$$\text{true} = \lambda t f.\ t$$

$$\text{false} = \lambda t f.\ f$$

$$\lambda f x y.\ f x y$$

$$\text{pair} = \lambda x y f.\ f x y$$

$$\text{first} = \lambda p.\ p\ \text{true}$$

$$\underbrace{(\lambda x.\ P)\, N}_{\beta\text{-редекс}} = P[x := N]$$

$$\text{second} = \lambda p.\ p\ \text{false}$$

```
struct Pair {
    field-1,
    field-2,
}
```

$$\text{first}\ (\text{pair}\ a\ b) = a$$

$$(\lambda p.\ p\ \text{true})\ (\text{pair}\ a\ b) =$$

$$(\text{pair}\ a\ b)\ \text{true} =$$

$$((\lambda x y f.\ f x y)\ a\ b)\ \text{true} =$$

$$= ((\lambda y f.\ f a y)\ b)\ \text{true} =$$

$$(\lambda f.\ f a b)\ \text{true} =$$

$$(\lambda t f.\ t)\ a\ b =$$

$$a$$

$0 = \lambda s z.\ z$

$1 = \lambda s z.\ s z$

$2 = \lambda s z.\ s(s z)$

$\vdots$

$n = \lambda s z.\ s^{n}(z)$

$F^{0}(X) = X$

$F^{n+1}(X) = F(F^{n}(X))$

$is\_zero = \lambda n.\ n\ (\lambda x.\ false)\ true$

$is\_zero\ 0 =$

$(\lambda n.\ n(\lambda x.\ false)\ true)\ 0 =$

$= 0(\lambda x.\ false)\ true =$

$(((\lambda s z. z)\ (\lambda x.\ false))\ true) =$

$(\lambda z. z)\ true =$

$= true$

$is\_zero\ 1 =$

$= 1\ (\lambda x.\ false)\ true =$

$= (\lambda s z. s z)\ (\lambda x.\ false)\ true =$

$= (\lambda z. (\lambda x.\ false) z)\ true =$

$= (\lambda x.\ false)\ true =$

$false$

$$succ = \lambda n s z.\ s(n s z)$$

$$plus = \lambda m n s z.\ m s (n s z)$$

$$mult = \lambda m n s z.\ m (n s) z$$

$$succ\ 1 =$$

$$(\lambda n s z.\ s(n s z))\ 1 =$$

$$\lambda s z.\ s(1 s z) =$$

$$(\lambda s z. s z)\ s z = s z$$

$$= \lambda s z.\ s (s z) = 2$$

$$plus\ 1\ 2 = 3$$

Fixed point (FP)

$(\lambda x. M) N = M[x := N]$

$F \in \Lambda : \forall M, N, L \in \Lambda \quad \lambda \vdash FMNL = ML(NL)$

?

$\lambda \vdash FM = MF$

$F = \lambda m. mF$

$FMNL = ML(NL)$

$FMN = \lambda l. Ml(Nl)$ L

$FM = \lambda nl. Ml(nl)$ NL

$F = (\lambda mnl. ml(nl))MNL$

F ?

$FMNL = Ml(Nl)$

(MN)

$P = \underline{\lambda n. M n}$

$PN = MN$

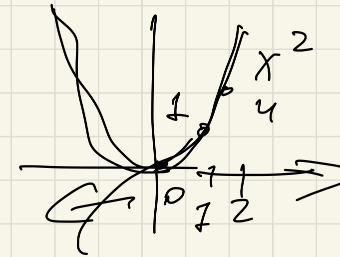
$P' L = FMNL$

$P' = \lambda l. FMNl$

Теорема о FP

$$\forall F \in \Lambda \quad \exists X \in \Lambda \quad \lambda \vdash FX = X$$

