PCA by hand.

Suppose we have the following dataset : $X = \begin{bmatrix} -1 & 0 \\ 1 & -2 \\ 6 & 5 \end{bmatrix}$ row-major format.

1) Center the data :

$\mu_1 = (-1+1+6)/3 = 2.$

$\mu_2 = (0-2+5)/3 = 1$

$$\tilde{X} = \begin{bmatrix} -1 & 0 \\ 1 & -2 \\ 6 & 5 \end{bmatrix} - \begin{bmatrix} 2 & 1 \\ 2 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} -3 & -1 \\ -1 & -3 \\ 4 & 4 \end{bmatrix}$$

2) Calculate the covariance Matrix :

$$\Sigma = \frac{1}{n} \tilde{X}^T \tilde{X}$$

$$= \frac{1}{3} \begin{bmatrix} -3 & -1 & 4 \\ -1 & -3 & 4 \end{bmatrix} \begin{bmatrix} -3 & -1 \\ -1 & -3 \\ 4 & 4 \end{bmatrix}$$

$$= \frac{1}{3} \begin{bmatrix} 9+1+16 & 3+3+16 \\ 3+3+16 & 1+9+16 \end{bmatrix}$$

$$= \frac{1}{3} \begin{bmatrix} 26 & 22 \\ 22 & 26 \end{bmatrix} = \begin{bmatrix} \frac{26}{3} & \frac{22}{3} \\ \frac{22}{3} & \frac{26}{3} \end{bmatrix}$$

3) Calculate the eigenvalues of the covariance matrix.

$\det(A - \lambda I) = 0.$

$$\det \begin{bmatrix} \frac{26}{3}-\lambda & \frac{22}{3} \\ \frac{22}{3} & \frac{26}{3}-\lambda \end{bmatrix} = 0$$

$$\left(\frac{26}{3}-\lambda\right)^2 - \left(\frac{22}{3}\right)^2 = 0.$$

$$\lambda^2 + \frac{676}{9} - \frac{52}{3}\lambda - \frac{484}{9} = 0.$$

$$\lambda^2 - \frac{52}{3}\lambda + \frac{192}{9} = 0.$$

$$\left(\lambda - \frac{48}{3}\right)\left(\lambda - \frac{4}{3}\right) = 0.$$

$$\lambda_1 = \frac{48}{3} = 16 \qquad \lambda_2 = \frac{4}{3}$$

4) Calculate the eigenvectors of Covariance matrix.

$$(A-\lambda I)\vec{v} = 0.$$

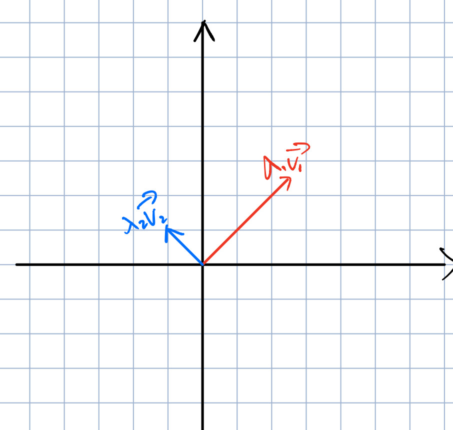
$$\begin{bmatrix} \frac{26}{3}-\frac{48}{3} & \frac{22}{3} \\ \frac{22}{3} & \frac{26}{3}-\frac{48}{3} \end{bmatrix} \begin{bmatrix} V_{11} \\ V_{12} \end{bmatrix} = 0$$

$$\begin{bmatrix} -\frac{22}{3} & \frac{22}{3} \\ \frac{22}{3} & -\frac{22}{3} \end{bmatrix} \begin{bmatrix} V_{11} \\ V_{12} \end{bmatrix} = 0.$$

$$-\frac{22}{3}V_{11} + \frac{22}{3}V_{12} = 0.$$

$$V_{11} = V_{12}$$

$$\vec{V_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \xRightarrow{\text{normalize}} \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}$$



$$\begin{bmatrix} \frac{26}{3}-\frac{4}{3} & \frac{22}{3} \\ \frac{22}{3} & \frac{26}{3}-\frac{4}{3} \end{bmatrix} \begin{bmatrix} V_{21} \\ V_{22} \end{bmatrix} = 0$$

$$\begin{bmatrix} \frac{22}{3} & \frac{22}{3} \\ \frac{22}{3} & \frac{22}{3} \end{bmatrix} \begin{bmatrix} V_{21} \\ V_{22} \end{bmatrix} = 0.$$

$$V_{21} = -V_{22}$$

$$\vec{V_2} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \xRightarrow{\text{normalize}} \begin{bmatrix} \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \end{bmatrix}$$

5) Project the data along the eigenvector corresponding to the largest eigenvalue.

$$Y = \tilde{X} \cdot C = \begin{bmatrix} -3 & -1 \\ -1 & -3 \\ 4 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \left(\frac{1}{\sqrt{2}}\right)$$

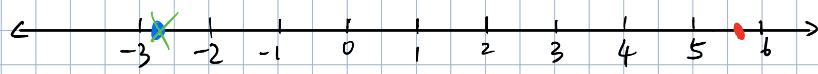
$$= \begin{bmatrix} -4 \\ -4 \\ 8 \end{bmatrix} \left(\frac{1}{\sqrt{2}}\right)$$

$$\approx \begin{bmatrix} -2.8 \\ -2.8 \\ 5.6 \end{bmatrix}$$

$C = d \times m$ matrix of eigenvectors, m = # of eigenvectors you want to keep.

(in our case we only want to keep 1)

6) Plot the projected data.


-3
-2
-1
0
1
2
3
4
5
6