1; m_2^{\rhd\alpha}; \mathscr{U}_\alpha\, m_3$ and $(m_1; m_2; \mathscr{U}_\alpha\, m_3)^{\lhd\alpha} = \mathscr{F}_\alpha\, m_1; m_2^{\lhd\alpha}; m_3$.

- We write $\mathscr{F}_e : \mathscr{F}_\beta A \longrightarrow \mathscr{F}_\alpha A$ and $\mathscr{U}_e : \mathscr{U}_\alpha A \longrightarrow \mathscr{U}_\beta A$ when e : $\alpha \Rightarrow \beta$ for the components of the two natural transformations in the adjunction morphism $S(e) : \mathscr{F}_\beta \dashv \mathscr{U}_\beta \longrightarrow \mathscr{F}_\alpha \dashv \mathscr{U}_\alpha$. Functoriality gives that $\mathscr{F}_{e\cdot e'} = \mathscr{F}_{e'}; \mathscr{F}_e$ and $\mathscr{F}_1 = 1$ and $\mathscr{U}_{e\cdot e'} = \mathscr{U}_e; \mathscr{U}_{e'}$ and $\mathscr{U}_1 = 1$. The conjugation property specifies that for any $m : \mathscr{F}_\alpha\, A \longrightarrow B$, we have $m^{\rhd\alpha}; \mathscr{U}_e =_{A \longrightarrow \mathscr{U}_\beta\, B} (\mathscr{F}_e; m)^{\rhd\beta}$, or equivalently that for any $m : A \longrightarrow \mathscr{U}_\alpha\, B$, we have $\mathscr{F}_e; m^{\lhd\alpha} =_{\mathscr{F}_\beta\, A \longrightarrow B} (m; \mathscr{U}_e)^{\lhd\beta}$.

- We write $\mathscr{F}^1 : \mathscr{F}_1\, A \cong A$ and $\mathscr{U}^1 : \mathscr{U}_1\, A \cong A$ for the components of the two natural isomorphisms in the adjunction isomorphism S^1 between $\mathscr{F}_1 \dashv \mathscr{U}_1$ and the identity adjunction. The conjugation property specifies that for any $m : A \longrightarrow B$, $({\mathscr{F}^1}_A; m)^{\rhd 1} = m; {\mathscr{U}^1}_B{}^{-1}$, and for any $m : A \longrightarrow B$, $(m; {\mathscr{U}^1}^{-1})^{\lhd 1} = \mathscr{F}^1; m$. In particular, taking $m = 1$ in the former, $({\mathscr{F}^1}_A)^{\rhd 1} = ({\mathscr{U}^1}_A)^{-1}$.

- We write $\mathscr{F}^\circ(\beta,\alpha) : \mathscr{F}_{\beta\circ\alpha}\, A \cong \mathscr{F}_\alpha\, (\mathscr{F}_\beta\, A)$ and $\mathscr{U}^\circ(\beta,\alpha) : \mathscr{U}_{\beta\circ\alpha}\, A \cong \mathscr{U}_\beta\, (\mathscr{U}_\alpha\, A)$ for the components of the two natural isomorphisms in the natural adjunction isomorphism S° between $\mathscr{F}_{\beta\circ\alpha} \dashv \mathscr{U}_{\beta\circ\alpha}$ and the composition of the adjunctions $\mathscr{F}_\alpha \dashv \mathscr{U}_\alpha$ and $\mathscr{F}_\beta \dashv \mathscr{U}_\beta$. Naturality in α, β means that for any $e_1 : \beta \Rightarrow \beta'$ and $e_2 : \alpha \Rightarrow \alpha'$, we have $\mathscr{F}_{e_1 \circ_2 e_2}; \mathscr{F}^\circ(\beta,\alpha) =_{\mathscr{F}_{\beta'\circ\alpha'}\, A \longrightarrow \mathscr{F}_\alpha\, \mathscr{F}_\beta\, A} \mathscr{F}^\circ(\beta',\alpha'); \mathscr{F}_{\alpha'}\, \mathscr{F}_{e_1}; \mathscr{F}_{e_2}$, or equivalently $\mathscr{F}^\circ(\beta',\alpha'); \mathscr{F}_{e_2}; \mathscr{F}_\alpha\, \mathscr{F}_{e_1}$, and similarly for $\mathscr{U}$. The conjugation property specifies that $(m; \mathscr{U}^\circ(\beta,\alpha)^{-1})^{\lhd\beta\circ\alpha} = \mathscr{F}^\circ(\beta,\alpha); m^{\lhd\beta\,\lhd\alpha}$ and similarly for $-^{\rhd}$.

- Using the fact that both $\mathscr{M}$ and **Adj** are strict 2-categories, and unpacking the definition of horizontal composition of natural transformations, the three coherence laws relating S° and S^1 specify the following:
  - $\mathscr{F}^\circ(1,\alpha)_A{}^{-1} =_{\mathscr{F}_\alpha\, \mathscr{F}_1\, A \longrightarrow \mathscr{F}_\alpha\, A} \mathscr{F}_\alpha\, ({\mathscr{F}^1}_A)$
  - $\mathscr{F}^\circ(\alpha,1)_A{}^{-1} =_{\mathscr{F}_1\, \mathscr{F}_\alpha\, A \longrightarrow \mathscr{F}_\alpha\, A} {\mathscr{F}^1}_{(\mathscr{F}_\alpha\, A)}$
  - $\mathscr{U}^\circ(\alpha,1)_A{}^{-1} =_{\mathscr{U}_\alpha\, \mathscr{U}_1\, A \longrightarrow \mathscr{U}_\alpha\, A} \mathscr{U}_\alpha\, ({\mathscr{U}^1}_A)$
  - $\mathscr{U}^\circ(1,\alpha)_A{}^{-1} =_{\mathscr{U}_1\, \mathscr{U}_\alpha\, A \longrightarrow \mathscr{U}_\alpha\, A} ({\mathscr{U}^1}_{\mathscr{U}_\alpha\, A})$
  - $\mathscr{F}^\circ(\gamma,\beta\circ\alpha); \mathscr{F}^\circ(\beta,\alpha) =_{\mathscr{F}_{\gamma\circ\beta\circ\alpha}\, A \longrightarrow \mathscr{F}_\alpha\, \mathscr{F}_\beta\, \mathscr{F}_\gamma\, A} \mathscr{F}^\circ(\gamma\circ\beta,\alpha); \mathscr{F}_\alpha\, \mathscr{F}^\circ(\gamma,\beta)$ and similarly for $\mathscr{U}$.

$[\![A \ \mathsf{type}_p]\!]$ is an object of $\mathscr{C}_p$; we assume an interpretation is given for each atomic proposition, and the basic rules of adjoint logic require only that we can interpret $F_\alpha\, A$ and $U_\alpha\, A$, which are interpreted as $\mathscr{F}_\alpha\, [\![A]\!]$ and $\mathscr{U}_\alpha\, [\![A]\!]$.

We can interpret the judgement $A\,[\alpha] \vdash B$ as either a morphism $\mathscr{F}_\alpha\, [\![A]\!] \longrightarrow [\![B]\!]$ or a morphism $[\![A]\!] \longrightarrow \mathscr{U}_\alpha\, [\![B]\!]$. We choose $\mathscr{F}_\alpha\, A \longrightarrow B$ because it seems like it will generalize better to a multiple-hypothesis sequent. This means that the interpretations of the rules for F do not use the adjunction structure, while the interpretations of the rules for U do. We now consider the interpretation of the sequent calculus rules:

THEOREM 4.1: SOUNDNESS OF THE SEQUENT CALCULUS. *There is a function $[\![-]\!]$ from derivations $D : A\,[\alpha] \vdash B$ to morphisms $\mathscr{F}_\alpha\, [\![A]\!] \longrightarrow [\![B]\!]$.*

Proof.

- For the hypothesis rule

$$\frac{e : 1 \Rightarrow \alpha}{P\,[\alpha] \vdash P}\ \mathsf{hyp}$$

we need a morphism $\mathscr{F}_\alpha\, [\![P]\!] \longrightarrow [\![P]\!]$, which we take to be the composite

$$\mathscr{F}_\alpha\, [\![P]\!] \xrightarrow{\mathscr{F}_e} \mathscr{F}_1\, [\![P]\!] \xrightarrow{\mathscr{F}^1} [\![P]\!]$$

- For FL

$$\frac{D : A\,[\alpha \circ \beta] \vdash C}{F_\alpha\, A\,[\beta] \vdash C}\ \mathsf{FL}$$

the premise is interpreted as $[\![D]\!] : \mathscr{F}_{\alpha\circ\beta}\, [\![A]\!] \longrightarrow [\![C]\!]$ and we want $\mathscr{F}_\beta\, \mathscr{F}_\alpha\, [\![A]\!] \longrightarrow [\![C]\!]$, so we take the interpretation to be

$$\mathscr{F}_\beta\, \mathscr{F}_\alpha\, [\![A]\!] \xrightarrow{\mathscr{F}^\circ(\alpha,\beta)^{-1}} \mathscr{F}_{\alpha\circ\beta}\, [\![A]\!] \xrightarrow{[\![D]\!]} [\![C]\!]$$

- For FR

$$\frac{\gamma : r \geq q \quad e : \gamma \circ \alpha \Rightarrow \beta \quad D : C\,[\gamma] \vdash A}{C\,[\beta] \vdash F_\alpha\, A}\ \mathsf{FR}$$

the premise $[\![D]\!]$ is $\mathscr{F}_\gamma\, [\![C]\!] \longrightarrow [\![A]\!]$, and we want $\mathscr{F}_\beta\, [\![C]\!] \longrightarrow \mathscr{F}_\alpha\, [\![A]\!]$. Using functoriality, we have $\mathscr{F}_\alpha\, [\![D]\!] : \mathscr{F}_\alpha\, \mathscr{F}_\gamma\, [\![C]\!] \longrightarrow \mathscr{F}_\alpha\, [\![A]\!]$, so

$$\mathscr{F}_\beta\, [\![C]\!] \xrightarrow{\mathscr{F}_e} \mathscr{F}_{\gamma\circ\alpha}\, [\![C]\!] \xrightarrow{\mathscr{F}^\circ(\gamma,\alpha)} \mathscr{F}_\alpha\, \mathscr{F}_\gamma\, [\![C]\!] \xrightarrow{\mathscr{F}_\alpha\, [\![D]\!]} \mathscr{F}_\alpha\, [\![A]\!]$$

- For UL
$$\frac{\gamma : q \geq p \quad e : \alpha \circ \gamma \Rightarrow \beta \quad A\,[\gamma] \vdash C}{U_\alpha\, A\,[\beta] \vdash C}\ \mathsf{UL}$$
The premise gives $[\![D]\!] : \mathscr{F}_\gamma\, [\![A]\!] \longrightarrow [\![C]\!]$, and we want $\mathscr{F}_\beta\, \mathscr{U}_\alpha\, [\![A]\!] \longrightarrow [\![C]\!]$. Using the adjunction, the premise gives $[\![D]\!]^{\rhd\gamma} : A \longrightarrow \mathscr{U}_\gamma\, C$ and it suffices to give $\mathscr{U}_\alpha\, [\![A]\!] \longrightarrow \mathscr{U}_\beta\, [\![C]\!]$. So we form the composite
$$\mathscr{U}_\alpha\, [\![A]\!] \xrightarrow{\mathscr{U}_\alpha\, ([\![D]\!]^{\rhd\gamma})} \mathscr{U}_\alpha\, \mathscr{U}_\gamma\, C \xrightarrow{\mathscr{U}^\circ(\alpha,\gamma)^{-1}} \mathscr{U}_{\alpha\circ\gamma}\, [\![C]\!] \xrightarrow{\mathscr{U}_e} \mathscr{U}_\beta\, [\![C]\!]$$
and then move it along the adjunction.

- For UR
$$\frac{C\,[\beta \circ \alpha] \vdash A}{C\,[\beta] \vdash U_\alpha\, A}\ \mathsf{UR}$$
The premise gives $\mathscr{F}_{\beta\circ\alpha}\, [\![C]\!] \longrightarrow [\![A]\!]$ and we want $\mathscr{F}_\beta\, [\![C]\!] \longrightarrow \mathscr{U}_\alpha\, A$. We have
$$\mathscr{F}_\alpha\, \mathscr{F}_\beta\, [\![C]\!] \xrightarrow{\mathscr{F}^\circ(\beta,\alpha)^{-1}} \mathscr{F}_{\beta\circ\alpha}\, [\![C]\!] \xrightarrow{[\![D]\!]} [\![A]\!]$$
so using the adjunction gives the result.

In summary, we have
$$\begin{array}{rcl}
[\![\mathsf{hyp}\, e]\!] & := & \mathscr{F}_e; \mathscr{F}^1 \\
[\![\mathsf{FL}(D)]\!] & := & \mathscr{F}^\circ(\alpha,\beta)^{-1}; [\![D]\!] \\
[\![\mathsf{FR}_e^\gamma(D)]\!] & := & \mathscr{F}_e; \mathscr{F}^\circ(\gamma,\alpha); \mathscr{F}_\alpha\, [\![D]\!] \\
[\![\mathsf{UR}(D)]\!] & := & (\mathscr{F}^\circ(\beta,\alpha)^{-1}; [\![D]\!])^{\rhd\alpha} \\
[\![\mathsf{UL}_e^\gamma(D)]\!] & := & (\mathscr{U}_\alpha\, ([\![D]\!]^{\rhd\gamma}); \mathscr{U}^\circ(\alpha,\gamma)^{-1}; \mathscr{U}_e)^{\lhd\beta}
\end{array}$$
□

In general, an admissible inference rule need not hold in all models. However, in this case, we are considering a class of models (pseudofunctors into **Adj**) in which the admissible rules (e.g. cut and identity) are true.

