Study Guide for the MATH 2331 Final

All content contained within is originally from Linear Algebra with Applications, 5th Ed. by Otto Brescher

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3 Subspaces of $\mathbb{R}^n$ and Their Dimensions

3.1 Image and Kernel of a Linear Transformation

The **image** of a function is the set of values that a function takes in its target space.

$$f : X \to Y \implies image(f) = \{f(x) \ \forall x \in X\}$$

The **span** of one or more vectors is the set of all linear combinations of those vectors.

$$\vec{v}_1 \dots \vec{v}_n \implies span(\{\vec{v}_1 \dots \vec{v}_n\}) = \{c_1 v_1 + \dots + c_n v_n \ \forall c_1 ... c_n \in \mathbb{R}\}$$

The image of a linear transformation is equal to the span of its columns.

$$T \text{ is linear}, \ T(\vec{x}) = A\vec{x} \implies image(T) = span(A)$$

The image of a linear transformation from m to n dimensions contains the zero vector in n, and is closed under addition and scalar multiplication.

The **kernel** or **nullspace** of a linear transformation is the set of all vectors that are "zeroes" of the transformation.

$$T : \mathbb{R}^m \to \mathbb{R}^n, \ T \text{ is linear} \implies kernel(T) = \{\vec{x} \in \mathbb{R}^m \mid T(\vec{x}) = \vec{0}_n\}$$

The kernel of a linear transformation from m to n dimensions contains the zero vector in m, and is closed under addition and scalar multiplication.

The kernel of a linear transformation from m to n dimensions contains only the zero vector if and only if the rank of its matrix is equal to m.

$$T : \mathbb{R}^m \to \mathbb{R}^n, \ T(\vec{x}) = A\vec{x} \implies kernel(T) = \{\vec{0}_m\} \iff rank(A) = m$$

If the kernel of a linear transformation contains only the zero vector, then the matrix has at most as many columns as rows.

$$A : \mathbb{R}^m \to \mathbb{R}^n,\ \ kernel(A) = \{\vec{0}_m\} \implies m \leq n$$

If an $n \times m$ matrix has more columns than rows, the kernel of the linear transformation contains nonzero vectors in $\mathbb{R}^m$.

$$A : \mathbb{R}^m \to \mathbb{R}^n,\ \ m > n \implies kernel(A) \neq \{\vec{0}_m\}$$

The kernel of a square matrix contains only the zero vector if and only if the matrix is invertible.

3.2 Subspaces of $\mathbb{R}^n$; Bases and Linear Independence

A subset of $\mathbb{R}^n$ is a (linear) **subspace** of $\mathbb{R}^n$ if it contains the zero vector in $\mathbb{R}^n$ and is closed under addition and multiplication.

For a linear transformation T from $\mathbb{R}^m$ to $\mathbb{R}^n$, $kernel(T)$ is a subspace of $\mathbb{R}^m$ and $image(T)$ is a subspace of $\mathbb{R}^n$.

A set of vectors $\vec{v}_1 \dots \vec{v}_n$ is **linearly independent** if none of them can be expressed as a linear combination of the others.

To determine that a vector $\vec{b}$ is linearly independent of some other vectors $\left[\vec{v}_1 \dots \vec{v}_n\right] = A$, use Gauss-Jordanian Elimination to show that the system $A\vec{x} = \vec{b}$ is inconsistent.

A set of vectors $\vec{v}_1 \dots \vec{v}_n \in V$ forms a **basis** of the subspace V if the vectors are linearly independent and $span(\{\vec{v}_1 \dots \vec{v}_n\}) = V$.

To construct the basis of $image(A)$, start with the column vectors of A and remove all the linearly dependent ("redundant") vectors.

