$\square$ — тип всех kinds

Coq, вселенные типов

$\square_i \subset \square_{i+1} \subset \square_{i+2} \ldots \subset \square_n$

$\lambda_{\underline{\omega}}$ — $\lambda_{\to}$ + типы, зависящие от типов

(sort) $\emptyset \vdash * : \square$ → sort rule

(intro-var) $$\frac{\Gamma \vdash A : {\color{blue}s}}{\Gamma, x : A \vdash x : A} \quad x \notin \Gamma$$

$s \equiv *$ → A — тип (не точный конструктор), x — терм

$s \equiv \square$ → A — kind, x — конструктор типа

$$(\text{weak}) \quad \frac{\Gamma \vdash A : B \qquad \Gamma \vdash C : s}{\Gamma, x : C \vdash A : B} \quad x \notin \Gamma$$

$$\alpha : *, \ x : * \vdash \alpha : *$$

$$\alpha : *, \beta : * \vdash \beta : *$$

$$\dfrac{\dfrac{\emptyset \vdash * : \square}{\alpha : * \vdash \alpha : *}\,(\text{intro-va}) \qquad \dfrac{\emptyset \vdash * : \square}{\alpha : * \vdash \alpha : *}\,(\text{intro-var})}{\alpha : *, x : \alpha \vdash \alpha : *}\,(\text{weak})$$

Weakening (Lemma)

Thinning (Lemma)

(intro →) $$\frac{\Gamma \vdash A : s \qquad \Gamma \vdash B : s}{\Gamma \vdash A \to B : s}$$ $\lambda \underline{\omega}$ $\cancel{\forall}$

$s \equiv *$ → A, B − типы ($\lambda \to$)

$s \equiv \square$ → A, B − kinds

(elim →) $$\frac{\Gamma \vdash M : A \to B \qquad \Gamma \vdash N : A}{\Gamma \vdash MN : B}$$

↙ работает и для термов, и для конструкторов типов

$$(\lambda \alpha : * .\ \alpha \to \alpha)\beta \to_{\beta} \beta \to \beta$$

(intro-abst) $$\frac{\Gamma, x:A \vdash M:B \qquad \Gamma \vdash A \to B : s}{\Gamma \vdash \lambda x:A.\ M : A \to B}$$

$s \equiv *$, $A \to B$ - тип, тоді правило еквівалентно правилу в $\lambda_{\to}$.

$s \equiv \Box$, $A \to B$ - kind, тоді получаєм спосіб формулювання типових конструкторів

(conv) $$\frac{\Gamma \vdash A:B \qquad \Gamma \vdash B':s}{\Gamma \vdash A:B'} \qquad B =_{\beta} B'$$

$$\beta:*,\ x:(\lambda\alpha:*.\ \alpha \to \alpha)\beta \vdash x:\beta \to \beta$$

$$\frac{\Gamma \vdash x:(\lambda\alpha:*.\ \alpha \to \alpha)\beta \qquad \Gamma \vdash \beta:*}{\Gamma \vdash x:\beta \to \beta}$$

Редукция субъекта

$$\frac{\begin{array}{c}\Gamma \vdash A : B \\ \big\downarrow_{\beta} \\ A'\end{array}}{\Gamma \vdash A' : B}$$

Редукция типа

$$\frac{\begin{array}{r}\Gamma \vdash A : B \\ \big\downarrow_{\beta} \\ B'\end{array}}{\Gamma \vdash A : B'}$$

(если $\Gamma \vdash B' : s$)

Преобразование типа (conv)

$$\frac{\begin{array}{r}\Gamma \vdash A : B \\ =_{\beta} \\ B'\end{array}}{\Gamma \vdash A : B'}$$

(если $\Gamma \vdash B' : s$)

λP

$\lambda x : A . M$ → конструктор типа $\qquad$ `std::array<10>`

Терм $\nearrow$ $\qquad$ Тип $\uparrow$

$\lambda P = \lambda_{\to}$ + типы, которые зависят от термов

$$\Gamma, x : A \vdash b(x) : B(x)$$

$\lambda 2 \qquad \forall$

$\lambda P \qquad \Pi$

$$\frac{\Gamma, x : A \vdash B(x) - \text{type}}{\Gamma \vdash \Pi_{(x:A)} B(x) - \text{type}} \quad \text{- - - - -} \quad \Pi x : A .\ B$$