LEMMA 4.2. *The admissible sequent calculus rules $e_*(D)$ and ident_A and $\mathsf{cut}\, D\, E$ are sound.*

Proof. • For $e_*(D)$, we have
$$\frac{e : \alpha \Rightarrow \beta \quad D : A\,[\alpha] \vdash C}{A\,[\beta] \vdash C}\ \mathsf{e_*(D)}$$

The premise gives $\mathscr{F}_\alpha\,[\![A]\!] \longrightarrow [\![C]\!]$, and we want $\mathscr{F}_\beta\,[\![A]\!] \longrightarrow [\![C]\!]$ so we have

$$\mathscr{F}_\beta\,[\![A]\!] \xrightarrow{\mathscr{F}_e} \mathscr{F}_\alpha\,[\![A]\!] \xrightarrow{[\![D]\!]} [\![C]\!]$$

- For identity

$$\frac{}{A\,[1] \vdash A}\ \mathsf{ident}$$

  we want $\mathscr{F}_1\,[\![A]\!] \longrightarrow [\![A]\!]$, so use $\mathscr{F}^1$.

- For cut

$$\frac{D : A\,[\beta] \vdash B \quad E : B\,[\alpha] \vdash C}{A\,[\beta \circ \alpha] \vdash C}\ \mathsf{cut}$$

  the interpretations of the premises gives $\mathscr{F}_\beta\,[\![A]\!] \longrightarrow [\![B]\!]$ and $\mathscr{F}_\alpha\,[\![B]\!] \longrightarrow [\![C]\!]$. To get $\mathscr{F}_{\beta\circ\alpha}\,[\![A]\!] \longrightarrow [\![C]\!]$, we compose as follows:

$$\mathscr{F}_{\beta\circ\alpha}\,[\![A]\!] \xrightarrow{\mathscr{F}^\circ(\beta,\alpha)} \mathscr{F}_\alpha\,\mathscr{F}_\beta\,[\![A]\!] \xrightarrow{\mathscr{F}_\alpha\,[\![D]\!]} \mathscr{F}_\alpha\,[\![B]\!] \xrightarrow{[\![E]\!]} [\![C]\!]$$

$\square$

We now have two possible interpretations for the admissible rules: first, the one given by expanding the definition in each instance, and second, the compositional definition given above. In the next few lemmas, we show that these agree:

$$\begin{array}{rcl} [\![e_*(D)]\!] & = & \mathscr{F}_e; [\![D]\!] \\ [\![\mathsf{ident}_A]\!] & = & \mathscr{F}^1 \\ [\![\mathsf{cut}\,D\,E]\!] & = & \mathscr{F}^\circ(\beta,\alpha); \mathscr{F}_\alpha\,[\![D]\!]; [\![E]\!] \end{array}$$

LEMMA 4.3. *For all $e : \beta \Rightarrow \beta'$ and derivations $D : A\,[\beta] \vdash B$, $[\![e_*(D)]\!] = \mathscr{F}_e; [\![D]\!]$.*

Proof. The proof is by induction on D, and in each case we can unfold the definition of $e_*(D)$, so we have to show:

- $\mathscr{F}_e;[\![\mathsf{hyp}\,e']\!] = [\![\mathsf{hyp}\,(e' \cdot e)]\!]$ After unfolding definitions, it suffices to use functoriality of $\mathscr{F}_e$ to show $\mathscr{F}_{e' \cdot e} = \mathscr{F}_e; \mathscr{F}_{e'}$.
- $\mathscr{F}_e;[\![\mathsf{FR}^\gamma_{e'}(D)]\!] = [\![\mathsf{FR}^\gamma_{e' \cdot e}(D)]\!]$ Again, $\mathscr{F}_{e' \cdot e} = \mathscr{F}_e; \mathscr{F}_{e'}$ suffices.
- $\mathscr{F}_e;[\![\mathsf{FL}(D)]\!] = [\![\mathsf{FL}((1 \circ_2 e)_*(D))]\!]$ After unfolding definitions and applying the IH, we need to know that $\mathscr{F}_{e(\mathscr{F}_\alpha A)}; \mathscr{F}^\circ(\alpha,\beta)^{-1} = \mathscr{F}^\circ(\alpha,\beta')^{-1}; \mathscr{F}_{(1_\alpha \circ_2 e)_A}$ as arrows $\mathscr{F}_{\beta'}\,\mathscr{F}_\alpha\,A \longrightarrow \mathscr{F}_{\alpha\circ\beta'}\,A$, which is true by naturality of $\mathscr{F}^\circ(-,-)$ and $\mathscr{F}_1 = 1$ and $\mathscr{F}_\beta\,1 = 1$.

- $\mathscr{F}_e;[\![\mathsf{UL}^{\gamma}_{e'}(D)]\!] = [\![\mathsf{UL}^{\gamma}_{e'\cdot e}(D)]\!]$ After unfolding the definitions, it suffices to use functoriality $\mathscr{U}_{e'\cdot e} = \mathscr{U}_{e'};\mathscr{U}_e$ and the conjugation property for $-^{\lhd}$.

- $\mathscr{F}_e;[\![\mathsf{UR}(D)]\!] = [\![\mathsf{UR}((e \circ_2 1)_*(D))]\!]$ After unfolding definitions and applying the IH, we need to know that $\mathscr{F}_e;(\mathscr{F}^{\circ}(\beta,\alpha)^{-1};[\![D]\!])^{\rhd} = (\mathscr{F}^{\circ}(\beta',\alpha)^{-1};\mathscr{F}_{e\circ_2 1_\alpha};[\![D]\!])^{\rhd}$ By naturality of the adjunction, the former is equal to $(\mathscr{F}_\alpha\ (\mathscr{F}_e);\mathscr{F}^{\circ}(\beta,\alpha)^{-1};[\![D]\!])^{\rhd}$ and then naturality of $\mathscr{F}^{\circ}(-,-)$ gives the result.

$\square$

LEMMA 4.4. *For all types A, $[\![\mathsf{ident}_A]\!] = \mathscr{F}^1{}_{[\![A]\!]}$.*

Proof. The proof is by induction on A. In each case, we can unfold the definition of ident_A, so we need to show:

- Case for P: $\mathscr{F}^1{}_P = [\![\mathsf{hyp}\,1]\!]$. Works because $\mathscr{F}_1 = 1$.

- Case for $F_\alpha\,A$: $\mathscr{F}^1{}_{\mathscr{F}_\alpha\,[\![A]\!]} = [\![\mathsf{FL}(\mathsf{FR}^1_1(\mathsf{ident}_A))]\!]$ After unfolding the definitions and using the IH, it suffices to show that the composite

$$\mathscr{F}_1\ \mathscr{F}_\alpha\ [\![A]\!] \xrightarrow{\mathscr{F}^{\circ}(\alpha,1)^{-1}} \mathscr{F}_\alpha\ [\![A]\!] \xrightarrow{\mathscr{F}^{\circ}(1,\alpha)} \mathscr{F}_\alpha\ \mathscr{F}_1\ [\![A]\!] \xrightarrow{\mathscr{F}_\alpha\ \mathscr{F}^1{}_{[\![A]\!]}} \mathscr{F}_\alpha\ [\![A]\!]$$

  is $\mathscr{F}_1\,{}_{\mathscr{F}_\alpha\,[\![A]\!]}$, which is true by the $\mathscr{F}^1/\mathscr{F}^{\circ}$ coherence laws.

- Case for $U_\alpha\,A$: $\mathscr{F}^1{}_{\mathscr{U}_\alpha\,[\![A]\!]} = [\![\mathsf{UR}(\mathsf{UL}^1_1(\mathsf{ident}_A))]\!]$

  Expanding the definitions and using the IH and using $\mathscr{U}_1 = 1$, the right-hand side is equal to

$$(\mathscr{F}^{\circ}(1,\alpha)^{-1};(\mathscr{U}_\alpha\ ((\mathscr{F}^1{}_{[\![A]\!]})^{\rhd_1});\mathscr{U}^{\circ}(\alpha,1)^{-1})^{\lhd_\alpha})^{\rhd_\alpha}$$

  By coherence $\mathscr{F}^{\circ}(1,\alpha)^{-1} = \mathscr{F}_\alpha\ (\mathscr{F}^1{}_{\mathscr{U}_\alpha\,[\![A]\!]})$, so by naturality of $-^{\rhd_\alpha}$, it's equal to

$$\mathscr{F}^1;((\mathscr{U}_\alpha\ ((\mathscr{F}^1{}_{[\![A]\!]})^{\rhd_1});\mathscr{U}^{\circ}(\alpha,1)^{-1})^{\lhd_\alpha})^{\rhd_\alpha}$$

  and canceling $-^{\lhd_\alpha \rhd_\alpha}$ gives

$$\mathscr{F}^1;\mathscr{U}_\alpha\ ((\mathscr{F}^1{}_{[\![A]\!]})^{\rhd_1});\mathscr{U}^{\circ}(\alpha,1)^{-1}$$

  so it suffices to show

$$\mathscr{U}_\alpha\ ((\mathscr{F}^1{}_{[\![A]\!]})^{\rhd_1});\mathscr{U}^{\circ}(\alpha,1)^{-1} = 1$$

  But $((\mathscr{F}^1)^{\rhd_1}) = \mathscr{U}^{1^{-1}}$ by conjugation, and $\mathscr{U}^{\circ}(\alpha,1)^{-1} =_{\mathscr{U}_\alpha\,[\![A]\!] \longrightarrow \mathscr{U}_\alpha\,\mathscr{U}_1\,[\![A]\!]} \mathscr{U}_\alpha\ \mathscr{U}^1$ by coherence, so this is true.

$\square$

To prove the cut lemma, it will helpful to use the following equivalent definition of $[\![\mathsf{UL}_e^\gamma(D)]\!]$, which uses only $\mathscr{F}$ operations, except for the counit $(1_{\mathscr{U}_\alpha\,[\![A]\!]})^{\lhd\alpha} : \mathscr{F}_\alpha\,\mathscr{U}_\alpha\,A \longrightarrow A$

LEMMA 4.5. *For any γ, e, D, $[\![\mathsf{UL}_e^\gamma(D)]\!] = \mathscr{F}_e\,; \mathscr{F}^\circ(\alpha,\gamma); \mathscr{F}_\gamma\,(1^{\lhd\alpha}_{\mathscr{U}_\alpha\,[\![A]\!]}); [\![D]\!]$*

Proof. By conjugation for $\mathscr{U}_e$ and $\mathscr{U}^\circ$, $[\![\mathsf{UL}_e^\gamma(D)]\!] = \mathscr{F}_e\,; \mathscr{F}^\circ(\alpha,\gamma); ((\mathscr{U}_\alpha\,([\![D]\!]^{\rhd\gamma}))^{\lhd\alpha})^{\lhd\gamma}$ By naturality of $-^{\lhd\alpha}$,

$$(\mathscr{U}_\alpha\,([\![D]\!]^{\rhd\gamma}))^{\lhd\alpha} = 1^{\lhd\alpha}; ([\![D]\!]^{\rhd\gamma})$$

and by naturality of $-^{\lhd\gamma}$,

$$(1^{\lhd\alpha}; ([\![D]\!]^{\rhd\gamma}))^{\lhd\gamma} = \mathscr{F}_\gamma\,(1^{\lhd\alpha}); ([\![D]\!]^{\rhd\gamma})^{\lhd\gamma}$$

so collapsing inverses gives the result. $\square$

LEMMA 4.6. *For all derivations $D : A\,[\beta] \vdash B$ and $E : B\,[\beta'] \vdash C$, we have*

$$[\![\mathsf{cut}\,D\,E]\!] = \mathscr{F}^\circ(\beta,\beta'); \mathscr{F}_{\beta'}\,[\![D]\!]; [\![E]\!].$$

Proof. The proof is by the same induction on A, D, E that defines cut, and in each case the cut reduces, so we need to show:

- $\mathscr{F}^\circ(\beta,\beta'); \mathscr{F}_{\beta'}\,[\![(\mathsf{hyp}\,e)]\!]; [\![(\mathsf{hyp}\,e')]\!] = [\![\mathsf{hyp}\,(e \circ_2 e')]\!]$

  After expanding definitions, we need to show that

  $$\mathscr{F}^\circ(\beta,\beta'); \mathscr{F}_{\beta'}\,(\mathscr{F}_e\,; \mathscr{F}^1); \mathscr{F}_{e'}\,; \mathscr{F}^1 = \mathscr{F}_{e\circ_2 e'}; \mathscr{F}^1$$

  We have $\mathscr{F}_{\beta'}\,(\mathscr{F}_e\,; \mathscr{F}^1) = \mathscr{F}_{\beta'}\,\mathscr{F}_e\,; \mathscr{F}_{\beta'}\,\mathscr{F}^1$ by functoriality, and $\mathscr{F}_{e'}\,; \mathscr{F}_1\,\mathscr{F}^1 = \mathscr{F}_{\beta'}\,\mathscr{F}^1; \mathscr{F}_{e'}$ by naturality of $\mathscr{F}_{e'}$, so the LHS equals

  $$(\mathscr{F}^\circ(\beta,\beta'); \mathscr{F}_{\beta'}\,\mathscr{F}_e; \mathscr{F}_{e'}\,); \mathscr{F}_1\,\mathscr{F}^1; \mathscr{F}^1$$

  By naturality of $\mathscr{F}^\circ$in α, β, this equals

  $$\mathscr{F}_{e\circ_2 e'}; \mathscr{F}^\circ(1,1); \mathscr{F}_1\,\mathscr{F}^1; \mathscr{F}^1$$

  Coherence implies that $\mathscr{F}^\circ(1,1)^{-1} = \mathscr{F}_1\,\mathscr{F}^1$, so collapsing inverses gives the result.