A linear relation between the column vectors of an $n \times m$ matrix $A = \left[\vec{v}_1 \dots \vec{v}_m\right]$ corresponds to an entry $kernel(A)$. This gives the following result:

$$\vec{v}_1 \dots \vec{v}_m \text{ are linearly independent } \iff (kernel(A) = \vec{0}_m \iff rank(A) = m)$$

Based on a theorem from 3.1 this means that there are only n linearly independent vectors in $\mathbb{R}^n$.

A set of vectors $\vec{v}_1 \dots \vec{v}_n \in V$ forms a **basis** of the subspace V iff all vectors in V can be expressed uniquely as a linear combination of those vectors:

$$\vec{v} = c_1\vec{v}_1 + \ldots + c_n\vec{v}_n \ \ \forall \vec{v} \in V$$

The coefficients $c_1 \dots c_n$ are known as the **coordinates** of $\vec{v}$ with respect to the basis.

3.3 The Dimension of a Subspace of $\mathbb{R}^n$

All bases of a subspace V of $\mathbb{R}^n$ have the same number of vectors, known as the **dimension $\boldsymbol{dim(V)}$** of V.

For any subspace V of $\mathbb{R}^n$ with $dim(V) = m$ the following is true:

- There are at most m linearly independent vectors in V.
- At least m vectors are required to span V.
- $\{\vec{v}_1 \dots \vec{v}_m\}$ are linearly independent $\iff$ $\{\vec{v}_1 \dots \vec{v}_m\}$ form a basis of V
- $span(\{\vec{v}_1 \dots \vec{v}_m\}) = V \iff \{\vec{v}_1 \dots \vec{v}_m\}$ form a basis of V

To find a basis of $\boldsymbol{kernel(A)}$ (and thus its dimension):

- Calculate $rref(A)$ and determine which columns don't have leading 1's.
- Parameterize those columns and solve for each entry in $\vec{x}$.
- Lastly, factor out the parameters. The resulting vectors form a basis of $kernel(A)$.

To find a basis of $\boldsymbol{image(A)}$ (and thus its dimension):

- Calculate $rref(A)$ and determine which columns *do* have leading 1's.
- Find the corresponding columns in A. Those column vectors form a basis of $image(A)$.

Theorem: For any matrix A, $dim(image(A)) = rank(A)$.

This all leads to the **Rank-Nullity Theorem**: for an $n \times m$ matrix A:

$$dim(image(A)) + dim(kernel(A)) = m$$

Theorem: The vectors $\vec{v}_1 \dots \vec{v}_n$ form a basis of $\mathbb{R}^n$ iff the matrix $\begin{bmatrix}\vec{v}_1 \dots \vec{v}_n\end{bmatrix}$ is invertible.

3.4 Coordinates

If we have a basis $\mathfrak{B} = (\vec{v}_1 \dots \vec{v}_m)$ of a subspace V of $\mathbb{R}^n$, then any vector in V can be written uniquely as $\vec{v} = c_1\vec{v}_1 + \dots + c_m\vec{v}_m$.

The vector $[\vec{x}]_{\mathfrak{B}} = \begin{bmatrix} c_1 \\ \dots \\ c_m \end{bmatrix}$ is then the $\mathfrak{B}$-coordinate vector of $\vec{x}$.

Then $\vec{x} = S[\vec{x}]_{\mathfrak{B}}$, where $S = \begin{bmatrix}\vec{v}_1 \dots \vec{v}_m\end{bmatrix}$ with dimensions $n \times m$.

For any linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ and a basis $\mathfrak{B} = (\vec{v}_1 \dots \vec{v}_n)$ there is a matrix B that represents the transformation T in $\mathfrak{B}$:

$$\exists B : n \times n \text{ s.t. } [T(\vec{x})]_{\mathfrak{B}} = B[\vec{x}]_{\mathfrak{B}}$$

B can be constructed column-by-column:

$$B = \big[[T(\vec{v}_1)]_{\mathfrak{B}} \dots [T(\vec{v}_n)]_{\mathfrak{B}} \big]$$

Or B can also be constructed in terms of A (where $T(\vec{x}) = A\vec{x}$):

$$AS = BS, \;\; B = S^{-1}AS, \;\; A = SBS^{-1}$$

The middle formula $\boldsymbol{B = S^{-1}AS}$ is the form most commonly seen, and can be interpreted best when right-to-left:

- take the vector out of the basis $\mathfrak{B}$ (S)
- apply the transformation (A)
- and bring the vector back into $\mathfrak{B}$ (S^{-1})

If the above relationship holds, we say that A and B are **similar matrices**.