$f(0) = 0$



$f(x) = x$

Док-во:

$$W = \lambda x.\ F(xx)$$

$$X = WW$$

$$X = (\lambda x.\ F(xx))\ W = F(WW) = F(X) = FX \quad \square$$

Равномерная теорема о FP

$$\exists X \in \Lambda \quad \forall F \in \Lambda \quad \lambda \vdash FX = X$$

$$X = YF \qquad F(YF) = YF$$

Док-во:

$$Y = \lambda f.\left[(\lambda x.\ f(xx))(\lambda x.\ f(xx))\right]$$

$$YF = (\lambda x.\ F(xx))\ (\lambda x.\ F(xx)) =$$

$$= F(\underbrace{\lambda x.F(xx)\ \lambda x.F(xx)}_{YF}) = F(YF)$$

$$Y = \lambda f.[(\lambda x. f(xx))(\lambda x. f(xx))] \qquad F$$

$$YF = (\lambda x. F(xx))(\lambda x. F(xx))$$

$$YF = (\lambda x. F(xx))(\lambda x. F(xx)) = F((\lambda x. F(xx))(\lambda x. F(xx))) =$$

$$= F(YF)$$

$$M \in \Lambda \qquad\qquad \exists ?\ M \in \Lambda \quad \lambda \vdash MX = XMX$$

$$M\cancel{X} = XMX$$

$$\Uparrow$$

$$M = \lambda x. xMx \qquad\qquad L = \lambda m.(\lambda x. xmx)$$

$$LM = M \leftarrow \exists ?\ M \qquad\qquad M$$

$$\underline{M = YL}$$

$$L(YL) = YL$$

$0 = \lambda sn.\ sn$ $\qquad MX = X$

$1 = \lambda sn.\ s(sn)$ $\qquad X = YM$

$fac = \lambda n.\ if\ (is\_zero\ n)\ 1\ (mult\ n\ (fac\ (pred\ n)))$

$fac = [\underbrace{\lambda f n.\ if (is\_zero\ n)\ 1\ (mult\ n\ (f\ (pred\ n)))}_{N}]\ fac$

$n - 1$

$fac = N\ fac$ $\qquad N(fac) = fac$

$fac = \underline{YN}$ $\qquad N$

$\underline{N(YN) = YN}$

$fac = YN$

$fac\ 3 = (Y\ N) 3 = \underbrace{N(YN)}\ 3 = [\lambda f n.\ if (n == 0) 1\ (n \cdot f(n-1))]\ (YN)\ 3$

$= if\ (3 == 0)\ 1\ (3 \cdot \underbrace{YN}(2)) =$

$= 3 \cdot [N(YN)\ 2] = 3 \cdot [\lambda f n.\ if (n == 0)\ 1\ (n \cdot f(n-1))\ (YN) 2] =$

$$3 \cdot 2 \cdot [\,(YN)\,1\,] =$$

$$3 \cdot 2 \cdot [\,N(YN)\,1\,] =$$

$$= 3 \cdot 2 \cdot [\,1\,0\,(YN)\,0\,] =$$

$$= 3 \cdot 2 \cdot 1 \cdot 1$$

β-редукция

One-step β-reduction, $\to_\beta$

1. (Basis) $(\lambda x.M)N \to_\beta M[x:=N]$
2. (Compatibility) если $M \to_\beta N$, то $\forall L \in \Lambda$:

$$ML \to_\beta NL$$

$$LM \to_\beta LN$$

$$\lambda x.M \to_\beta \lambda x.N$$

Редекс (redex)

подтерм следующего вида $(\lambda x.M)N$

$\Omega = \omega\omega = (\lambda x.xx)(\lambda x.xx) \to_\beta \omega\omega \to_\beta \omega\omega$

Нормальная форма

$M \in \Lambda$ в НФ $\Leftrightarrow$ $\nexists$ β-редексов

$\lambda xy.y$, $\lambda xy.y(\lambda z.zx)y$

$KI\Omega =$

$= (\lambda xy.x)(\lambda x.x)[\underbrace{(\lambda x.xx)(\lambda x.xx)}]$

$$\begin{tikzcd}
 & KI\Omega \arrow[dl, "\beta"'] \arrow[dr, "\beta"] & & \\
I \arrow[ddrr] & & KI\Omega \arrow[ll, "\beta"'] \arrow[dr, "\beta"] & \\
 & & & KI\Omega \arrow[ulll, "\beta"] \arrow[dl, "\beta\cdot\infty"] \\
 & & N &
\end{tikzcd}$$