- $\mathscr{F}^\circ(\beta,\beta');\mathscr{F}_{\beta'}\,[\![(\mathsf{FR}^\gamma_e(D))]\!];[\![(\mathsf{FL}(E))]\!] = [\![(e\circ_2 1)_*(\mathsf{cut}\,D\,E)]\!]$

  After unfolding the definitions on the left-hand side, and using Lemma 4.3 and the IH on the right-hand side, the calculation consists of using naturality of $\mathscr{F}^{\circ-1}_A$ in A to show that $\mathscr{F}_{\beta'}\,\mathscr{F}_\alpha\,[\![D]\!];\mathscr{F}^\circ(\alpha,\beta')^{-1} = \mathscr{F}^\circ(\alpha,\beta')^{-1};\mathscr{F}_{\alpha\circ\beta'}\,[\![D]\!]$, using naturality of $\mathscr{F}^\circ(\alpha,\beta)$ in α,β to show $\mathscr{F}^\circ(\beta,\beta');\mathscr{F}_{\beta'}\,\mathscr{F}_e = \mathscr{F}_{e\circ_2 1_{\beta'}};\mathscr{F}^\circ(\gamma\circ\alpha,\beta')$, and using the associativity coherence to show $\mathscr{F}^\circ(\gamma\circ\alpha,\beta');\mathscr{F}_{\beta'}\,(\mathscr{F}^\circ(\gamma,\alpha)) = \mathscr{F}^\circ(\gamma,\alpha\circ\beta');\mathscr{F}^\circ(\alpha,\beta')$.

- $\mathscr{F}^\circ(\beta,\beta');\mathscr{F}_{\beta'}\,[\![(\mathsf{UR}(D))]\!];[\![(\mathsf{UL}^\gamma_e(E))]\!] = [\![(1\circ_2 e)_*(\mathsf{cut}\,D\,E)]\!]$

  Unfolding the definitions, using Lemma 4.3 and the IH, we need to show $LHS = RHS$, where

  $$LHS := \mathscr{F}^\circ(\beta,\beta');\mathscr{F}_\beta\,((\mathscr{F}^\circ(\beta,\alpha)^{-1};[\![D]\!])^{\rhd\alpha});(\mathscr{U}_\alpha\,([\![E]\!]^{\rhd\gamma});\mathscr{U}^\circ(\alpha,\gamma)^{-1};\mathscr{U}_e\,)^{\lhd\beta'}$$
  $$RHS := \mathscr{F}_{1\circ_2 e}\,;\mathscr{F}^\circ(\beta\circ\alpha,\gamma);\mathscr{F}_\gamma\,[\![D]\!];[\![E]\!]$$

  Using conjugation to move $\mathscr{U}_e$ *a*nd $\mathscr{U}^\circ(\alpha,\gamma)^{-1}$ outside of the $-^{\lhd\beta'}$, we have

  $$(\mathscr{U}_\alpha\,([\![E]\!]^{\rhd\gamma});\mathscr{U}^\circ(\alpha,\gamma)^{-1};\mathscr{U}_e\,)^{\lhd\beta'} = \mathscr{F}_e\,;\mathscr{F}^\circ(\alpha,\gamma);(\mathscr{U}_\alpha\,[\![E]\!]^{\rhd\gamma})^{\lhd\alpha\,\lhd\gamma}$$

  By naturality of $-^{\lhd\alpha}$, the right-hand side of that is equal to

  $$\mathscr{F}_e\,;\mathscr{F}^\circ(\alpha,\gamma);(1^{\lhd\alpha};[\![E]\!]^{\rhd\gamma})^{\lhd\alpha\,\lhd\gamma}$$

  and then by naturality of $-^{\lhd\gamma}$, that is equal to

  $$\mathscr{F}_e\,;\mathscr{F}^\circ(\alpha,\gamma);\mathscr{F}_\gamma\,(1^{\lhd\alpha});([\![E]\!]^{\rhd\gamma})^{\lhd\gamma}$$

  so collapsing inverses, we have overall that

  $$(\mathscr{U}_\alpha\,([\![E]\!]^{\rhd\gamma});\mathscr{U}^\circ(\alpha,\gamma)^{-1};\mathscr{U}_e\,)^{\rhd\beta'} = \mathscr{F}_e\,;\mathscr{F}^\circ(\alpha,\gamma);\mathscr{F}_\gamma\,(1^{\lhd\alpha});[\![E]\!]$$

  Therefore

  $$LHS = \mathscr{F}^\circ(\beta,\beta');\mathscr{F}_\beta\,((\mathscr{F}^\circ(\beta,\alpha)^{-1};[\![D]\!])^{\rhd\alpha});\mathscr{F}_e\,;\mathscr{F}^\circ(\alpha,\gamma);\mathscr{F}_\gamma\,(1^{\lhd\alpha});[\![E]\!]$$

  Moving $\mathscr{F}_e$ to the left using its naturality, this is equal to

  $$\mathscr{F}^\circ(\beta,\beta');\mathscr{F}_e\,;\mathscr{F}_{\alpha\circ\gamma}\,((\mathscr{F}^\circ(\beta,\alpha)^{-1};[\![D]\!])^{\rhd\alpha});\mathscr{F}^\circ(\alpha,\gamma);\mathscr{F}_\gamma\,(1^{\lhd\alpha});[\![E]\!]$$

  and then moving it to the left again using naturality of $\mathscr{F}^\circ(\alpha,\gamma)$ in α,β gives

  $$\mathscr{F}_{(1\circ_2 e)}\,;\mathscr{F}^\circ(\beta,\alpha\circ\gamma);\mathscr{F}_{\alpha\circ\gamma}\,((\mathscr{F}^\circ(\beta,\alpha)^{-1};[\![D]\!])^{\rhd\alpha});\mathscr{F}^\circ(\alpha,\gamma);\mathscr{F}_\gamma\,(1^{\lhd\alpha});[\![E]\!]$$

and moving $\mathscr{F}^{\circ}_{A}(\alpha,\beta)$ to the left using naturality in A gives

$$\mathscr{F}_{(1\circ_2 e)}\,;\mathscr{F}^{\circ}(\beta,\alpha\circ\gamma);\mathscr{F}^{\circ}(\alpha,\gamma);\mathscr{F}_{\gamma}\,\mathscr{F}_{\alpha}\,((\mathscr{F}^{\circ}(\beta,\alpha)^{-1};[\![D]\!])^{\rhd\alpha});\mathscr{F}_{\gamma}\,(1^{\lhd\alpha});[\![E]\!]$$

By the associativity coherence, this is equal to

$$\mathscr{F}_{(1\circ_2 e)}\,;\mathscr{F}^{\circ}(\beta\circ\alpha,\gamma);\mathscr{F}_{\gamma}\,\mathscr{F}^{\circ}(\beta,\alpha);\mathscr{F}_{\gamma}\,\mathscr{F}_{\alpha}\,((\mathscr{F}^{\circ}(\beta,\alpha)^{-1};[\![D]\!])^{\rhd\alpha});\mathscr{F}_{\gamma}\,(1^{\lhd\alpha});[\![E]\!]$$

so collecting the three terms that are under $\mathscr{F}_{\gamma}$, to show that $LHS = RHS$, it suffices to show that

$$\mathscr{F}^{\circ}(\beta,\alpha);\mathscr{F}_{\alpha}\,((\mathscr{F}^{\circ}(\beta,\alpha)^{-1};[\![D]\!])^{\rhd\alpha});(1^{\lhd\alpha}) = [\![D]\!]$$

By naturality of the adjunction,

$$\begin{aligned}\mathscr{F}_{\alpha}\,((\mathscr{F}^{\circ}(\beta,\alpha)^{-1};[\![D]\!])^{\rhd\alpha});(1^{\lhd\alpha}) &= (((\mathscr{F}^{\circ}(\beta,\alpha)^{-1};[\![D]\!])^{\rhd\alpha});1)^{\lhd\alpha}\\ &= (\mathscr{F}^{\circ}(\beta,\alpha)^{-1};[\![D]\!])\end{aligned}$$

so collapsing inverses gives the result.

- $\mathscr{F}^{\circ}(\beta,\beta');\mathscr{F}_{\beta'}\,[\![D]\!];[\![(\mathsf{FR}^{\gamma}_{e}(E))]\!] = [\![\mathsf{FR}^{\beta\circ\gamma}_{1\circ_2 e}(\mathsf{cut}\,D\,E)]\!]$

  After unfolding the definitions and using the IH on the right-hand side, the proof uses naturality of $\mathscr{F}_{e\,A}$ in A to show $\mathscr{F}_{\beta'}\,[\![D]\!];\mathscr{F}_{e}\, = \mathscr{F}_{e}\,;\mathscr{F}_{\gamma\circ\alpha}\,[\![D]\!]$, naturality of $\mathscr{F}^{\circ}_{A}$ in A to show $\mathscr{F}_{\gamma\circ\alpha}\,[\![D]\!];\mathscr{F}^{\circ}(\gamma,\alpha) = \mathscr{F}^{\circ}(\gamma,\alpha);\mathscr{F}_{\alpha}\,\mathscr{F}_{\gamma}\,[\![D]\!]$, naturality of $\mathscr{F}^{\circ}(\alpha,\beta)$ in α,β to equate $\mathscr{F}^{\circ}(\beta,\beta');\mathscr{F}_{e}\, = \mathscr{F}_{1\circ_2 e}\,;\mathscr{F}^{\circ}(\beta,\gamma\circ\alpha)$ and the associativity coherence to equate $\mathscr{F}^{\circ}(\beta,\gamma\circ\alpha);\mathscr{F}^{\circ}(\gamma,\alpha) = \mathscr{F}^{\circ}(\beta\circ\gamma,\alpha);\mathscr{F}_{\alpha}\,(\mathscr{F}^{\circ}(\beta,\gamma))$.

- $\mathscr{F}^{\circ}(\beta,\beta');\mathscr{F}_{\beta'}\,[\![D]\!];[\![(\mathsf{UR}(E))]\!] = [\![\mathsf{UR}(\mathsf{cut}\,D\,E)]\!]$

  After expanding definitions and using the IH, we need to show

  $$\begin{aligned}&\mathscr{F}^{\circ}(\beta,\beta');\mathscr{F}_{\beta'}\,[\![D]\!];(\mathscr{F}^{\circ}(\beta',\alpha)^{-1};[\![E]\!])^{\rhd\alpha}\\ &\qquad = (\mathscr{F}^{\circ}(\beta\circ\beta',\alpha)^{-1};\mathscr{F}^{\circ}(\beta,\beta'\circ\alpha);\mathscr{F}_{\beta'\circ\alpha}\,[\![D]\!];[\![E]\!])^{\rhd\alpha}.\end{aligned}$$

  By the associativity coherence,

  $$\mathscr{F}^{\circ}(\beta\circ\beta',\alpha)^{-1} = \mathscr{F}_{\alpha}\,(\mathscr{F}^{\circ}(\beta,\beta'));\mathscr{F}^{\circ}(\beta',\alpha)^{-1};\mathscr{F}^{\circ}(\beta,\beta'\circ\alpha)^{-1}$$

  and plugging this in to the RHS and then collapsing inverses gives

  $$(\mathscr{F}_{\alpha}\,\mathscr{F}^{\circ}(\beta,\beta');\mathscr{F}^{\circ}(\beta',\alpha)^{-1};\mathscr{F}_{\beta'\circ\alpha}\,[\![D]\!];[\![E]\!])^{\rhd\alpha}.$$

By naturality of ${}^{\rhd\alpha}$, this is the same as

$$\mathscr{F}^{\circ}(\beta,\beta');(\mathscr{F}^{\circ}(\beta',\alpha)^{-1};\mathscr{F}_{\beta'\circ\alpha}\,[\![D]\!];[\![E]\!])^{\rhd\alpha}$$

By naturality of $\mathscr{F}^{\circ}(\beta',\alpha)^{-1}$, this is the same as

$$\mathscr{F}^{\circ}(\beta,\beta');(\mathscr{F}_{\alpha}\,\mathscr{F}_{\beta'}\,[\![D]\!];\mathscr{F}^{\circ}(\beta',\alpha)^{-1};[\![E]\!])^{\rhd\alpha}$$

so using naturality of ${}^{\rhd\alpha}$ again gives the result.

- $\mathscr{F}^{\circ}(\beta,\beta');\mathscr{F}_{\beta'}\,[\![(\mathsf{FL}(D))]\!];[\![E]\!] = [\![\mathsf{FL}(\mathsf{cut}\,D\,E)]\!]$ (note: we could assume that E is not a right rule, but this assumption is not necessary). After expanding the definitions and using the IH, the main step is to use the associativity coherence to show $\mathscr{F}^{\circ}(\alpha,\beta\circ\beta');\mathscr{F}^{\circ}(\beta,\beta') = \mathscr{F}^{\circ}(\alpha\circ\beta,\beta');\mathscr{F}_{\beta'}\,(\mathscr{F}^{\circ}(\alpha,\beta))$.

- $\mathscr{F}^{\circ}(\beta,\beta');\mathscr{F}_{\beta'}\,[\![(\mathsf{UL}^{\gamma}_{e}(D))]\!];[\![E]\!] = [\![\mathsf{UL}^{\gamma\circ\beta'}_{e\circ_2 1}(\mathsf{cut}\,D\,E)]\!]$ (note: we could assume that E is not a right rule, but this assumption is not necessary).