5 Orthogonality and Least Squares

5.1 Orthogonal Projections and Orthonormal Bases

Two vectors $\vec{u}, \vec{v}$ are **orthogonal** (perpendicular) if their dot product is equal to zero: $\vec{u} \cdot \vec{v} = 0$.

The **length** or **norm** of a vector $\vec{v}$ is the square root of the dot product with itself: $|\vec{v}| = \sqrt{\vec{v} \cdot \vec{v}}$.

A **unit vector** $\vec{u}$ has length equal to 1. A unit vector can be obtained by multiplying by the reciprocal of the norm:

$$\vec{u} = \frac{1}{|\vec{v}|}\vec{v}$$

A set of vectors $\vec{u}_1 \dots \vec{u}_n$ is said to be **orthonormal** if they are all unit vectors and all orthogonal to one another. Put another way: $\vec{u}_i \cdot \vec{u}_j = 0 \;\; \forall i \neq j$

Orthonormal vectors are independent, and n orthonormal vectors will form an **orthonormal basis** of $\mathbb{R}^n$.

If V is a subspace of $\mathbb{R}^n$ with orthonormal basis $\vec{u}_1 \dots \vec{u}_m$, the orthogonal projection onto V is equivalent to projecting onto each component of V and adding the results:

$$proj_V(\vec{x}) = x^{\|} = (\vec{u}_1 \cdot \vec{x})\vec{u}_1 + \ldots + (\vec{u}_m \cdot \vec{x})\vec{u}_m$$

A special case of this occurs when projecting onto $\mathbb{R}^n$ itself:

$$proj_{\mathbb{R}^n}(\vec{x}) = \vec{x} = (\vec{u}_1 \cdot \vec{x})\vec{u}_1 + \ldots + (\vec{u}_n \cdot \vec{x})\vec{u}_n$$

This tells us that the coordinates of $\vec{x}$ in an orthonormal basis can be found through the dot product: $c_i = \vec{u}_i \cdot \vec{x}$

The **orthogonal complement** of a subspace V is the set of vectors in $\mathbb{R}^n$ perpendicular to all vectors in V. It can be thought of as $V^\perp = kernel(proj_V(\vec{x}))$, and from the rank-nullity theorem we get $dim(V^\|) + dim(V^\perp) = n$.

5.2 Gram-Schmidt Process and QR Factorization

The **Gram-Schmidt Process** is an algorithm for taking any basis of V and transforming it into an orthonormal basis for the same subspace V.

We start with the first vector $\vec{v}_1$:

$$\vec{u}_1 = \frac{1}{|\vec{v}_1|}\vec{v}_1$$

Then for each vector following $(\vec{v}_2 \dots \vec{v}_n)$, we find the component of that vector perpendicular to all unit vectors found so far:

$$\vec{v}_n^\perp = \vec{v}_n - (\vec{u}_1 \cdot \vec{v}_n)\vec{u}_1 - \dots - (\vec{u}_{n-1} \cdot \vec{v}_n)\vec{u}_{n-1}$$

And then reduce that component to a unit vector by dividing out its magnitude:

$$\vec{u}_n = \frac{1}{|\vec{v}_n^\perp|}\vec{v}_n^\perp$$

This is known as **QR Factorization** because the matrix $\begin{bmatrix}\vec{v}_1 \dots \vec{v}_n\end{bmatrix}$ is factored into two matrices Q and R, where $Q = \begin{bmatrix}\vec{u}_1 \dots \vec{u}_n\end{bmatrix}$ and $R_{i,j} = \begin{cases} (\vec{u}_i \cdot \vec{v}_j), & i \neq j \\ |\vec{v}_j^\perp|, & i = j \end{cases}$.

