

Why Bayes

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Introduction

Bayesian approaches

- Typically contrasted with **frequentist** approaches
- Treat parameters as uncertain, data as fixed

Bayes' Rule

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$$p(\theta|x) = \frac{\overset{\text{likelihood}}{p(x|\theta)} \overset{\text{prior}}{p(\theta)}}{\underset{\text{evidence}}{p(x)}} = \frac{p(x|\theta)p(\theta)}{\int p(x|\theta)p(\theta) d\theta}$$

Posterior

The posterior distribution is proportional to the prior times the likelihood:

$$p(\theta|x) \propto p(x|\theta)p(\theta)$$

The posterior distribution *is a distribution* over θ .

Evidence

The evidence, or *marginal likelihood*, can be used for model comparison.

Motivations

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Avoiding overfitting

Maximum likelihood and overfitting

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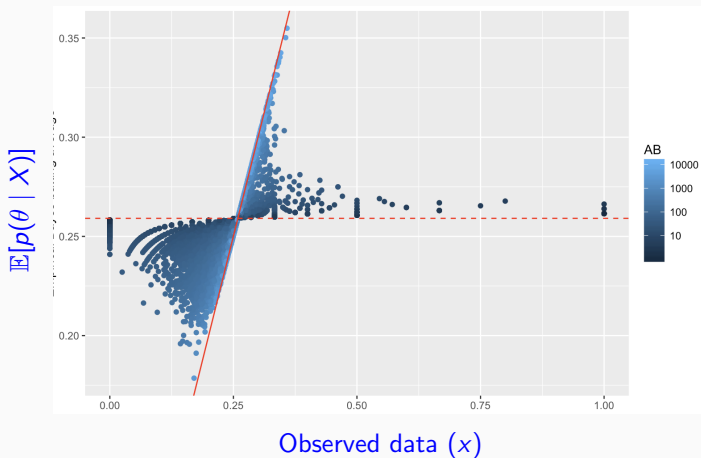
The approach can be seen as *over-committing* to a single, fixed parameter value.

Bayesian methods can correct this by treating parameters as random variables.

Bayesian estimation of batting averages

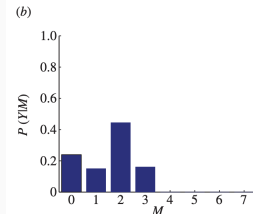
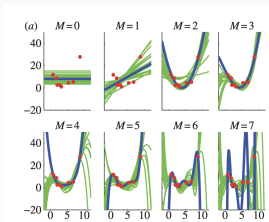
Let

- x be observed data (batting average after n at bats)
- θ be parameters (a player's 'true' batting average)



Bayesian Occam's Razor

Maximum Likelihood (ML) solutions tend to overfit. Bayesian marginalization reduces overfitting.

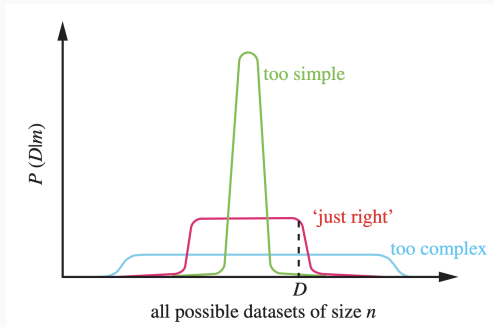


Models $y = f(x) + \epsilon$ of various complexity (polynomials of various order, M) were fit to 8 data points.

- Plotted are **ML polynomials** (least squares fits to the data under Gaussian noise) and **posterior samples from a Bayesian model** (which used a Gaussian prior for the coefficients, and an inverse gamma prior on the noise).
- The ML estimate can look very different from a typical sample from the posterior!

The evidence is plotted as a function of model order. Model orders $M=0$ to $M=3$ have considerably higher evidence than other model orders. We see that Bayesian marginalization has reduced overfitting. (The maximum likelihood model, the $M = 7$ model, fits the data perfectly, but overfits wildly, predicting the function will shoot up or down between neighboring data points.)