  Applying Lemma 4.5, to the left and the right sides, and using the IH, we need to show $LHS = RHS$, where

  $$\begin{array}{l} LHS := \mathscr{F}^{\circ}(\beta,\beta');\mathscr{F}_{\beta'}\,\mathscr{F}_{e};\mathscr{F}_{\beta'}\,\mathscr{F}^{\circ}(\alpha,\gamma);\mathscr{F}_{\beta'}\,\mathscr{F}_{\gamma}\,(1^{\lhd\alpha});\mathscr{F}_{\beta'}\,[\![D]\!];[\![E]\!] \\ RHS := \mathscr{F}_{e\circ_2 1};\mathscr{F}^{\circ}(\alpha,\gamma\circ\beta');\mathscr{F}_{\gamma\circ\beta'}\,(1^{\lhd\alpha});\mathscr{F}^{\circ}(\gamma,\beta');\mathscr{F}_{\beta'}\,[\![D]\!];[\![E]\!] \end{array}$$

  By naturality of $\mathscr{F}^{\circ}(\alpha,\beta)$ in α,β

  $$\mathscr{F}^{\circ}(\beta,\beta');\mathscr{F}_{\beta'}\,\mathscr{F}_{e} = \mathscr{F}_{e\circ_2 1};\mathscr{F}^{\circ}(\alpha\circ\gamma,\beta')$$

  and by the associativity coherence,

  $$\mathscr{F}^{\circ}(\alpha\circ\gamma,\beta');\mathscr{F}_{\beta'}\,\mathscr{F}^{\circ}(\alpha,\gamma) = \mathscr{F}^{\circ}(\alpha,\gamma\circ\beta');\mathscr{F}^{\circ}(\gamma,\beta')$$

  so

  $$LHS = \mathscr{F}_{e\circ_2 1};\mathscr{F}^{\circ}(\alpha,\gamma\circ\beta');\mathscr{F}^{\circ}(\gamma,\beta');\mathscr{F}_{\beta'}\,\mathscr{F}_{\gamma}\,(1^{\lhd\alpha});\mathscr{F}_{\beta'}\,[\![D]\!];[\![E]\!]$$

  Therefore using naturality of $\mathscr{F}^{\circ}(\gamma,\beta')_{A}$ in A to move it to the right gives the result.

□

Next, we validate the rules for $\approx$.

THEOREM 4.7: SOUNDESS OF THE EQUATIONAL THEORY.. *If $D \approx D'$ then $[\![D]\!] = [\![D']\!]$.*

Proof. Because the goal is equality of morphisms, the congruence (equivalence relation, compatibility for each derivation constructor) rules are all true. It remains to validate the axioms:

- $[\![D]\!] = [\![\mathsf{FL}(\mathsf{cut}\ (\mathsf{FR}^1_1(\mathsf{ident}_A))\ D)]\!]$ when $D : F_\alpha\, A\, [\beta] \vdash C$

  After expanding the definitions and using Lemmas 4.4, 4.6, it suffices to show

  $$\mathscr{F}^\circ(\alpha,\beta)^{-1};\mathscr{F}^\circ(\alpha,\beta);\mathscr{F}_\beta\ (\mathscr{F}_1;\mathscr{F}^\circ(1,\alpha);\mathscr{F}_\alpha\ \mathscr{F}^1);[\![D]\!] = [\![D]\!]$$

  This is true because $\mathscr{F}_1 = 1$ and because $\mathscr{F}^\circ(1,\alpha)^{-1} = \mathscr{F}_\alpha\ \mathscr{F}^1$, so canceling identities and inverses gives the result.

- $[\![D]\!] = [\![\mathsf{UR}(\mathsf{cut}\ D\ (\mathsf{UL}^1_1(\mathsf{ident}_A)))]\!]$ when $D : C\,[\beta] \vdash U_\alpha\, A$

  After expanding the definitions and using Lemmas 4.4, 4.6, it suffices to show

  $$(\mathscr{F}^\circ(\beta,\alpha)^{-1};\mathscr{F}^\circ(\beta,\alpha);\mathscr{F}_\alpha\ [\![D]\!];(\mathscr{U}_\alpha\ (\mathscr{F}^{1^{\rhd_1}});\mathscr{U}^\circ(\alpha,1)^{-1};\mathscr{U}_1)^{\lhd_\alpha})^{\rhd_\alpha} = [\![D]\!]$$

  Canceling the $\mathscr{F}^\circ(\beta,\alpha)$ and using naturality of $-^{\rhd_\alpha}$, this is equal to

  $$[\![D]\!];((\mathscr{U}_\alpha\ (\mathscr{F}^{1^{\rhd_1}});\mathscr{U}^\circ(\alpha,1)^{-1};\mathscr{U}_1)^{\lhd_\alpha})^{\rhd_\alpha}$$

  so it suffices to show that the later is the identity. Canceling the adjunction round-trip and $\mathscr{U}_1$, it is equal to

  $$\mathscr{U}_\alpha\ (\mathscr{F}^{1^{\rhd_1}});\mathscr{U}^\circ(\alpha,1)^{-1}$$

  By conjugation for $\mathscr{F}^1$, we have $\mathscr{F}^{1^{\lhd_1}} = \mathscr{U}^1$, and by the associativity coherence, we have $\mathscr{U}^\circ(\alpha,1)^{-1} = \mathscr{U}_\alpha\ \mathscr{U}^1$, so canceling inverses gives the result.

- $[\![\mathsf{FR}^\gamma_e(e_{2*}(D))]\!] = [\![\mathsf{FR}^{\gamma'}_{((e_2\circ_2 1)\cdot e)}(D)]\!]$ when $e : \gamma\circ\alpha \Rightarrow \beta$ and $D : C\,[\gamma'] \vdash A$ and $e_2 : \gamma' \Rightarrow \gamma$.

  After expanding the definitions and using Lemma 4.3, we need to show

  $$\mathscr{F}_e;\mathscr{F}^\circ(\gamma,\alpha);\mathscr{F}_\alpha\ \mathscr{F}_{e_2};\mathscr{F}_\alpha\ [\![D]\!] = \mathscr{F}_{(e_2\circ_2 1)\cdot e};\mathscr{F}^\circ(\gamma',\alpha);\mathscr{F}_\alpha\ [\![D]\!]$$

  This is true using functoriality to show $\mathscr{F}_{(e_2\circ_2 1)\cdot e} = \mathscr{F}_e;\mathscr{F}_{(e_2\circ_2 1)}$, and naturality of $\mathscr{F}^\circ(\alpha,\beta)$ in α,β to show $\mathscr{F}_{(e_2\circ_2 1)};\mathscr{F}^\circ(\gamma',\alpha) = \mathscr{F}^\circ(\gamma,\alpha);\mathscr{F}_\alpha\ \mathscr{F}_{e_2}$.

- $[\![\mathsf{UL}^{\gamma}_{e}(e_{2*}(D))]\!] = [\![\mathsf{UL}^{\gamma'}_{((1\circ_2 e_2)\cdot e)}(D)]\!]$ when $e : \gamma \circ \alpha \Rightarrow \beta$ and $D : C[\gamma'] \vdash A$ and $e_2 : \gamma' \Rightarrow \gamma$.

  Using Lemmas 4.5 and 4.3, we need to show

  $$\mathscr{F}_e; \mathscr{F}^\circ(\alpha,\gamma); \mathscr{F}_\gamma\,(1^{\rhd\alpha}); \mathscr{F}_{e_2}; [\![D]\!] = \mathscr{F}_{(1\circ_2 e_2)\cdot e}; \mathscr{F}^\circ(\alpha,\gamma'); \mathscr{F}_{\gamma'}\,(1^{\rhd\alpha}); [\![D]\!]$$

  By functoriality, $\mathscr{F}_{(1\circ_2 e_2)\cdot e} = \mathscr{F}_e\,; \mathscr{F}_{1\circ_2 e_2}$, and by naturality of $\mathscr{F}^\circ(\alpha,\beta)$ in α,β, $\mathscr{F}_{(1\circ_2 e_2)}; \mathscr{F}^\circ(\alpha,\gamma') = \mathscr{F}^\circ(\alpha,\gamma); \mathscr{F}_{e_2}$, so the right-hand side is equal to

  $$\mathscr{F}_e; \mathscr{F}^\circ(\alpha,\gamma); \mathscr{F}_{e_2}; \mathscr{F}_{\gamma'}\,(1^{\rhd\alpha}); [\![D]\!]$$

  so using naturality of $\mathscr{F}_{e_2}$ gives the result.

- $[\![\mathsf{UL}^{\beta}_{e_2}(\mathsf{FR}^{\gamma}_{e_1}(D))]\!] = [\![\mathsf{FR}^{\delta_3}_{e_4}(\mathsf{UL}^{\gamma}_{e_3}(D))]\!]$ when $(1\circ_2 e_1)\cdot e_2 = (e_3\circ_2 1)\cdot e_4$, where $e_1 : (\gamma\circ\alpha) \Rightarrow \beta$ and $e_2 : (\delta_1\circ\beta) \Rightarrow \delta_2$ and $e_3 : (\delta_1\circ\gamma) \Rightarrow \delta_3$ and $e_4 : (\delta_3\circ\alpha) \Rightarrow \delta_2$ and

  Expanding the definitions and using Lemma 4.5, we need to show

  $$\begin{array}{rl} & \mathscr{F}_{e_2}; \mathscr{F}^\circ(\delta_1,\beta); \mathscr{F}_\beta\,(1^{\lhd\delta_1}); \mathscr{F}_{e_1}; \mathscr{F}^\circ(\gamma,\alpha); \mathscr{F}_\alpha\,[\![D]\!] \\ = & \mathscr{F}_{e_4}; \mathscr{F}^\circ(\delta_3,\alpha); \mathscr{F}_\alpha\,\mathscr{F}_{e_3}; \mathscr{F}_\alpha\,\mathscr{F}^\circ(\delta_1,\gamma); \mathscr{F}_\alpha\,\mathscr{F}_\gamma\,(1^{\rhd\delta_1}); \mathscr{F}_\alpha\,[\![D]\!] \end{array}$$

  Using naturality of $\mathscr{F}_{e_1}$ and $\mathscr{F}^\circ(\gamma,\alpha)$, the left-hand side is equal to

  $$\mathscr{F}_{e_2}; \mathscr{F}^\circ(\delta_1,\beta); \mathscr{F}_{e_1}; \mathscr{F}^\circ(\gamma,\alpha); \mathscr{F}_\alpha\,\mathscr{F}_\gamma\,(1^{\lhd\delta_1}); \mathscr{F}_\alpha\,[\![D]\!]$$

  so it suffices to show

  $$\mathscr{F}_{e_2}; \mathscr{F}^\circ(\delta_1,\beta); \mathscr{F}_{e_1}; \mathscr{F}^\circ(\gamma,\alpha) = \mathscr{F}_{e_4}; \mathscr{F}^\circ(\delta_3,\alpha); \mathscr{F}_\alpha\,\mathscr{F}_{e_3}; \mathscr{F}_\alpha\,\mathscr{F}^\circ(\delta_1,\gamma)$$

  Using naturality of $\mathscr{F}^\circ(\alpha,\beta)$ in α,β the LHS equals

  $$\mathscr{F}_{e_2}; \mathscr{F}_{1\circ_2 e_1}; \mathscr{F}^\circ(\delta_1,\gamma\circ\alpha); \mathscr{F}^\circ(\gamma,\alpha)$$

  and the RHS equals

  $$\mathscr{F}_{e_4}; \mathscr{F}_{e_3\circ_2 1}; \mathscr{F}^\circ(\delta_1\circ\gamma,\alpha); \mathscr{F}_\alpha\,\mathscr{F}^\circ(\delta_1,\gamma)$$

  But $\mathscr{F}^\circ(\delta_1,\gamma\circ\alpha); \mathscr{F}^\circ(\gamma,\alpha) = \mathscr{F}^\circ(\delta_1\circ\gamma,\alpha); \mathscr{F}_\alpha\,\mathscr{F}^\circ(\delta_1,\gamma)$ by the associativity coherence, and $\mathscr{F}_{e_2}; \mathscr{F}_{1\circ_2 e_1} = \mathscr{F}_{e_4}; \mathscr{F}_{e_3\circ_2 1}$ by functoriality using the assumption that $(1\circ_2 e_1)\cdot e_2 = (e_3\circ_2 1)\cdot e_4$.

$\square$

Just as the class of models we are considering supported interpretations of $e_*(D)$ and ident_A and $\mathsf{cut}\ D\ E$ in general, the admissible rules for $\approx$ hold in general:

THEOREM 4.8: SOUNDNESS OF ADMISSIBLE EQUATIONAL RULES. *The admissible rules for $D \approx D'$ in Section 2.3 are true in the semantics.*

Proof.

- For $1_*(D) = D$ and $(e_1 \cdot e_2)_*(D) = e_{2*}(e_{1*}(D))$, by Lemma 4.3, we need to show $\mathscr{F}_e; [\![D]\!] = [\![D]\!]$ and $\mathscr{F}_{e_1 \cdot e_2}; [\![D]\!] = \mathscr{F}_{e_2}; \mathscr{F}_{e_1}; [\![D]\!]$, which are true by functoriality.

- For the congruence rules:

$$\frac{D \approx D'}{e_*(D) \approx e_*(D')} \quad \frac{D \approx D'}{\mathsf{cut}\ D\ E \approx \mathsf{cut}\ D'\ E} \quad \frac{E \approx E'}{\mathsf{cut}\ D\ E \approx \mathsf{cut}\ D\ E'}$$

  By assumption, $[\![D]\!] = [\![D']\!]$ or $[\![E]\!] = [\![E']\!]$. By Lemma 4.3 and 4.6, $[\![e_*(D)]\!]$ and $[\![\mathsf{cut}\ D\ E]\!]$ are compositional in $[\![D]\!]$ and $[\![E]\!]$, so the conclusions are equal as well.

- For $(e \circ_2 e')_*(\mathsf{cut}\ D\ D') \approx \mathsf{cut}\ (e_*(D))\ (e'_*(D'))$ where $e : \alpha \Rightarrow \beta$ and $e' : \alpha' \Rightarrow \beta'$, by Lemmas 4.3 and 4.6, we need to show

$$\mathscr{F}_{(e \circ_2 e')}; \mathscr{F}^\circ(\alpha, \alpha'); \mathscr{F}_{\alpha'}\ ([\![D]\!]); [\![D']\!] = \mathscr{F}^\circ(\beta, \beta'); \mathscr{F}_{\beta'}\ \mathscr{F}_e; \mathscr{F}_{\beta'}\ [\![D]\!]; \mathscr{F}_{e'}; [\![D']\!]$$

  Using naturality for $\mathscr{F}_{e'}$, the right-hand side equals

$$\mathscr{F}^\circ(\beta, \beta'); \mathscr{F}_{\beta'}\ \mathscr{F}_e; \mathscr{F}_{e'}; \mathscr{F}_{\alpha'}\ [\![D]\!]; [\![D']\!]$$

  so naturality of $\mathscr{F}^\circ(\alpha, \beta)$ in α, β gives the result.

