week-1-problem-1g

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1 Week 1, Problem 1(g)

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1.1 Introduction

This notebook uses Macaulay2 to solve selected problems from Week 1.

1.2 Problem 1: (g)

Compute by hand the Hilbert series of the (quotient by the) monomial ideal

$$I = \langle a^2, ab, b^3, b^2c \rangle$$

1.2.1 Solution by Hand

The Hilbert series is an invariant that can be computed for a graded R-module M where R is some commutative unital ring. Here the ring is the coordinate ring defined by I which is an ideal in the polynomial ring $S = k[a, b, c]$ where k is some field. For computations, take $k = \mathbb{Q}$.

$$\begin{aligned} S &= k[a, b, c] \\ R &= S/I \end{aligned}$$

The polynomial ring S is generated by the monomials and is naturally graded by the total degree d of the monomials.

$$S = \bigoplus_{d=0}^{\infty} S_d$$

Since I is a monomial ideal, the quotient ring R is also a graded ring.

$$R = \bigoplus_{d=0}^{\infty} R_d$$

A graded ring R is itself a graded R-module M, i.e. $M_d = R_d$. Each M_d is a finite-dimensional k-vector space. The Hilbert series $P(M, t)$ is therefore

$$P(M,t) = \sum_{d=0}^{\infty} \dim_k(M_d)$$

To compute $\dim_k(M_d)$, enumerate all the monomials in S_d and exclude those that are divisible by any of the generators of I. The following table lists the monomials in M_d.

d	basis of M_d	$\dim_k(M_d)$
0	1	1
1	a, b, c	3
2	ac, b^2, bc, c^2	4
3	ac^2, bc^2, c^3	3
4	ac^3, bc^3, c^4	3
5	ac^4, bc^4, c^5	3
6	ac^5, bc^5, c^6	3
d	ac^{d-1}, bc^{d-1}, c^d	3

The dimension stabilizes at 3 for $d \geq 3$

The Hilbert series is therefore

$$\begin{aligned}
P(M,t) &= 1 + 3t + 4t^2 + 3t^3 + 3t^4 + ... \\
&= -2 + t^2 + 3(1 + t + t^2 + t^3 + ...) \\
&= -2 + t^2 + \frac{3}{1-t} \\
&= \frac{(1-t)(-2+t^2)+3}{1-t} \\
&= \frac{-2+t^2+2t-t^3+3}{1-t} \\
&= \frac{1+2t+t^2-t^3}{1-t}
\end{aligned}$$

1.2.2 Solution by M2

Verify this calculation using M2.

Create the polynomial ring S over $\mathbb{Q}$

[1]:
```
S = QQ[a,b,c]
```

```
o1 = S

o1 : PolynomialRing
```

Create the ideal I.

```
[2]: I = ideal(a^2, a*b, b^3, b^2*c)
```

```
              2         3   2
o2 = ideal (a , a*b, b , b c)

o2 : Ideal of S
```

Create the quotient ring R.

```
[3]: R = S/I
```

```
o3 = R

o3 : QuotientRing
```

Compute the Hilbert series of R.

```
[4]: hilbertSeries R
```

```
            2    3     4    5
     1 - 2T  - T  + 3T  - T
o4 = -----------------------
                       3
             (1 - T)

o4 : Expression of class Divide
```

Reduce the expression so we can compare it with the manual calculation.

```
[5]: reduceHilbert o4
```

```
                2    3
     1 + 2T + T  - T
o5 = ----------------
         (1 - T)

o5 : Expression of class Divide
```

The expressions agree. The manual calculation is correct.