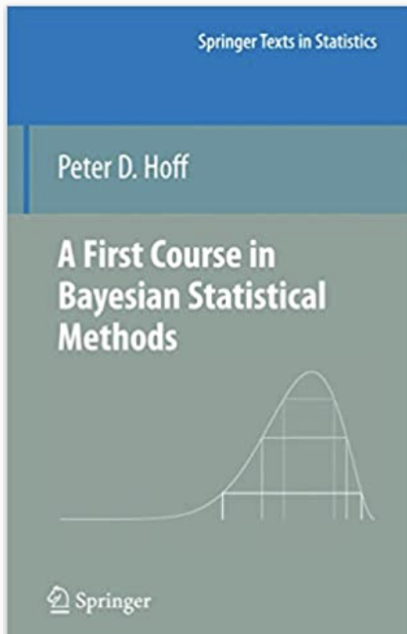
Bayesian Occam's Razor



Competing probabilistic models correspond to alternative distributions over the datasets. Here, we have illustrated three possible models that spread their probability mass in different ways over these possible datasets. A *complex* model (shown in blue) spreads its mass over many more possible datasets, whereas a *simple* model (shown in green) concentrates its mass on a smaller fraction of possible data. Because probabilities have to sum to one, the complex model spreads its mass at the cost of not being able to model simple datasets as well as a simple model—this normalization is what results in an automatic Occam razor. Given any particular dataset, here indicated by the dotted line, we can use the marginal likelihood to reject both overly simple models, and overly complex models.

Motivations

Estimating the probability of a rare event



Description of problem

- Want to estimate the prevalence of an infectious disease in a small town.
- The higher the prevalence, the more public health precautions will be recommended.
- A small random sample of 20 individuals are checked for infection.

Description of problem

- **Parameter** θ , the fraction of infected individuals in the city.
- **Parameter space**: $\Theta = [0, 1]$
- **Sample**: Y the number of infected individuals in the sample
- **Sample space**: $\mathcal{Y} = \{0, 1, \dots, 20\}$

Sampling model

If the value of θ were known, a reasonable sampling model for Y would be

$$Y \mid \theta \sim \text{Binomial}(20, \theta)$$

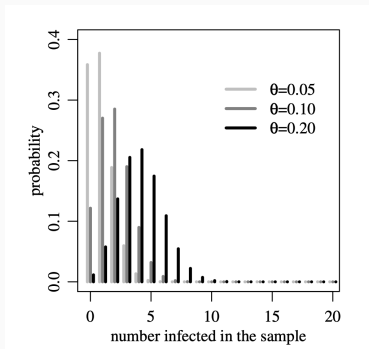


Figure 1: Binomial($20, \theta$) distributions for three values of θ .

Prior distribution

Other studies from various parts of the country indicate that the infection rate in comparable cities range from about 0.05 to 0.20, with an average prevalence of 0.10.

Prior distribution

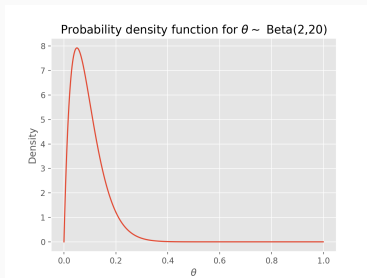
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We can encode this prior information using

$$\theta \sim \text{Beta}(2, 20)$$

From prior to posterior

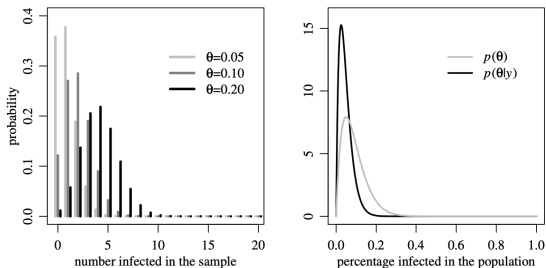


Fig. 1.1. Sampling model, prior and posterior distributions for the infection rate example. The plot on the left-hand side gives binomial(20, θ) distributions for three values of θ . The right-hand side gives prior (gray) and posterior (black) densities of θ .

Prior

$$\theta \sim \text{Beta}(2, 20)$$

$$\mathbb{E}[\theta] = 0.09$$

$$\text{mode}[\theta] = 0.05$$

$$P(\theta < 0.10) = 0.64$$

$$P(0.05 < \theta < 0.20) = 0.66$$

Posterior

$$\theta \mid \{Y = 0\} \sim \text{Beta}(4, 20)$$

$$\mathbb{E}[\theta \mid \{Y = 0\}] = 0.048$$

$$\text{mode}[\theta \mid \{Y = 0\}] = 0.025$$

$$P(\theta < 0.10 \mid \{Y = 0\}) = 0.93$$

Sensitivity analysis

Suppose we consider beliefs represented by $\text{Beta}(a, b)$ distributions for (a, b) other than $(2, 20)$.