- For $\mathsf{cut}\ D_1\ (\mathsf{cut}\ D_2\ D_3) \approx \mathsf{cut}\ (\mathsf{cut}\ D_1\ D_2)\ D_3$, by Lemma 4.6, we need to show

$$\begin{array}{rl} & \mathscr{F}^\circ(\beta_1, \beta_2 \circ \beta_3); \mathscr{F}_{\beta_2 \circ \beta_3}\ [\![D_1]\!]; \mathscr{F}^\circ(\beta_2, \beta_3); \mathscr{F}_{\beta_3}\ [\![D_2]\!]; [\![D_3]\!] \\ = & \mathscr{F}^\circ(\beta_1 \circ \beta_2, \beta_3); \mathscr{F}_{\beta_3}\ \mathscr{F}^\circ(\beta_1, \beta_2); \mathscr{F}_{\beta_3}\ \mathscr{F}_{\beta_2}\ [\![D_1]\!]; \mathscr{F}_{\beta_3}\ [\![D_2]\!]; [\![D_3]\!] \end{array}$$

  By naturality of $\mathscr{F}^\circ(\beta_2, \beta_3)_A$ in A, the LHS equals

$$\mathscr{F}^\circ(\beta_1, \beta_2 \circ \beta_3); \mathscr{F}^\circ(\beta_2, \beta_3); \mathscr{F}_{\beta_2}\ \mathscr{F}_{\beta_3}\ [\![D_1]\!]; \mathscr{F}_{\beta_3}\ [\![D_2]\!]; [\![D_3]\!]$$

  so the associativity coherence gives the result.

- For $\mathsf{cut}\ D\ \mathsf{ident} \approx D$, by Lemmas 4.6 and 4.4, it suffices to show

$$\mathscr{F}^\circ(\beta, 1); \mathscr{F}_1\ [\![D]\!]; \mathscr{F}^1 = [\![D]\!]$$

  By naturality of $\mathscr{F}^1$, the left-hand side equals $\mathscr{F}^\circ(\beta, 1); \mathscr{F}^1; [\![D]\!]$, so coherence (and a unit law for $\mathscr{C}_p$—interpreting a unit law for the syntax involves a unit law for the semantics) gives the result.

- For cut ident $D \approx D$, by by Lemmas 4.6 and 4.4, it suffices to show

$$\mathscr{F}^{\circ}(1,\beta);\mathscr{F}_{\beta}\,\mathscr{F}^{1};[\![D]\!] = [\![D]\!]$$

which is true by coherence (and a unit law for $\mathscr{C}_p$).

- The unrestricted left-commutative rules $\mathsf{cut}\ (\mathsf{FL}(D))\ E \approx \mathsf{FL}(\mathsf{cut}\ D\ E)$ and $\mathsf{cut}\ (\mathsf{UL}^{\gamma}_{e}(D))\ E \approx \mathsf{UL}^{\gamma\circ\alpha}_{e\circ_2 1}(\mathsf{cut}\ D\ E)$ were checked as part of the proof of Lemma 4.6 above.

$\square$

LEMMA 4.9: INTERPRETATION OF COPRODUCTS. *If each $\mathscr{C}_p$ has coproducts, then Theorem 4.1 and Lemmas 4.3 and 4.4 and 4.6 and Theorem 4.7 are true when the rules for coproducts in Figure 1 are added to the logic.*

Proof. Write $inl : A \longrightarrow A+B$ and $inr : B \longrightarrow A+B$ and $[m_1,m_2]$ for the coproduct maps.

- First, we show how to interpret the sequent calculus rules. For $\mathsf{Inl}(D)$ and $\mathsf{Inr}(D)$, define

$$\begin{array}{rcl} [\![\mathsf{Inl}(D)]\!] & := & [\![D]\!];inl \\ [\![\mathsf{Inr}(D)]\!] & := & [\![D]\!];inr \end{array}$$

  The left case makes sense because $[\![D]\!] : [\![C]\!] \longrightarrow [\![A]\!]$, so post-composing with *inl* has the right codomain; the other case is analogous.

  For $[\![\mathsf{Case}(D_1,D_2)]\!]$, we essentially need to do the proof that left adjoints preserve coproducts: we have $[\![D_1]\!] : \mathscr{F}_{\alpha}\,A \longrightarrow C$ and $[\![D_2]\!] : \mathscr{F}_{\alpha}\,B \longrightarrow C$ and we want a map $\mathscr{F}_{\alpha}\,(A+B) \longrightarrow C$, which we define as follows:

$$[\![\mathsf{Case}(D_1,D_2)]\!] \quad := \quad [[\![D_1]\!]^{\rhd\alpha},[\![D_2]\!]^{\rhd\alpha}]^{\lhd\alpha}$$

- Next, we give the new cases of Lemma 4.3, where $e : \alpha \Rightarrow \beta$ and the given derivation has mode α. For $\mathsf{Inl}(D)$, we need to show

$$[\![e_*(D)]\!];inl = \mathscr{F}_e;[\![D]\!];inl$$

  which is immediate by the IH (and associativity). The $\mathsf{Inr}(D)$ case is analogous. For $\mathsf{Case}(D_1,D_2)$, after expanding the definitions and using the IH, we need to show

$$[(\mathscr{F}_e;[\![D_1]\!])^{\rhd\beta},(\mathscr{F}_e;[\![D_2]\!])^{\rhd\beta}]^{\lhd\beta} = \mathscr{F}_e;([[\![D_1]\!]^{\rhd\alpha},[\![D_2]\!]^{\rhd\alpha}])^{\lhd\alpha}$$

By conjugation, the right-hand side equals

$$([\llbracket D_1 \rrbracket^{\rhd\alpha}, \llbracket D_2 \rrbracket^{\rhd\alpha}]; \mathscr{U}_e)^{\lhd\beta}$$

and the left-hand side equals

$$[\llbracket D_1 \rrbracket^{\rhd\beta}; \mathscr{U}_e, \llbracket D_2 \rrbracket^{\rhd\beta}; \mathscr{U}_e]^{\lhd\beta}$$

and these are equal by the uniqueness part of the universal property for coproducts.

- Next, we give the new case of Lemma 4.4: $\llbracket \mathsf{Case}(\mathsf{Inl}(\mathsf{ident}_A), \mathsf{Inr}(\mathsf{ident}_B)) \rrbracket = \mathscr{F}^1$. After expanding the definitions and using the IH, we need to show

  $$[(\mathscr{F}^1; inl)^{\rhd 1}, (\mathscr{F}^1; inr)^{\rhd 1}]^{\lhd 1}$$

  By conjugation for $\mathscr{F}^1$, this is equal to

  $$[inl; \mathscr{U}^{1^{-1}}, inr; \mathscr{U}^{1^{-1}}]^{\lhd 1}$$

  By uniqueness for coproducts, this is equal to

  $$([inl, inr]; \mathscr{U}^{1^{-1}})^{\lhd 1}$$

  By conjugation for $\mathscr{U}^1$, this is equal to

  $$\mathscr{F}^1; [inl, inr]$$

  and by uniqueness for coproducts, $[inl, inr] = 1_{A+B}$.

- Next, we give the new cases of Lemma 4.6. There are 5 reductions; we show the $\mathsf{Inl}(-)$ cases of the principal and right-commutative cuts, and the left-commutative cut case; the $\mathsf{Inr}(-)$ cases are analogous.

  – For $\mathsf{cut}\ (\mathsf{Inl}(D))\ (\mathsf{Case}(E_1, E_2)) := \mathsf{cut}\ D\ E_1$, by the IH we need to show that

    $$\mathscr{F}^\circ(\beta, \beta'); \mathscr{F}_{\beta'}\ \llbracket D \rrbracket; \llbracket E_1 \rrbracket = \mathscr{F}^\circ(\beta, \beta'); \mathscr{F}_{\beta'}\ \llbracket D \rrbracket; \mathscr{F}_{\beta'}\ inl; [\llbracket E_1 \rrbracket^{\rhd\beta'}, \llbracket E_2 \rrbracket^{\rhd\beta'}]^{\lhd\beta'}$$

    By naturality of $\beta'^{\lhd}$, $\mathscr{F}_{\beta'}\ inl; [\llbracket E_1 \rrbracket^{\rhd\beta'}, \llbracket E_2 \rrbracket^{\rhd\beta'}]^{\lhd\beta'} = (inl; [\llbracket E_1 \rrbracket^{\rhd\beta'}, \llbracket E_2 \rrbracket^{\rhd\beta'}])^{\lhd\beta'}$, which by the universal property for coproducts equals $(\llbracket E_1 \rrbracket^{\rhd\beta'})^{\lhd\beta'}$, which equals $\llbracket E_1 \rrbracket$ by collapsing inverses.

  – For $\mathsf{cut}\ D\ (\mathsf{Inl}(E)) := \mathsf{Inl}(\mathsf{cut}\ D\ E)$, the result is immediate by the IH.

– For cut $(\mathsf{Case}(D_1, D_2))\, E := \mathsf{Case}(\mathsf{cut}\, D_1\, E, \mathsf{cut}\, D_2\, E)$, by the IH we need to show that

$$\begin{array}{rl} & [(\mathscr{F}^\circ(\beta,\beta');\mathscr{F}_{\beta'}\,[\![D_1]\!];[\![E]\!])^{\rhd_{\beta\circ\beta'}},(\mathscr{F}^\circ(\beta,\beta');\mathscr{F}_{\beta'}\,[\![D_2]\!];[\![E]\!])^{\rhd_{\beta\circ\beta'}}]^{\lhd_{\beta\circ\beta'}} \\ = & \mathscr{F}^\circ(\beta,\beta');\mathscr{F}_{\beta'}\,[[\![D_1]\!]^{\rhd_\beta},[\![D_2]\!]^{\rhd_\beta}]^{\lhd_\beta};[\![E]\!] \end{array}$$

Conjugating the $\mathscr{F}^\circ(\beta,\beta')$ outside the $-^{\lhd_{\beta\circ\beta'}}$, and then conjugating the resulting $\mathscr{U}^\circ(\beta,\beta')$ outside the $-^{\rhd_{\beta\circ\beta'}}$, the left-hand side is equal to

$$\mathscr{F}^\circ(\beta,\beta');([((\mathscr{F}_{\beta'}\,[\![D_1]\!];[\![E]\!])^{\rhd_{\beta'}})^{\rhd_\beta},((\mathscr{F}_{\beta'}\,[\![D_2]\!];[\![E]\!])^{\rhd_{\beta'}})^{\rhd_\beta}]^{\lhd_\beta})^{\lhd_{\beta'}}$$

By naturality of the adjunction, this is the same as

$$\mathscr{F}^\circ(\beta,\beta');([([\![D_1]\!];[\![E]\!]^{\rhd_{\beta'}})^{\rhd_\beta},([\![D_2]\!];[\![E]\!]^{\rhd_{\beta'}})^{\rhd_\beta}]^{\lhd_\beta})^{\lhd_{\beta'}}$$

and then

$$\mathscr{F}^\circ(\beta,\beta');([[\![D_1]\!]^{\rhd_\beta};\mathscr{U}_\beta\,([\![E]\!]^{\rhd_{\beta'}}),[\![D_2]\!]^{\rhd_\beta};\mathscr{U}_\beta\,([\![E]\!]^{\rhd_{\beta'}})]^{\lhd_\beta})^{\lhd_{\beta'}}$$

By uniqueness for coproducts, this is

$$\mathscr{F}^\circ(\beta,\beta');(([[\![D_1]\!]^{\rhd_\beta},[\![D_2]\!]^{\rhd_\beta}];\mathscr{U}_\beta\,([\![E]\!]^{\rhd_{\beta'}}))^{\lhd_\beta})^{\lhd_{\beta'}}$$

By naturality of the adjunction, that is

$$\mathscr{F}^\circ(\beta,\beta');(([[\![D_1]\!]^{\rhd_\beta},[\![D_2]\!]^{\rhd_\beta}])^{\lhd_\beta};([\![E]\!]^{\rhd_{\beta'}}))^{\lhd_{\beta'}}$$

and then

$$\mathscr{F}^\circ(\beta,\beta');\mathscr{F}_{\beta'}\,(([[\![D_1]\!]^{\rhd_\beta},[\![D_2]\!]^{\rhd_\beta}])^{\lhd_\beta});([\![E]\!]^{\rhd_{\beta'}})^{\lhd_{\beta'}}$$

so collapsing inverses gives the result.

- For $D \approx \mathsf{Case}(\mathsf{cut}\,(\mathsf{Inl}(\mathsf{ident}_A))\, D, \mathsf{cut}\,(\mathsf{Inr}(\mathsf{ident}_B))\, D)$ (where $\mathrm{D} : A + B\,[\alpha] \vdash C$), we need to show that

$$[(\mathscr{F}^\circ(1,\alpha);\mathscr{F}_\alpha\,\mathscr{F}^1;\mathscr{F}_\alpha\,inl;[\![D]\!])^{\rhd_\alpha},(\mathscr{F}^\circ(1,\alpha);\mathscr{F}_\alpha\,\mathscr{F}^1;\mathscr{F}_\alpha\,inr;[\![D]\!])^{\rhd_\alpha}]^{\lhd_\alpha}$$

By coherence, $\mathscr{F}^\circ(1,\alpha) = \mathscr{F}_\alpha\,\mathscr{F}^{1^{-1}}$, so this is

$$[(\mathscr{F}_\alpha\,inl;[\![D]\!])^{\rhd_\alpha},(\mathscr{F}_\alpha\,inr;[\![D]\!])^{\rhd_\alpha}]^{\lhd_\alpha}.$$

By naturality of the adjunction, this is

$$[inl;[\![D]\!]^{\rhd_\alpha},inr;[\![D]\!]^{\rhd_\alpha}]^{\lhd_\alpha}.$$

which by uniqueness for coproducts is

$$([\![D]\!]^{\rhd_\alpha})^{\lhd_\alpha}.$$

so collapsing inverses gives the result.

