Question 1

You have one type of chicken wire to build a temporary enclosure for housing chickens in your backyard. Your plan is to build a triangular enclosure with sides of lengths x, y and z respectively. See Figure 1:

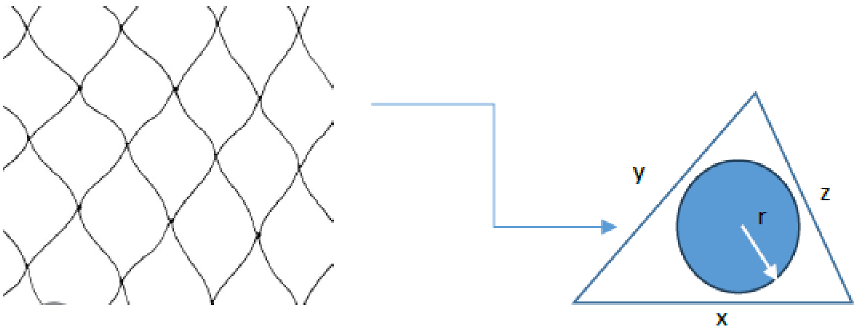

Figure 1 – Triangular enclosure (chicken house) and the inner circle.

You have 100m of chicken wire, your goal is to maximise the area of the circle inside the triangular enclosure for your given materials.

Assuming that two sides of the triangle are equal in length (x = y), you are tasked with determining the lengths of the three sides (x, y, and z) using different optimisation methods.

a) **Apply the Lagrange multiplier method. In your implementation, you must use three variables: x, y, and z. If fewer unknown variables are used, no marks will be awarded. There is no requirement for MATLAB code in this part of the question.**

To apply the Lagrange multiplier method, the constraint and object functions must first be defined. The constraint function will relate to the sum of lengths x, y and z being equal to 100m.

Constraint Function:

$$x + y + z = 100\ m$$

$$g(x, y, z) = x + y + z - 100$$

Given the above constraint function, the partial derivatives with respect to x, y and z are then taken. These PDEs will be used in the Lagrange multiplier method.

Constraint Function PDEs:

$$\frac{dg}{dx} = 1\,,\quad \frac{dg}{dy} = 1\,,\quad \frac{dg}{dz} = 1$$

The object function is then defined based on the relationship between the side-lengths of a triangle and the radius of the in-circle contained within the triangle.

In-radius Equation:

$$r = \sqrt{\frac{(s-x)(s-y)(s-z)}{s}}$$

From the above equation, “s” is the semi-perimeter given by half of the triangle perimeter. In this case, the total perimeter is 100m, therefore the semi-perimeter is 50m.

Semi-perimeter Equation:

$$s = \frac{1}{2}(x+y+z) = \frac{1}{2} \times 100 = 50\ m$$

Given the in-radius and semi-perimeter equations, the object function can now be defined given that the area of a circle is directly proportional to the radius of the circle squared.

Area of a circle:

$$A = \pi r^2$$

Therefore, maximising the radius squared (r^2) will also maximise the area of the circle (A). Given this, the object function can be defined as equal to the squared radius of the circle.

Object Function:

$$f(x,y,z) = r^2 = \frac{(s-x)(s-y)(s-z)}{s}$$

Substituting in “s = 50” from the semi-perimeter equation and expanding the function gives the following expression. (Full working is given in Appendix A)

Object Function (expanded form):

$$f(x,y,z) = 2500 - 50x - 50y - 50z + xy + xz + yz - \frac{xyz}{50}$$

The partial derivatives of the object function with respect to x, y and z can now be taken. These PDEs will be used in the Lagrange multiplier method.

Object Function PDE’s:

$$\frac{df}{dx} = y + z - 50 - \frac{yz}{50}$$

$$\frac{df}{dy} = x + z - 50 - \frac{xz}{50}$$

$$\frac{df}{dz} = x + y - 50 - \frac{xy}{50}$$

Given the PDEs of the constrain and object functions, the Lagrange multiplier method can now be used.