Π - all

Σ - any

(sort) $\emptyset \vdash * : \square$

(intro-var) $$\frac{\Gamma \vdash A : s}{\Gamma, x : A \vdash x : A} \quad x \notin \Gamma$$

(weak) $$\frac{\Gamma \vdash A : B \qquad \Gamma \vdash C : s}{\Gamma, x : C \vdash A : B} \quad x \notin \Gamma$$

$s \equiv *$

$s \equiv \square$

(intro-Π) $$\frac{\Gamma \vdash A : * \qquad \Gamma, x : A \vdash B : s}{\Gamma \vdash \Pi_{(x:A)} B : s}$$

(elim-Π) $$\frac{\Gamma \vdash M : \Pi_{(x:A)} B \qquad \Gamma \vdash N : A}{\Gamma \vdash MN : B[x := N]}$$

(intro-abst) $$\frac{\Gamma, x : A \vdash M : B \qquad \Gamma \vdash \Pi_{(x:A)} B : s}{\Gamma \vdash \lambda x : A . M : \Pi_{(x:A)} B}$$

(conv) $$\frac{\Gamma \vdash A : B \qquad \Gamma \vdash B' : s}{\Gamma \vdash A : B'} \quad B =_\beta B'$$

П - тип - это подмножество стрелочных типов

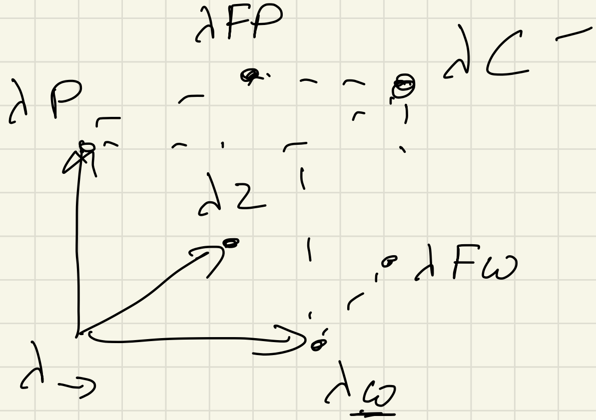
$$\begin{tikzcd} \Pi_{(x:A)} B \arrow[d, dash, dashed] \\ \forall x.\ B(x) \arrow[d] \\ A \end{tikzcd}$$

$A \to B$ (B в нет свободных x)

$x \to B$

$[x \to isPrime(x)]\ 5$

x: nat



Calculus of constructions

(intro →)
$\lambda\underline{\omega}$

$$\frac{\Gamma \vdash A : s \qquad \Gamma \vdash B : s}{\Gamma \vdash A \to B : s}$$

(intro Π)
λP

$$\frac{\Gamma \vdash A : * \qquad \Gamma, x : A \vdash B : s}{\Gamma \vdash \Pi_{x:A} B : s}$$

(intro- Π)
λC

$$\frac{\Gamma \vdash A : s_1 \qquad \Gamma, x : A \vdash B : s_2}{\Gamma \vdash \Pi_{x:A} B : s_2}$$

$\leftarrow s_3$ – pure type system

$$(s_1, s_2) = \underset{\lambda_{\to}}{\{*, *\}}, \quad \underset{\lambda P}{\{*, \square\}}, \quad \underset{\lambda 2}{\{\square, *\}}, \quad \underset{\lambda \underline{\omega}}{\{\square, \square\}}$$

$x : \Gamma \vdash x : \Gamma$ — $\lambda_{\to}$ / $\lambda 2$

$\underbrace{\emptyset \vdash * : \square}$ — $\lambda \underline{\omega}$ / λP

$\lambda\underline{\omega}$ $\quad \alpha : *, \beta : * \vdash \beta : *$

$\alpha : *, \beta : * \to * \vdash \beta(\beta\alpha) : *$

(1) $\boxed{\alpha : *}$

(2) $\boxed{\beta : * \to *}$

(3) $\beta\alpha : *$ (elim → on 1, 2)

$\beta(\beta\alpha) : *$ (elim → on 2, 3)

Очевидные правила вывода:

— sort, $\overset{\text{intro}}{}$ var, weak

—

$$\frac{\Gamma \vdash A : s \qquad \Gamma \vdash B : s}{\Gamma \vdash A \to B : s}$$

$$\frac{\Gamma, x : A \vdash M : B \qquad \Gamma \vdash A \to B : s}{\Gamma \vdash \lambda x : A .\ M : A \to B}$$