- The rule $\mathsf{Inl}(\mathsf{UL}^{\gamma}_{e}(D)) \approx \mathsf{UL}^{\gamma}_{e}(\mathsf{Inl}(D))$ (and the analogous rule for $\mathsf{Inr}(D)$) is immediate by Lemma 4.5, because $[\![\mathsf{UL}^{\gamma}_{e}(D)]\!]$ precomposes $[\![D]\!]$ with something, and $[\![\mathsf{Inl}(D)]\!]$ postcomposes $[\![D]\!]$ with *inl*.

$\square$

5. Adjoint Triples

In this section, we consider two mode theories for adjoint triples $L \dashv M \dashv R$. The first corresponds to a general adjoint triple, while the second adds some additional properties motivated by the triple $\Delta \dashv \Gamma \dashv \nabla$ in axiomatic cohesion.

5.1. Walking Adjunction

Our first mode 2-category is the walking adjunction $\mathsf{d} \dashv \mathsf{n}$, which is generated by

- objects c and s
- 1-cells $\mathsf{d} : \mathsf{s} \geq \mathsf{c}$ and $\mathsf{n} : \mathsf{c} \geq \mathsf{s}$
- 2-cells $\mathsf{unit} : 1_{\mathsf{c}} \Rightarrow \mathsf{n} \circ \mathsf{d}$ and $\mathsf{counit} : \mathsf{d} \circ \mathsf{n} \Rightarrow 1_{\mathsf{s}}$ satisfying $(1_{\mathsf{d}} \circ_2 \mathsf{unit}) \cdot (\mathsf{counit} \circ_2 1_{\mathsf{d}}) = 1$ and $(\mathsf{unit} \circ_2 1_{\mathsf{n}}) \cdot (1_{\mathsf{n}} \circ_2 \mathsf{counit}) = 1$.

The 1-cells specify two adjunctions $F_{\mathsf{d}} \dashv U_{\mathsf{d}}$ and $F_{\mathsf{n}} \dashv U_{\mathsf{n}}$. However, the natural transformations specified by the 2-cells also give adjunctions $F_{\mathsf{d}} \dashv F_{\mathsf{n}}$ and $U_{\mathsf{d}} \dashv U_{\mathsf{n}}$ (using the unit/counit definition of adjunction). Since a right or left adjoint of a given functor is unique up to isomorphism, it follows that the two functors $U_{\mathsf{d}}, F_{\mathsf{n}} : \mathsf{c} \to \mathsf{s}$ are isomorphic, resulting in an adjoint triple $F_{\mathsf{d}} \dashv (U_{\mathsf{d}} \cong F_{\mathsf{n}}) \dashv U_{\mathsf{n}}$. However, rather than proving $F_{\mathsf{d}} \dashv F_{\mathsf{n}}$ or $U_{\mathsf{d}} \dashv U_{\mathsf{n}}$ and then concluding $U_{\mathsf{d}} \cong F_{\mathsf{n}}$ from uniqueness of adjoints, we can construct the isomorphism directly:

LEMMA 5.1. *$U_{\mathsf{d}}\, A \cong F_{\mathsf{n}}\, A$ naturally in A.*

Proof. One way to define the maps is to use the constructions of Theorem 3.1 and Lemma 3.2 (the adjunction, the isomorphisms for F/U on 1 and $\circ$, and the action of F/U on 2-cells, the comonad structure):

$$U_{\mathsf{d}}\, A \longrightarrow F_1\, U_{\mathsf{d}}\, A \xrightarrow{F_{\mathsf{counit}}} F_{\mathsf{d}\circ\mathsf{n}}\, U_{\mathsf{d}}\, A \longrightarrow F_{\mathsf{n}}\, (F_{\mathsf{d}}\, U_{\mathsf{d}}\, A) \longrightarrow F_{\mathsf{n}}\, (A)$$

For $F_{\mathsf{n}}\, A \to U_{\mathsf{d}}\, A$, transpose $A \longrightarrow U_1\, A \xrightarrow{U_{\mathsf{unit}}} U_{\mathsf{n}\circ\mathsf{d}}\, A \longrightarrow U_{\mathsf{n}}\, U_{\mathsf{d}}\, A$

However, we can also write the maps directly as follows:

$$\dfrac{\mathsf{d} : \mathsf{s} \geq \mathsf{c} \quad \mathsf{counit} : \mathsf{d} \circ \mathsf{n} \Rightarrow 1 \quad \dfrac{1 : \mathsf{c} \geq \mathsf{c} \quad 1 : \mathsf{d} \Rightarrow \mathsf{d} \quad \dfrac{}{A\,[1] \vdash A}\,\mathsf{ident}}{U_{\mathsf{d}}\,A\,[\mathsf{d}] \vdash A}\,\mathsf{UL}}{U_{\mathsf{d}}\,A\,[1] \vdash F_{\mathsf{n}}\,A}\,\mathsf{FR}$$

$$\dfrac{\dfrac{\dfrac{\mathsf{unit} : 1 \Rightarrow \mathsf{n} \circ \mathsf{d} \quad \dfrac{}{A\,[1] \vdash A}\,\mathsf{ident}}{A\,[\mathsf{n} \circ \mathsf{d}] \vdash A}\,-_*(-)}{A\,[\mathsf{n}] \vdash U_{\mathsf{d}}\,A}\,\mathsf{UR}}{F_{\mathsf{n}}\,A\,[1] \vdash U_{\mathsf{d}}\,A}\,\mathsf{FL}$$

In the Agda code, we verify that these are inverse and natural. □

We can develop some of the expected properties of an adjoint triple $L \dashv M \dashv R$, such as the fact that the "left" comonad LM is itself left adjoint to the "right" monad RM, and consequently, LM preserves colimits. In this case, we have $L = F_{\mathsf{d}}$, $M = U_{\mathsf{d}} \cong F_{\mathsf{n}}$, and $R = U_{\mathsf{n}}$, and we write $\Box_{\mathsf{d}} A := F_{\mathsf{d}}\, U_{\mathsf{d}}\, A$ and $\bigcirc_{\mathsf{n}} A := U_{\mathsf{n}}\, F_{\mathsf{n}}\, A$.

THEOREM 5.2: PROPERTIES OF AN ADJOINT TRIPLE.

1. $\Box_{\mathsf{d}} \dashv \bigcirc_{\mathsf{n}}$
2. $\Box_{\mathsf{d}}\,(A+B) \cong \Box_{\mathsf{d}} A + \Box_{\mathsf{d}} B$

Proof. Using the fact that functors preserve natural isomorphisms, Lemma 5.1, and properties of U and F from Theorem 3.1, we can prove that $\Box_{\mathsf{d}} A$ and $\bigcirc_{\mathsf{n}} A$ are isomorphic to a single F and U, respectively:

$$\begin{aligned} \Box_{\mathsf{d}} A &= F_{\mathsf{d}}\, U_{\mathsf{d}}\, A \cong F_{\mathsf{d}}\, F_{\mathsf{n}}\, A \cong F_{\mathsf{n}\circ\mathsf{d}}\, A \\ \bigcirc_{\mathsf{n}} A &= U_{\mathsf{n}}\, F_{\mathsf{n}}\, A \cong U_{\mathsf{n}}\, U_{\mathsf{d}}\, A \cong U_{\mathsf{n}\circ\mathsf{d}}\, A \end{aligned}$$

This implies the above properties because $F_{\mathsf{n}\circ\mathsf{d}} \dashv U_{\mathsf{n}\circ\mathsf{d}}$ (Theorem 3.1) and $F_{\mathsf{n}\circ\mathsf{d}}$ preserves coproducts (Lemma 3.2) and these facts respect natural isomorphism. □

From a polarity point of view, it is unusual for a comonad $F\,U\,A$ to preserve positives, because the negative connective U interrupts focus/inversion phases. Here, this behavior is explained by the fact that $F_{\mathsf{d}}\, U_{\mathsf{d}}\, A$ is isomorphic to a single positive connective $F_{\mathsf{n}\circ\mathsf{d}}\, A$. The ambipolar middle connective in an adjoint triple thus emerges from the presence of two isomorphic connectives, one positive and one negative.

5.2. Walking Reflection

In our motivating example of axiomatic cohesion, the adjoint triple $\Delta \dashv \Gamma \dashv \nabla$ has some additional properties. We now write $\flat$ for the comonad $\Delta\Gamma$ and $\sharp$ for the monad $\nabla\Gamma$. $\flat$ takes a cohesive space and "retopologizes" it with the discrete cohesion, while $\sharp$ takes a cohesive space and retopologizes it with the codiscrete cohesion. Intuitively, retopologizing twice should be the same as retopologizing once, because each retopologization forgets the existing cohesive structure; that is, we want $\flat\flat A \cong \flat A$ and $\sharp\sharp A \cong \sharp A$ and $\flat\sharp A \cong \flat A$ and $\sharp\flat A \cong \sharp A$. Moreover, Δ and ∇ should be full and faithful, because a map between discrete or codiscrete spaces is exactly a map of sets.

Recalling that a right (resp. left) adjoint is full and faithful exactly when the counit (resp. unit) of the adjunction is an isomorphism, we can capture these properties by considering a different mode 2-category, the "walking reflection". This has the same objects and This has the same objects and generating morphisms as the walking adjunction, but we now take $\mathsf{d} \circ \mathsf{n} = 1$, with the counit being just the identity 2-cell, and the equations simplify to $\mathsf{unit} \circ_2 1_{\mathsf{n}} = 1$ and $1_{\mathsf{d}} \circ_2 \mathsf{unit} = 1$. Note that the only non-identity morphisms of this mode category are d, n, and $\mathsf{n} \circ \mathsf{d}$.

We write $\Delta := F_{\mathsf{d}}$, $\Gamma := (U_{\mathsf{d}} \ \cong F_{\mathsf{n}}$), and $\nabla := U_{\mathsf{n}}$, so $\flat = \Box_{\mathsf{d}}$ and $\sharp = \bigcirc_{\mathsf{n}}$. Since in particular we still have an adjunction, this mode theory inherits all the theorems from the previous section; it also has the following additional properties:

THEOREM 5.3: PROPERTIES OF THE WALKING REFLECTION.

1. *$\flat\flat A \cong \flat A$ and $\sharp\sharp A \cong \sharp A$ naturally in A.*
2. *$\sharp\flat A \cong \sharp A$ and $\flat\sharp A \cong \flat A$ naturally in A.*
3. *F_{d} and U_{n} are full and faithful.*

Proof. The first two parts say that "retopologizing" twice is the same as the "outer" retopologization. To prove them, using Theorem 3.1, the equality of morphisms $\mathsf{d} \circ \mathsf{n} = 1$ implies that

$$F_{\mathsf{n}}\, F_{\mathsf{d}}\, A \cong F_{\mathsf{d}\circ\mathsf{n}}\, A = F_1\, A \cong A$$
$$U_{\mathsf{d}}\, U_{\mathsf{n}}\, A \cong U_{\mathsf{d}\circ\mathsf{n}}\, A = U_1\, A \cong A$$

Consequently, by Lemma 5.1, the other (co)monads besides $\flat$ and $\sharp$ are trivial:

$$\bigcirc_{\mathsf{d}} A = U_{\mathsf{d}}\, F_{\mathsf{d}}\, A \cong F_{\mathsf{n}}\, F_{\mathsf{d}}\, A \cong A$$
$$\Box_{\mathsf{n}} A = F_{\mathsf{n}}\, U_{\mathsf{n}}\, A \cong U_{\mathsf{d}}\, U_{\mathsf{n}}\, A \cong A$$

Thus, we have idempotence:

$$\flat\flat A = F_{\mathsf{d}}\, (U_{\mathsf{d}}\, F_{\mathsf{d}}\, (U_{\mathsf{d}}\, A)) \cong F_{\mathsf{d}}\, U_{\mathsf{d}}\, A = \flat A$$
$$\sharp\sharp A = U_{\mathsf{n}}\, (F_{\mathsf{n}}\, U_{\mathsf{n}}\, (F_{\mathsf{n}}\, A)) \cong U_{\mathsf{n}}\, F_{\mathsf{n}}\, A = \sharp A$$

and that composing discrete and codiscrete retopologization is the same as the outer one:

$$\flat\sharp A = F_{\mathsf{d}}\ (U_{\mathsf{d}}\ U_{\mathsf{n}}\ (F_{\mathsf{n}}\ A)) \cong F_{\mathsf{d}}\ F_{\mathsf{n}}\ A \cong F_{\mathsf{n}\circ\mathsf{d}}\ A \cong \flat A$$
$$\sharp\flat A = U_{\mathsf{n}}\ (F_{\mathsf{n}}\ F_{\mathsf{d}}\ (U_{\mathsf{d}}\ A)) \cong U_{\mathsf{n}}\ U_{\mathsf{d}}\ A \cong U_{\mathsf{n}\circ\mathsf{d}}\ A \cong \sharp A$$

Finally, we check that F_{d} and U_{n} are full and faithful. This follows by general category-theoretic arguments from the triviality of $\bigcirc_{\mathsf{d}}$ and $\Box_{\mathsf{n}}$ (see (Mac Lane, 1998, §IV.3)), but to avoid needing to prove the Yoneda lemma in our formalization, we give an explicit argument instead. Consider F_{d}; an analogous argument works for U_{n}. We want to show that the functoriality on derivations $F_{\mathsf{d}}\ (D : A\,[1] \vdash B) : F_{\mathsf{d}}\ A\,[1] \vdash F_{\mathsf{d}}\ A$ is a bijection. Above, we showed that F_{n} is post-inverse to F_{d}—we have a natural isomorphism $i : F_{\mathsf{n}}\ F_{\mathsf{d}}\ A \cong A$. Therefore we can map a derivation $D : F_{\mathsf{d}}\ A\,[1] \vdash F_{\mathsf{d}}\ B$ to a derivation of $A\,[1] \vdash B$ by

$$A \xrightarrow{\quad i^{-1}\quad} F_{\mathsf{n}}\ F_{\mathsf{d}}\ A \xrightarrow{F_{\mathsf{n}}\ D} F_{\mathsf{n}}\ F_{\mathsf{d}}\ B \xrightarrow{\ i\ } B$$