This allows the first column of R to be computed, then the first column of Q, followed by the second column of R, then the second column of Q, ... Note that R is upper triangular.

5.3 Orthogonal Transformations and Orthogonal Matrices

A linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^n$ is **orthogonal** if it preserves the length of its inputs:

$$|T(\vec{x})| = |\vec{x}| \ \ \forall \vec{x} \in \mathbb{R}^n$$

The matrix of an orthogonal linear transformation is called an **orthogonal matrix**.

An orthogonal transformation T preserves orthogonality: if two vectors $\vec{v}, \vec{w}$ are orthogonal, so are $T(\vec{v}), T(\vec{w})$.

A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is orthogonal iff the column vectors of A form an orthonormal basis of $\mathbb{R}^n$.

The product of two orthogonal matrices is also orthogonal. The inverse of an orthogonal matrix is also orthogonal.

A matrix is **symmetric** if $A^T = A$, and **skew-symmetric** if $A^T = -A$

The dot product can be expressed as a matrix product using the transpose: $\vec{v} \cdot \vec{w} = \vec{v}^T \vec{w}$

A matrix A is orthogonal iff $A^T A = I_n$ or $A^T = A^{-1}$.

If V is a subspace of $\mathbb{R}^n$ with orthonormal basis $\vec{u}_1 \dots \vec{u}_n$, the matrix for the orthogonal projection onto V is $P = QQ^T$ where $Q = \left[\vec{u}_1 \dots \vec{u}_n\right]$

5.4 Least Squares and Data Fitting

TODO – write me!

6 Determinants

6.1 Introduction to Determinants

The **determinant** of a matrix tells if that matrix is invertible: for a matrix A, A^{-1} exists iff $det(A) \neq 0$.

The determinant of a triangular matrix (upper or lower) is the product of its diagonal entries.

Recall that the determinant of a 2×2 matrix is $ad - bc$, and its inverse is $\frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

6.2 Properties of the Determinant

If A is a square matrix then $det(A^T) = det(A)$

The following properties of the determinant are useful when doing Gauss-Jordanian Elimination:

- If B comes from dividing a row of A by a scalar k then $det(B) = \frac{1}{k} det(A)$
- If B comes from a row swap in A then $det(B) = -det(A)$
- If B comes from adding a multiple of a row to another in A then $det(B) = det(A)$

To calculate the determinant of an $n \times n$ matrix A:

- Perform row reductions on A until you reach $rref(A)$, keeping a careful tally of which operations you perform at each step.

- Calculate $det(rref(A))$ by multiplying the diagonal entries of $rref(A)$
- Using the rules from above, work backwards from $det(rref(A))$ to reach $det(A)$.

For two $n \times n$ matrices A, B it is true that $det(AB) = det(A)det(B)$, and for a positive integer m it is true that $det(A^m) = det(A)^m$

The determinant is independent of basis: similar matrices have the same determinant.

If A is invertible then $det(A^{-1}) = \frac{1}{det(A)}$

7 Eigenvalues and Eigenvectors

7.1 Diagonalization

A matrix is **diagonal** if there are nonzero entries only along the diagonal of the matrix.

The matrix A of the linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is said to be **diagonalizable** if the matrix B of T with respect to $\mathfrak{B}$ is diagonal.

To diagonalize a matrix, one finds an invertible matrix S and a diagonal matrix B such that the change-of-bases formula holds: $B = SAS^{-1}$.