If $\theta \sim \text{Beta}(a, b)$, then $\theta \mid Y = y \sim \text{Beta}(a + y, b + n - y)$.

The posterior expectation is

$$\begin{aligned}\mathbb{E}[\theta \mid Y = y] &= \frac{a + y}{a + b + n} \\ &= \frac{n}{a + b + n} \frac{y}{n} + \frac{a + b}{a + b + n} \frac{a}{a + b} \\ &= \frac{n}{w + n} \bar{y} + \frac{w}{w + n} \theta_0\end{aligned}$$

where $\theta_0 = a/(a + b)$ is the prior expectation of θ and $w = a + b$.

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where $\theta_0 = a/(a + b)$ is the prior expectation of θ and $w = a + b$.

So the posterior expectation is a compromise between the prior expectation θ_0 and sample mean \bar{y} . The weights on each depend on the sample size, n , and our prior confidence in this guess, w .

Sensitivity analysis

If someone provides us with a prior guess θ_0 and degree of confidence w , then we can approximate their prior beliefs about θ with

$$\text{Beta}\left(a = w\theta_0, \quad b = w(1 - \theta_0)\right)$$

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We can compute such a posterior distribution for a wide range of θ_0 and w values to perform a *sensitivity analysis*, an exploration of how posterior information is affected by differences in prior opinion.

Sensitivity analysis

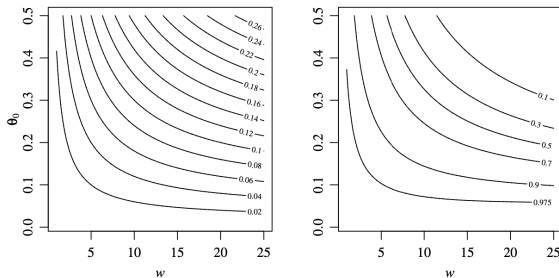


Fig. 1.2. Posterior quantities under different beta prior distributions. The left- and right-hand panels give contours of $E[\theta|Y=0]$ and $\Pr(\theta < 0.10|Y=0)$, respectively, for a range of prior expectations and levels of confidence.

The second plot may be of use if, e.g., city officials would like to recommend a vaccine to the general public unless they were reasonably sure the current infection rate was less than 0.10.

Sensitivity analysis

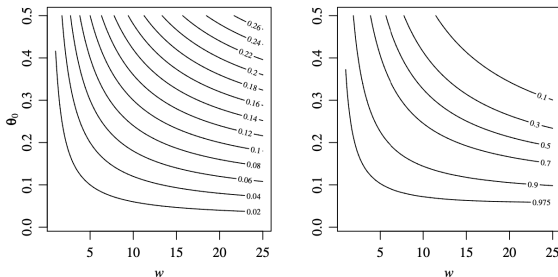


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A high degree of certainty (say 97.5%) is only achieved by people who already thought the infection rate was lower than the average of other cities.

Comparison to non-Bayesian methods

A 95% confidence interval for population proportion θ is the *Wald interval*, given by

$$\bar{y} \pm 1.96 \sqrt{\bar{y}(1 - \bar{y})/n}$$

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In fact, the 99.99% Wald interval also comes out to be zero.

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The “adjusted” Wald interval suggested by Agresti and Coull (1998) is given by

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While not motivated as such, the interval is clearly related to Bayesian inference: $\hat{\theta}$ is equivalent to the posterior mean for θ under a Beta(2, 2) prior, which represents weak prior information centered around $\theta = 1/2$.

Comparison to non-Bayesian methods

Compared to the post-hoc “adjustment” approach, the Bayesian formalism provides

- Reasonable conclusions which fall naturally out of the framework
- Flexibility to other choice of priors than $\text{Beta}(2, 2)$
- Sensitivity analysis to consider the sets of conclusions that would be reached by people with different priors.
- Simultaneous access to various functionals of the posterior – not just $\mathbb{E}[\theta \mid Y = y]$ but also $\mathbb{P}[\theta < 0.10 \mid Y = 0]$.

Extensions: Hierarchical models