By naturality of i, this function is post-inverse to F_{d} (i.e. $(i^{-1} \bullet (F_{\mathsf{n}}\ F_{\mathsf{d}}\ D) \bullet i) \approx D$), which shows that F_{d} is faithful. To show that it is full, we need to check the other composite, which simplifies to checking that for $D' : F_{\mathsf{d}}\ A\,[1] \vdash F_{\mathsf{d}}\ B$,

$$(F_{\mathsf{d}}\ F_{\mathsf{n}}\ D') \bullet F_{\mathsf{d}}\ i \approx F_{\mathsf{d}}\ i \bullet D'$$

(i.e. that $F_{\mathsf{d}}\ i$ is a natural isomorphism between the inclusion functor from the full subcategory whose objects are of the form $F_{\mathsf{d}}\ A$, and the functor $F_{\mathsf{d}}\ F_{\mathsf{n}}$ restricted to this subcategory). To show this, we prove that $F_{\mathsf{d}}\ i$ is equal to the $F_{\mathsf{d}}\ A$ component of the following natural transformation j, which uses $\mathsf{unit} : 1 \Rightarrow \mathsf{n}\circ\mathsf{d}$

$$F_{\mathsf{d}}\ F_{\mathsf{n}}\ A \longrightarrow F_{\mathsf{n}\circ\mathsf{d}}\ A \xrightarrow{F_{\mathsf{unit}}} F_1\ A \longrightarrow A$$

The fact that $F_{\mathsf{d}}\ i \approx j_{F_{\mathsf{d}}\ A}$ follows from the pseudofunctor associativity/unit coherences (used to show $F_{\mathsf{d}}\ i \approx (F^{\circ}(\mathsf{n},\mathsf{d}) \bullet F^{\circ}(\mathsf{d},\mathsf{n}\circ\mathsf{d}))$ and $F^{\circ}(\mathsf{d},1) = F^{1}_{F_{\mathsf{d}}\ A}$) and naturality of the composition isomorphism (used to show $F_{\mathsf{unit}} \bullet F^{\circ}(\mathsf{d},1) \approx F^{\circ}(\mathsf{d},\mathsf{n}\circ\mathsf{d})$). Then $(F_{\mathsf{d}}\ F_{\mathsf{n}}\ D') \bullet j \approx j \bullet D'$ is exactly the naturality square for j.

□

6. Spatial Type Theory

Next, we consider a further refinement of the walking reflection mode theory, and connect it to the rules for spatial type theory used in Shulman (2015b). The walking reflection mode theory allows us to work with cohesive types (which have mode c) and non-cohesive types (which have mode s). However, because Δ and ∇

are full and faithful, it is not strictly necessary to ever work in s itself—we could equivalently work in the image of Δ or ∇ in c. If we wish to restrict ourselves to constructions in c, we can simplify the mode theory to the (strictly) idempotent monad:

- object c
- one generating 1-cell $\mathsf{r} : \mathsf{c} \geq \mathsf{c}$ such that $\mathsf{r} \circ \mathsf{r} = \mathsf{r}$.
- one generating 2-cell $\mathsf{unit} : 1 \Rightarrow \mathsf{r}$ satisfying $\mathsf{t} \circ_2 \mathsf{unit} = 1$ and $\mathsf{unit} \circ_2 \mathsf{r} = 1$.

Observe that the only 1-cells are 1 and r, and the only 2-cells are 1_1, 1_{r}, and unit. This mode theory embeds in the walking reflection, with $\mathsf{r} := \mathsf{n} \circ \mathsf{d}$, so we could equivalently work in the c-types above.

For this mode theory, we define $\flat := F_{\mathsf{r}}$ and $\sharp := U_{\mathsf{r}}$. In the walking reflection, we defined $\flat := \Box_{\mathsf{d}}$ and $\sharp := \bigcirc_{\mathsf{n}}$ and then proved (in the proof of Theorem 5.2) that $\flat \cong F_{\mathsf{n} \circ \mathsf{d}}$ and $\sharp \cong U_{\mathsf{n} \circ \mathsf{d}}$. Here, we take the other side of this isomorphism as the definition, so we immediately have $\flat \dashv \sharp$ and $\flat$ preserves coproducts by Theorem 3.1 and Lemma 3.2, but we must prove that they are (co)monads. A simple route to this is to prove absorption, because $\flat \sharp A = F_{\mathsf{r}}\, U_{\mathsf{r}}\, A$ which is a monad by Lemma 3.2, and dually.

THEOREM 6.1: IDEMPOTENCE AND ABSORPTION. *$\flat\flat A \cong \flat A$ and $\sharp\sharp A \cong \sharp A$ and $\sharp\flat A \cong \sharp A$ and $\flat\sharp A \cong \flat A$ naturally in A.*

Proof. Because $\mathsf{r} \circ \mathsf{r} = \mathsf{r}$, idempotence is just the composition isomorphisms F° and U° from Theorem 3.1. The absorption isomorphisms are constructed directly; see our Agda formalization. $\square$

In the remainder of this section, we explore some alternative proof theories for this mode theory. If we think of these alternative proof theories as other ways to write derivations in adjoint logic with the walking idempotent-monad mode theory, then we would like them to be sound (can be translated into adjoint logic) and complete (adjoint logic can be translated into them), and for soundness-after-completeness to be the identity up to $\approx$. This way, any construction we do in the alternative proof theory could be done in adjoint logic, and any derivation in adjoint logic an be translated into a derivation in the alternative proof theory that represents its $\approx$-equivalence class. We give three alternative proof theories, two sequent calculi and one natural deduction system, all of which have these properties. The first specialized calculus eliminates the choices of γ and e in FL and UR, which simplifies the construction of proofs, and has a simple generalization to multi-assumption sequents. The second specialized calculus, and an equivalent natural deduction system, treat positive types more similarly to how they are traditionally handled in intensional type theory, and corresponds closely to the rules for spatial type theory used in (Shulman, 2015b).

6.1. *Keep assumptions crisp as long as possible*

For this mode theory, there are two 1-cells r and 1, the general sequent calculus rules allow choices of γ and e in $\mathsf{FR}^{\gamma}_{e}(D)$ and $\mathsf{UL}^{\gamma}_{e}(D)$. However, it turns out that we can without loss of generality always take γ to be r and e to be 1 in these rules, and use only the following instances of the rules:

$$\frac{}{P\,[1] \vdash P}\ \mathsf{hyp}\,1_1 \qquad \frac{}{P\,[\mathsf{r}] \vdash P}\ \mathsf{hyp\,unit}$$

$$\frac{A\,[\mathsf{r}] \vdash B}{A\,[\_] \vdash \sharp B}\ \mathsf{UR}(-) \qquad \frac{A\,[\mathsf{r}] \vdash B}{\sharp A\,[\mathsf{r}] \vdash B}\ \mathsf{UL}^{\mathsf{r}}_{1}(-) \qquad \frac{A\,[\mathsf{r}] \vdash B}{A\,[\mathsf{r}] \vdash \flat B}\ \mathsf{FR}^{\mathsf{r}}_{1}(-) \qquad \frac{A\,[\mathsf{r}] \vdash B}{\flat A\,[\_] \vdash B}\ \mathsf{FL}(-)$$

$$\frac{A\,[\alpha] \vdash B}{A\,[\alpha] \vdash U_1\,B}\ \mathsf{UR}(-) \qquad \frac{A\,[\alpha] \vdash B}{U_1\,A\,[\alpha] \vdash B}\ \mathsf{UL}^{\alpha}_{1}(-) \qquad \frac{A\,[\alpha] \vdash B}{A\,[\alpha] \vdash F_1\,B}\ \mathsf{FR}^{\alpha}_{1}(-) \qquad \frac{A\,[\alpha] \vdash B}{F_1\,A\,[\alpha] \vdash B}\ \mathsf{FL}(-)$$

Intuitively, we have two kinds of assumptions, which we call *cohesive* ($A[1_{\mathsf{c}}]$) and *crisp* ($A[\mathsf{r}]$, r for cRisp). A crisp assumption is the judgemental analogue of $\flat A$—i.e. it means we know A retopologized with the discrete cohesion. The admissible principle $\mathsf{unit}_{*}(\mathsf{ident}_A) : A\,[\mathsf{r}] \vdash A$ says that a crisp assumption of A can be used to prove A itself, so a crisp assumption is stronger than a cohesive one. The $\flat$ left rule says that if we know $\flat A$ either crisply or cohesively, then we know A crisply. The $\flat$ right rule says that we can map into $\flat B$ by mapping into B, as long as the assumption is already crisp (this constraint prevents using a cohesive variable to map into a discrete type). The added restriction relative to the general adjoint logic rules is that we can, without loss of generality, always keep the crispness of A in the premise—there is never any reason to demote it to a cohesive variable at this time. The $\sharp$ right rule says that if we are mapping into the codiscretization of B, then we can make the assumption crisp. The $\sharp$ left rule says that knowing the codiscretization crisply is the same as knowing A crisply, because in either case we know A under the discrete retopologization. Here the added restriction relative to the general adjoint logic rules is that we always assume A crisply in the premise; we could instead assume it cohesively (take $\gamma = 1$), but we can without loss of generality keep it crisp.

These rules are clearly sound (they are a subset of adjoint logic) and are also complete:

THEOREM 6.2. *For all $D : A\,[\alpha] \vdash B$, there is a derivation $D' : A\,[\alpha] \vdash B$ that uses only the above rules and satisfies $D \approx D'$.*

Proof. For this mode theory, the only mode is c, the only 1-cells are 1 and r, and the only 2-cells are 1_1 and 1_{r} and unit. The case for $\mathsf{hyp}\,e$ is immediate, since the only 2-cells from 1 are 1_1 and unit, and we have included both of those.

We have included rules for F and U for both 1-cells 1 and r; we discuss why the rules for $\sharp$ are complete ($\flat$ and F_1 and U_1 are analogous). For UR, a general instance of the rule will have some β in the conclusion, and pass to $\mathsf{r} \circ \beta$ in the premise. But β must be 1 or r, and in either case the premise is r, so the IH gives the result. For UL, a general instance will have the form

$$\frac{\mathsf{r} \circ \gamma \Rightarrow \beta \quad A\,[\gamma] \vdash C}{U_{\mathsf{r}}\,A\,[\beta] \vdash C}\ \mathsf{UL}$$

Here γ can be 1 or r, but in either case β must have been r, because for either value of γ we have $\mathsf{r} \circ \gamma = \mathsf{r}$ and there is no 2-cell $\mathsf{r} \Rightarrow 1$. Moreover, the only 2-cell $\mathsf{r} \Rightarrow \mathsf{r}$ is 1, so the derivation must be of the form

$$\frac{A\,[\gamma] \vdash C}{U_{\mathsf{r}}\,A\,[\mathsf{r}] \vdash C}\ \mathsf{UL}_1^{\gamma}(-)$$

By the IH we have an equivalent derivation $D' : A\,[\gamma] \vdash C$ that uses only the restricted rules. If γ is r, this gives the result. If γ is 1, then we can make a derivation

$$\cfrac{\cfrac{D' : A\,[1] \vdash C}{A\,[\mathsf{r}] \vdash C}\ \mathsf{unit}_*(-)}{U_{\mathsf{r}}\,A\,[\mathsf{r}] \vdash C}\ \mathsf{UL}_1^{\mathsf{r}}(-)$$

and show using the equational rules that $\mathsf{UL}_1^{\gamma}(D') \approx \mathsf{UL}_1^{\mathsf{r}}(\mathsf{unit}_*(D'))$. It remains to show a lemma that $\mathsf{unit}_*(D)$ is closed under the restricted rules, which can be proved by induction on D. The complete proof of this theorem is about 250 lines of Agda. □

For this restricted sequent calculus, there is a natural generalization to a multiple assumption sequent, where each of the assumptions is either cohesive or crisp. We write Γ for a context consisting of assumptions $A[1]$ or $A[\mathsf{r}]$. We write $\Gamma{\uparrow}^{\mathsf{crisp}}$ for a context where each cohesive assumption $A[1]$ has been promoted into a crisp assumption $A[\mathsf{r}]$, and each $A[\mathsf{r}]$ has been kept the same. We write $\Gamma{\downarrow}_{\mathsf{coh}}$ for a context where each cohesive assumption $A[1]$ has been dropped, and each crisp assumption $A[\mathsf{r}]$ has been kept the same. Then the multi-assumptioned rules are as follows:

$$\frac{P[1] \in \Gamma}{\Gamma \vdash P} \quad \frac{P[\mathsf{r}] \in \Gamma}{\Gamma \vdash P} \quad \frac{\Gamma{\uparrow}^{\mathsf{crisp}} \vdash B}{\Gamma \vdash \sharp B} \quad \frac{\Gamma, \sharp A[\mathsf{r}], A[\mathsf{r}] \vdash B}{\Gamma, \sharp A[\mathsf{r}] \vdash B} \quad \frac{\Gamma{\downarrow}_{\mathsf{coh}} \vdash B}{\Gamma \vdash \flat B} \quad \frac{\Gamma, \flat A[\alpha], A[\mathsf{r}] \vdash B}{\Gamma, \flat A[\alpha] \vdash B}$$

The left rules are just the single-assumption left rules in context (with the principle formula preserved in the premise to make contraction admissible). The single-assumption $\sharp$ right rule promotes the one assumption to crisp; here, we promote all assumptions. The single-assumption $\flat$ right rule insists that the one assumption be crisp; here, we drop all non-crisp assumptions. The rules for $\flat$ are reminiscent of the $\Box A$ modality in Pfenning and Davies (2001), with crisp corresponding to the valid judgement, except here we can apply a left rule to a valid/crisp assumption of $\Box A$, because $\flat$ is an F which is a positive-to-positive connective. The right rule for $\sharp$ is reminiscent of the right rule for the proof irrelevance/erasability modality $[A]$ in Pfenning (2001, 2008), though the left rule is different.[7] Like in Figure 1, the left rule for coproducts would allow elimination on a coproduct in either mode:

$$\frac{\Gamma \vdash A}{\Gamma \vdash A+B} \quad \frac{\Gamma \vdash B}{\Gamma \vdash A+B} \quad \frac{\Gamma,(A+B)[\alpha],A[\alpha] \vdash C \quad \Gamma,(A+B)[\alpha],B[\alpha] \vdash C}{\Gamma,(A+B)[\alpha] \vdash C}$$

6.2. Restricted Rules for Positives

The treatment of positive types (such as $\flat$ and $+$) in sequent calculi is a bit different than how positives are typically treated in intensional type theory. In sequent calculi, the goal is generally to have the subformula property, and for the identity and cut rules to be admissible (if not for the entire logic, then for as large a fragment as possible). To achieve this, certain left commutative cuts are used as part of the cut admissibility algorithm, and are therefore definitional equalities; e.g. in defining cut we said

$$\mathsf{cut}\ (\mathsf{Case}(D_1,D_2))\ E = \mathsf{Case}(\mathsf{cut}\ D_1\ E, \mathsf{cut}\ D_2\ E) \text{ if } E \text{ is not a right rule}$$

However, these left-commutative cuts for positives are not typically taken as definitional equalities in intensional dependent type theories. From a sequent calculus point of view, a positive type in intensional type theory is treated more like a (positive) base type/atomic proposition that is equipped with an elimination constant of function type. For example, for coproducts, the elimination constant is

$$\mathsf{case} : (A \to C) \to (B \to C) \to ((A+B) \to C)$$

and this is often used via (iterated) implication-left:

$$\frac{\Gamma \vdash A \to C \quad \Gamma \vdash B \to C \quad A+B \in \Gamma \quad \Gamma, C \vdash D}{\Gamma \vdash D}$$

[7] The $[A]$ type in that work is an applicative functor, so it cannot be mapped into adjoint logic in a straightforward way, because it does not have an adjoint on either side.