Lagrange Equalities:

$$\frac{df}{dx} + \lambda \frac{dg}{dx} = 0\,,\;\; y + z - 50 - \frac{yz}{50} + \lambda = 0 \tag{1}$$

$$\frac{df}{dy} + \lambda \frac{dg}{dy} = 0\,,\;\; x + z - 50 - \frac{xz}{50} + \lambda = 0 \tag{2}$$

$$\frac{df}{dz} + \lambda \frac{dg}{dz} = 0\,,\;\; x + y - 50 - \frac{xy}{50} + \lambda = 0 \tag{3}$$

By rearranging the above equations, it is possible to remove the “lambda” terms leaving only the x, y and z variables. Simplifying further gives the following equalities. (Full working is given in appendix B)

Resultant system of equations:

$$x = y\,,\;\; x = z\,,\;\; y = z$$

Given the above system of equations, coupled with the original constraint function, the following results can be concluded.

Given:

$$x = y = z$$

$$x + y + z = 100$$

Results:

$$x = \frac{100}{3}\,,\;\; y = \frac{100}{3}\,,\;\; z = \frac{100}{3}$$

Given the above results, it can be concluded that an equilateral triangle will give the largest in-circle area.

b) Employ the Golden Section Search and Newton's methods. Please transform it into a one-dimensional optimisation problem and include your MATLAB code.

In order to use the Golden Section Search and Newton's Method algorithms, the object function must first be transformed into a 1D optimisation problem. This can be done in the following steps.

Recall the object function:

$$f(x,y,z) = r^2 = \frac{(s-x)(s-y)(s-z)}{s}$$

Given in the question we can assume that x = y. Substituting this assumption into the object function is given as follows.

Updated object function (assume x = y):

$$f(x,z) = r^2 = \frac{(s-x)^2(s-z)}{s}$$

Additionally, it is known that the total perimeter of the triangle is 100m. This is expressed in the constraint function, which can be rearranged to get an expression for z in terms of x and y. Again, we will make use of the assumption that x = y.

Rearranged constraint function:

$$x + y + z = 100$$

$$z = 100 - x - y\,,\ \ x = y$$

$$z = 100 - 2x$$

Finally, the rearranged constraint expression can be substituted into the updated object function. Furthermore, the semi-perimeter is known to be half of the total perimeter length at s = 50.

Final 1D object function:

$$f(x) = \frac{(s-x)^2(s-(100-2x))}{s}\ ,\ \ s = 50$$

$$f(x) = \frac{(50-x)^2(50-100+2x)}{50}$$

$$f(x) = \frac{(50-x)^2(2x-50)}{50}$$

Plotting the resultant 1D optimisation function gives the following curve:

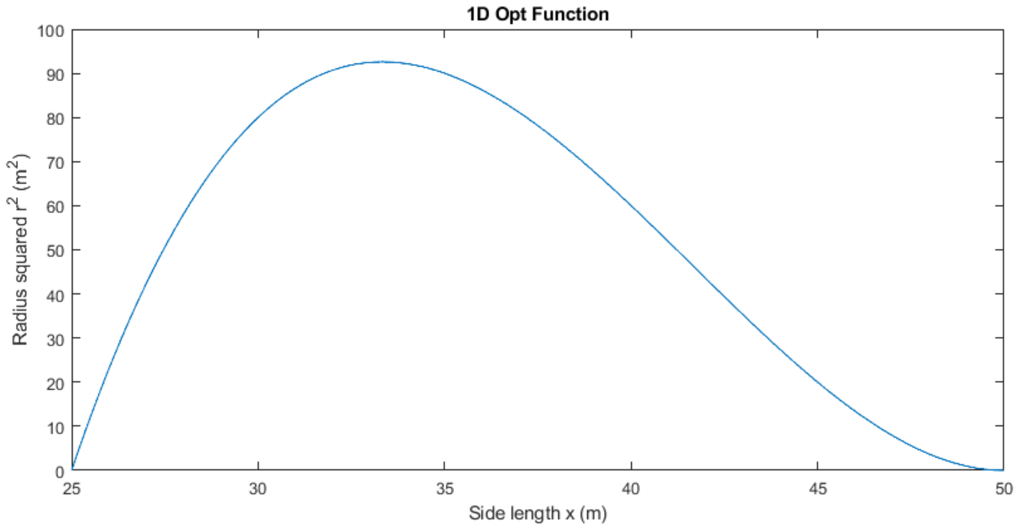

Figure 1: Plot of the 1D optimisation function

Note that the bounds of side length "x" are limited to between 25m and 50m:

$$25\ m < x < 50\ m$$

This is because, assuming that x = y, the side length value "x" that is outside of this range will result in an impossible triangle. Both extremes, x = 25m and x = 50m, are listed as follows:

- At x = 25m, y = 25m, z = 50m
- At x = 50m, y = 50m, z = 0m

Given this, the search range will be limited to "x" between 25m and 50m to avoid erroneous resultant critical points. Implemented in MATLAB, the resultant 1D optimisation methods gave the following results for the maxima x-value.

- Golden Section Search: x = 33.333976 m
- Newton's Method (n = 100): x = 33.334572 m

As shown, these values are essentially identical to the theoretical values of x = 100/3 m calculated in part a. This confirms the conclusion that an equilateral triangle gives the largest in-circle area.

Appendix A – Object Function (expanded form)

Initial object function:

$$f(x,y,z) = r^2 = \frac{(s-x)(s-y)(s-z)}{s}$$

Substitute the semi-perimeter value “s = 50”:

$$f(x,y,z) = \frac{(50-x)(50-y)(50-z)}{50}$$

Expand the numerator:

$$f(x,y,z) = \frac{125000 - 2500x - 2500y - 2500z + 50xy + 50xz + 50yz - xyz}{50}$$

Divide through by 50.

$$f(x,y,z) = 2500 - 50x - 50y - 50z + xy + xz + yz - \frac{xyz}{50}$$

Appendix B – **Lagrange Equalities simplification**

Initial Lagrange Equalities:

$$y + z - 50 - \frac{yz}{50} + \lambda = 0 \tag{1}$$

$$x + z - 50 - \frac{xz}{50} + \lambda = 0 \tag{2}$$

$$x + y - 50 - \frac{xy}{50} + \lambda = 0 \tag{3}$$

Move all constants to the RHS:

$$y + z - \frac{yz}{50} = 50 - \lambda \tag{1a}$$

$$x + z - \frac{xz}{50} = 50 - \lambda \tag{2a}$$

$$x + y - \frac{xy}{50} = 50 - \lambda \tag{3a}$$

Equate RHS of equations:

(1a) = (2a)
$$y + z - \frac{yz}{50} = x + z - \frac{xz}{50} \tag{4a}$$

(1a) = (3a)
$$z + y - \frac{yz}{50} = x + y - \frac{xy}{50} \tag{5a}$$

(2a) = (3a)
$$z + x - \frac{xz}{50} = y + x - \frac{xy}{50} \tag{6a}$$

Remove cancelled out terms and multiply through by 50.

$$50y - yz = 50x - xz \tag{4b}$$

$$50z - yz = 50x - xy \tag{5b}$$

$$50z - xz = 50y - xy \tag{6b}$$

Extract out common terms.

$$y(50 - z) = x(50 - z) \tag{4c}$$

$$z(50 - y) = x(50 - y) \tag{5c}$$

$$z(50 - x) = y(50 - x) \tag{6c}$$

Remove cancelled out terms.

$$y = x \tag{7}$$

$$z = x \tag{8}$$

$$z = y \tag{9}$$