$\boxed{x : \alpha}$

(1) $x : \alpha$

$\alpha : * ,\ x : \alpha \vdash \lambda y^{\alpha} . x : (\lambda \beta : * .\ \beta \to \beta)\ \alpha$

(0) $\boxed{* : \square}$

(a) $\boxed{\alpha : *}$ (context)

(b) $\boxed{x : \alpha}$ (context)

(c) $\boxed{\alpha \to \alpha : *}$ (intro → on a, a)

(d) $\boxed{y : \alpha}$ (intro var on a)

(e) $\boxed{x : \alpha}$ (intro var on a)

(1) $\lambda y . x : \alpha \to \alpha$ (intro abst d, e, c)

(4) $\boxed{\beta : *}$ (intro var on 0)

(5) $\boxed{\beta \to \beta : *}$ (intro → on 4, 4)

(3) $\lambda \beta : * .\ \beta \to \beta : * \to *$ (intro abst 4, 5)

(2) $\underline{(\lambda \beta : * .\ \beta \to \beta)}\ \alpha : *$ (elim → on 3, a)

$\lambda y^{\alpha} . x : (\lambda \beta : * .\ \beta \to \beta)\ \alpha$ (conv 1, 2) (sort) $\emptyset \vdash * : \square$

(intro-var) $$\frac{\Gamma \vdash A : s}{\Gamma, x : A \vdash x : A} \qquad x \notin \Gamma$$

(weak) $$\frac{\Gamma \vdash A : B \qquad \Gamma \vdash C : s}{\Gamma, x : C \vdash A : B}$$

(intro $\to$) $$\frac{\Gamma \vdash A : s \qquad \Gamma \vdash B : s}{\Gamma \vdash A \to B : s}$$

(elim $\to$) $$\frac{\Gamma \vdash M : A \to B \qquad \Gamma \vdash N : A}{\Gamma \vdash MN : B}$$

(intro-abst) $$\frac{\Gamma, x : A \vdash M : B \qquad \Gamma \vdash A \to B : s}{\Gamma \vdash \lambda x^{A} . M : A \to B}$$

(conv) $$\frac{\Gamma \vdash A : B \qquad \Gamma \vdash B : s'}{\Gamma \vdash A : B'} \qquad B =_{\beta} B'$$

$\boxed{\lambda P}$

(1) $A : *$

(2) $* : \square$

$A \to * : \square$ (intro Π on 1, 2)

$A \to * \equiv \Pi_{x:A} *$

кайнд, зависящий от терма

$P : A \to *$

Тип, зависящий от терма

$$\forall x:S\,(P(x) \to Q(x)) \to (\forall y:S\,P(y) \to \forall u:S\,Q(u))$$

$$\lambda P : P \equiv \Big(\prod_{x:S} \underbrace{Px \to Qx}\Big) \to \prod_{y:S} Py \to \prod_{z:S} Qz$$

$$\lambda u^{\prod_{x:S} Px \to Qx}\; v^{\prod_{y:S} Py}\; z^{S} \,.\; \underbrace{\overbrace{\underbrace{u\,z}_{P_z \to Q_z}\,\underbrace{(v\,z)}_{Pz}}^{Qz}}_{Qz} : P$$

(1) $\boxed{S : *}$ $\boxed{y : S}$ $(S \Rightarrow *)$

(2) $\boxed{P, Q : S \rightarrow *}$ (intro → 1)

(3) $\boxed{u : \prod_{x:S} Px \rightarrow Qx}$ (intro → 2)

(4) $\boxed{v : \prod_{y:S} Py}$ (intro Π 2)

(5) $\boxed{z : S}$ (intro var 1)

(6) $uz : Pz \rightarrow Qz$ (elim Π 3, 5)

(7) $vz : Pz$ (elim Π 4, 5)

(8) $uz(vz) : Qz$ (elim → 6, 7)

(9) $\lambda z^{S} . uz(vz) : \prod_{x:S} Qz$ (intro → 8, 5)

(10) $\lambda v^{\prod_{y:S} Py} z^{S} . uz(vz) : \prod_{y:S} Py \rightarrow \prod_{z:S} S . Qz$ (intro → 9, 4)

$\lambda u^{\prod_{x:S} Px \rightarrow Qx} v^{\prod_{y:S} Py} z^{S} . uz(vz) : P$ (intro → 10, 3)