This builds in a cut from the result type of the case C to the overall goal D, so we do not need to reduce left-commutative cuts.

In adjoint logic specifically, there is another difference between the left rules for positives that we have been using and the use of an elimination constant. The adjoint logic left rules allow eliminating a positive assumption such as $(A+B)[\alpha]$ for any α, keeping $A[\alpha]$ and $B[\alpha]$ in the premises. For example, we can eliminate a crisp assumption of $A+B$, getting crisp assumptions in the premises. As we have seen in Lemma 3.2, this allows us to prove that F_α preserves coproducts using only the judgemental structure of the calculus, without mentioning the right adjoint connective U_α (indeed, we must be able to do this if the logic is to have the subformula property)—essentially because some of the proof that F preserves coproducts goes into the semantic interpretation (see Lemma 4.9). On the other hand, if we instead specify a positive type using the usual elimination constant in adjoint logic, then it is not automatic that we can eliminate on $(A+B)[\mathsf{r}]$ preserving the $A[\mathsf{r}]$ and $B[\mathsf{r}]$ in the premises—using the right rule for $A \to C$ would give an assumption $A[1]$, not $A[\mathsf{r}]$. However, as we will see below, for coproducts this can be proved using U.

There are reasons to explore a calculus where all positives are treated via the elimination constant approach. First, if we do not include the left-commutative reductions definitionally, then the syntax can be interpreted in models where they do not hold strictly (such as ones where positives involve a fibrant replacement). Second, the elimination constant approach seems necessary for infinite types such as the natural numbers, to allow strengthening the induction formula (unless one uses an infinitely wide proof tree with an ω-rule). Moreover, the elimination constant approach seems necessary for certain higher inductive types, which semantically should *not* be preserved by $\flat$. Thus, treating all positives with elimination constants is more uniform. (On the other hand, there are also arguments for having the general left rules as rules for the connectives where they do make sense, such as finite colimits, since this gives the subformula property for as large a sublogic as possible.)

We use the following sequent calculus $A\,[\alpha] \Vdash B$ to illustrate the idea of specifying all positives by elimination constants. For simplicity, we make all cuts derivable, and defer to future work an investigation of cut reduction for only principal and right-commutative cuts (the stuck left-commutative cuts prevent using the usual structural argument (Pfenning, 1994)). The rules are as follows:

$$\frac{}{A\,[\mathsf{r}] \Vdash A} \quad \frac{}{A\,[1] \Vdash A} \quad \frac{A\,[\beta] \Vdash B \quad B\,[\alpha] \Vdash C}{A\,[\beta \circ \alpha] \Vdash C}$$

$$\frac{A\,[\mathsf{r}] \Vdash B}{A\,[\alpha] \Vdash \sharp B} \quad \frac{A\,[\mathsf{r}] \Vdash B}{\sharp A\,[\mathsf{r}] \Vdash B} \quad \frac{A\,[\mathsf{r}] \Vdash B}{A\,[\mathsf{r}] \Vdash \flat B} \quad \frac{A\,[\mathsf{r}] \Vdash C}{\flat A\,[1] \Vdash C}$$

$$\frac{C\,[\alpha] \Vdash A}{C\,[\alpha] \Vdash A+B} \quad \frac{C\,[\alpha] \Vdash B}{C\,[\alpha] \Vdash A+B} \quad \frac{A\,[1] \Vdash C \quad B\,[1] \Vdash C}{(A+B)\,[1] \Vdash C}$$

On the first line, we have identity (for both crisp and cohesive variables) and cut as derivable rules; the identity rules could be restricted to positives and atoms and made admissible for negatives in order to force η-expansion. The rules for $\sharp$ and the right rules for $\flat$ and $+$ are the same as in Section 6.1. The left rules for positives are restricted to cohesive variables, and the left rule for coproducts binds cohesive variables in the branches.

However, in the presence of cut and identity rules, we can derive the more general left rules for crisp positive assumptions $(A+B)[\mathsf{r}]$ and $\flat A[\mathsf{r}]$. For $\flat A$, this is just precomposition with using a crisp variable:

$$\frac{\flat A\,[\mathsf{r}] \Vdash \flat A \quad \dfrac{A\,[\mathsf{r}] \Vdash C}{\flat A\,[1] \Vdash C}}{\flat A\,[\mathsf{r}] \Vdash C}$$

The fact that this works is quite specific to this mode theory, where forgetting the r-ness of the assumption and then reintroducing it arrives at the same place, because of idempotence. For $A+B$, we need to use $\sharp$ on the right to derive the more general rule:

$$\frac{\dfrac{\dfrac{A\,[\mathsf{r}] \Vdash C}{A\,[1] \Vdash \sharp C} \quad \dfrac{B\,[\mathsf{r}] \Vdash C}{B\,[1] \Vdash \sharp C}}{(A+B)\,[1] \Vdash \sharp C} \quad \dfrac{\dfrac{}{C\,[\mathsf{r}] \Vdash C}}{\sharp C\,[\mathsf{r}] \Vdash C}}{(A+B)\,[\mathsf{r}] \Vdash C}$$

(A similar move is necessary to prove crisp $\flat$-*induction* from $\flat$-induction—that is, in a dependently typed theory where C itself might depend either crisply or cohesively on $\flat A$—even though it was not necessary to get crisp $\flat$-recursion from $\flat$-recursion.) Observe that, if we were to apply a left-commutative cut reduction to this derivation, it would reduce to an instance of the $(A+B)\,[\mathsf{r}] \vdash C$ left-rule (crisp coproduct elimination).

This sequent calculus is sound and complete for the original one:[8]

THEOREM 6.3. *There are functions* $(A\,[\alpha] \vdash B) \leftrightarrow (A\,[\alpha] \Vdash B)$ *and the composite from* $A\,[\alpha] \vdash B$ *to itself is the identity up to* $\approx$.

Proof. The proof is about 150 lines of Agda. □

6.3. Natural deduction

Finally, to connect to the rules used in (Shulman, 2015b), we convert the previous sequent calculus to a natural deduction system:

$$\frac{}{A\,[\mathsf{r}] \vdash^{\mathsf{nd}} A} \qquad \frac{}{A\,[1] \vdash^{\mathsf{nd}} A}$$

$$\frac{A\,[\mathsf{r}] \vdash^{\mathsf{nd}} B}{A\,[\alpha] \vdash^{\mathsf{nd}} \sharp B} \quad \frac{C\,[\mathsf{r}] \vdash^{\mathsf{nd}} \sharp A}{C\,[\mathsf{r}] \vdash^{\mathsf{nd}} A} \quad \frac{C\,[\mathsf{r}] \vdash^{\mathsf{nd}} A}{C\,[\mathsf{r}] \vdash^{\mathsf{nd}} \flat A} \quad \frac{C\,[\alpha] \vdash^{\mathsf{nd}} \flat A \quad A\,[r] \vdash^{\mathsf{nd}} B}{C\,[\alpha] \vdash^{\mathsf{nd}} B}$$

$$\frac{C\,[\alpha] \vdash^{\mathsf{nd}} A}{C\,[\alpha] \vdash^{\mathsf{nd}} A+B} \quad \frac{C\,[\alpha] \vdash^{\mathsf{nd}} B}{C\,[\alpha] \vdash^{\mathsf{nd}} A+B} \quad \frac{C\,[\alpha] \vdash^{\mathsf{nd}} A+B \quad A\,[1] \vdash^{\mathsf{nd}} D \quad A\,[1] \vdash^{\mathsf{nd}} D}{C\,[\alpha] \vdash^{\mathsf{nd}} D}$$

The cut/substitution rule

$$\frac{A\,[\beta] \vdash^{\mathsf{nd}} B \quad B\,[\alpha] \vdash^{\mathsf{nd}} C}{A\,[\beta \circ \alpha] \vdash^{\mathsf{nd}} C}$$

is not included, but it is admissible, because each rule allows for precomposition.

This natural deduction system is also sound and complete for adjoint logic with this mode theory, and the the proof factors through the above sequent calculus:

THEOREM 6.4. *There are functions* $(A\,[\alpha] \vdash B) \leftrightarrow (A\,[\alpha] \Vdash B) \leftrightarrow (A\,[\alpha] \vdash^{\mathsf{nd}} B)$ *and back, and the composite from* $A\,[\alpha] \vdash B$ *to itself is the identity up to* $\approx$.

Proof. The proof is about 150 lines of Agda. □

We can generalize these natural deduction rules to multiple assumptions as follows, writing Δ for the crisp assumptions $A[\mathsf{r}]$ and Γ for the cohesive assumptions

[8] removing F_1 and U_1 from the original calculus or adding rules for them here

$A[1]$:

$$\frac{A \in \Gamma}{\Delta;\Gamma \vdash A} \quad \frac{A \in \Delta}{\Delta;\Gamma \vdash A}$$

$$\frac{\Delta,\Gamma;\cdot \vdash B}{\Delta;\Gamma \vdash \sharp B} \quad \frac{\Delta;\cdot \vdash \sharp A}{\Delta;\Gamma \vdash A} \quad \frac{\Delta;\cdot \vdash A}{\Delta;\Gamma \vdash \flat A} \quad \frac{\Delta;\Gamma \vdash \flat A \quad \Delta,A;\Gamma \vdash B}{\Delta;\Gamma \vdash B}$$

$$\frac{\Delta;\Gamma \vdash A}{\Delta;\Gamma \vdash A+B} \quad \frac{\Delta;\Gamma \vdash B}{\Delta;\Gamma \vdash A+B} \quad \frac{\Delta;\Gamma \vdash A+B \quad \Delta;\Gamma,A \vdash C \quad \Delta;\Gamma,B \vdash C}{\Delta;\Gamma \vdash C}$$

These rules, generalized to dependent types, are exactly the rules used in an investigation of cohesive type theory in Shulman (2015b). Here, we have given a proof theoretic explanation for them, by connecting them to a particular mode theory in adjoint logic. A next step for future work is to analyze the normal forms of the β-only equational theory for positives (avoiding the left-commutative equations that are obtained by the translation to adjoint logic). It is unclear whether structural cut elimination/hereditary substitution (Watkins et al., 2002) can be used to do this, because the unreduced left-commutative cuts break the subformula property; however, it should be possible to prove normalization using a logical relations argument (an equality algorithm for positives with only β-rules was considered in Licata and Harper (2005)).

7. Conclusion

In this paper, we have defined an adjoint logic that allows multiple different adjunctions between the same categories, shown soundness and completeness of the logic in pseudofunctors into the 2-category of adjunctions, and used some specific mode theories to model adjoint triples and the $\flat$ and $\sharp$ modalities of axiomatic cohesion. While we have considered only a single-hypothesis sequent through most of the paper, we discussed a generalization to multiple hypotheses for the specific mode theory in Sections 6, and the generalization of these rules to dependent types is discussed in Shulman (2015b).

One area for future work is to extend the general adjoint logic with multiple assumptions and dependent types. This would provide a context for investigating the shape modality $\int \dashv \flat$. We could certainly give a mode theory with one mode and $\int \dashv \flat \dashv \sharp$, or with two modes and $\int \dashv \Delta \dashv \Gamma \dashv \nabla$, but it remains to be investigated whether this can provide the right properties for $\int$ beyond adjointness. On the one hand, too much might be true: $\int$ does not preserve identity types, and the general dependently typed rules for F might force it to. On the other, too little might be true: for applications such as relating the shape of the topological circle to the homotopical circle, extra properties are needed, such as $\int\mathbb{R} \cong 1$. Both of these issues can be addressed as in Shulman (2015b) by treating $\int$ not as an abstract adjoint, of the kind we can represent using the mode 2-category, but as a defined type (specifically, a higher inductive), which among other things has the property that it is adjoint to $\flat$ (adjoint logic is still essential for representing $\flat$ and $\sharp$ themselves). Another area for future work is to consider ∞-category semantics, rather than the 1-categorical semantics of derivations that we have considered here. A final area for future work is to investigate applications of other mode theories in our generalized adjoint logic, beyond the motivating example of cohesive homotopy type theory.

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