$\Omega_3 = \omega_3 \, \omega_3 = (\lambda x. xxx)(\lambda x. xxx) =_\beta$

$=_\beta (\lambda x. xxx)(\lambda x. xxx)(\lambda x. xxx)$

$=_\beta \omega_3 \, \omega_3 \, \omega_3 \, \omega$

β-reduction (zero-or-more steps) $\twoheadrightarrow_\beta$

$M \twoheadrightarrow_\beta N \quad \exists\, n \in \mathbb{N} : M_0, M_1, \dots, M_n, \quad M_0 = M, \quad M_n = N,$

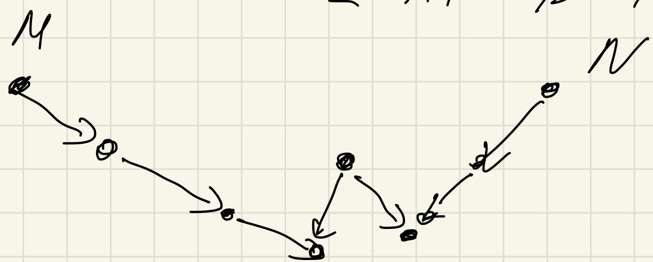
$\forall i \quad M_i \rightarrow_\beta M_{i+1}$

$M \equiv M_0 \rightarrow_\beta M_1 \rightarrow_\beta M_2 \rightarrow_\beta \dots \rightarrow_\beta N$

β- equality

$M =_\beta N \quad \exists\, n \in \mathbb{N}$ и $M_0, \ldots, M_n,\ M_0 = M,\ M_n = N$:

$$\forall i \in [0, n] : \quad \begin{cases} M_i \to_\beta M_{i+1} \\ M_{i+1} \to_\beta M_i \end{cases}$$



$$KI =_\beta IIK_* = K_*$$

$$KI = (\lambda xy.\,x)(\lambda x.\,x) =_\beta \lambda y.(\lambda x.\,x) = \lambda yx.\,x = K_*$$

$$IIK_* = (\lambda x.\,x)(\lambda x.\,x)(\lambda xy.\,y) = \underline{(\lambda x.\,x)}(\lambda xy.\,y) = K_*$$

$$\begin{array}{ccccc} KI & & & & IIK_* \\ & \searrow_\beta & & & \swarrow \\ & & K_* & \leftarrow & IK_* \end{array}$$

Λ:

$$v \in V$$

$$M, N \in \Lambda \Rightarrow (MN) \in \Lambda$$

$$M \in \Lambda \quad x \in V \Rightarrow (\lambda x. M) \in \Lambda$$

$$(\ldots ((v N_1) N_2 \ldots N_k)$$

$$(\ldots ((\lambda x. M) N_1) N_2 \ldots N_k)$$

Нормальная

$$(\lambda x. M) \quad N_1)$$

Аппликативная

сначала N_1 и только потом $\lambda x. M$

Теорема о нормализации

Если $M \in \Lambda$ имеет НФ, то применяя норм-ю стратегию, придём к нормальной форме

Church-Rosser (the diamond rule) Confluence, CR

Пусть $M \in \Lambda$ и $M \twoheadrightarrow_\beta N_1$, $M \twoheadrightarrow_\beta N_2$, тогда $\exists\, N_3$:

$N_1 \twoheadrightarrow_\beta N_3$, $N_2 \twoheadrightarrow_\beta N_3$

$$\begin{tikzcd}
 & M \arrow[dl, two heads] \arrow[dr, two heads] & \\
N_1 \arrow[dr, dotted, two heads] & & N_2 \arrow[dl, dotted, two heads] \\
 & N_3 &
\end{tikzcd}$$

MM

XX