A nonzero vector $\vec{v}$ is an **eigenvector** of A if $A\vec{v} = \lambda\vec{v}$. The value λ is the **eigenvalue** of eigenvector $\vec{v}$. One or more eigenvectors $\vec{v}_1 \dots \vec{v}_n$ of A form an **eigenbasis** of A.

If 0 is an eigenvalue of A then $kernel(A) \neq \{\vec{0}\}$ and A is not invertible.

A **discrete linear dynamical system** is a system where the vector $\vec{x}$ is a function of time t. Specifically $\vec{x}(t)$ is a linear transformation from $t-1$ to t such that:

$$\vec{x}(t) = A\vec{x}(t-1) = A^t\vec{x}_0$$

.
When dealing with discrete linear dynamical systems the goal is often to find a nonrecursive or **closed formula** for $\vec{x}(t)$: that is, a formula for $\vec{x}(t)$ in terms of t alone.

To solve for the closed formula of a discrete linear dynamical system:

- Find an eigenbasis $\vec{v}_1 \dots \vec{v}_n$ of the transformation matrix A.
- Express $\vec{x}_0$ as coordinates in that eigenbasis: $c_1 \dots c_n$.
- $\vec{x}(t) = c_1\lambda_1^t\vec{v}_1 + \ldots + c_n\lambda_n^t\vec{v}_n$

7.2 Finding the Eigenvalues of a Matrix

To determine the eigenvalues of a matrix, solve the following equation:

$$det(A - \lambda I_n) = 0$$

The above is known as the **characteristic equation** of the matrix A. It gives rise to the **characteristic polynomial** of the matrix, the roots of which are the eigenvalues of the matrix.

The **algebraic multiplicity** of an eigenvalue is equal to the multiplicity of its root of the characteristic polynomial.

The eigenvalues of a triangular matrix are its diagonal entries.

7.3 Finding the Eigenvectors of a Matrix

An **eigenspace** associated with eigenvalue λ is:

$$E_\lambda = kernel(A - \lambda I_n)$$

The eigenvectors with eigenvalue λ are the nonzero vectors in E_λ. Thus finding the eigenvectors of λ is equivalent to finding a basis of the eigenspace E_λ.

To determine the eigenvectors of a matrix:

- For each eigenvalue λ find the eigenspace for λ by solving for the vectors that span $kernel(A - \lambda I_n)$.
- Recall that this means solving for $rref(A)$ and parameterizing the non-leading columns.

So to review:

- Solve for eigenvalues: $det(A - \lambda I_n) = 0$
- For each eigenvalue, find a basis of the eigenspace $kernel(A - \lambda I_n)$
- Determine that the dimensions of the bases of all the eigenspaces add up to n. If so, concatenate those bases together to form S.

7.4 Dynamical Systems

TODO – write me!

8 Symmetric Matrices and Quadratic Forms

8.1 Symmetric Matrices

A matrix is **orthogonally diagonalizable** if it has an eigenbasis that is orthonormal.

A matrix is orthogonally diagonalizable if and only if it is symmetric.

$$\exists S \mid S^{-1}AS \text{ is diagonal } \iff A^T = A$$

If A is a symmetric matrix, and A has eigenvectors $\vec{v}_1$ and $\vec{v}_2$ with eigenvalues $\lambda_1 \neq \lambda_2$, then $\vec{v}_2$ is orthogonal to $\vec{v}_1$.

A symmetric $n \times n$ matrix A has n real eigenvalues if they are counted with their algebraic multiplicities.

To orthogonally diagonalize a symmetric matrix A, perform the following steps on A:

- Find the eigenvalues of A.
- For each eigenvalue, find a basis of its eigenspace.
- Use the Gram-Schmidt Process to find an orthonormal basis of each eigenspace.
- Concatenate the orthonormal bases into the matrix to make an orthonormal eigenbasis of A called S: the matrix S will be orthogonal and $S^{-1}AS$ will be diagonal.

8.2 Quadratic Forms

TODO – write me!

8.3 Singular Values

TODO – write me!