Wednesday 08/27/2025: Proofs

CSCI 246: Discrete Structures

Textbook reference: Sec. 5, Scheinerman

Announcements

- **Forming groups** - If you end up in a group with one person, feel free to join another one (preferably one with 2 people).
- **Recruiting help during group exercises** - Please grab me or Paul (the TA) if you'd like help. Also feel free to reach out to another group. Finally, feel free to consult the textbook.
- **Weekly quiz this Friday** - It will cover definitions (Sec 3), theorems (Sec 4), and proofs (Sec 5). Be sure that you've done the readings and know how to do the group exercises.

Today's Agenda

- Theorems: additional thoughts, review group ex. ($\approx$ 15 mins)
- Proofs: practice quiz, mini-lecture ($\approx$ 10 mins)
- Proofs: Group exercises ($\approx$ 15 mins) and review ($\approx$ 10 mins)

Study Guide For Quiz 1

Readings (1 question)

Sec. 3 (Definitions)

- Be familiar with the definitions (understand what they say and how to apply them).

Sec. 4 (Theorems)

- Understand these truth tables: `if-then`, `iff`, `and`, `or`, `not`.
- Understand vacuous truths.

Sec 5 (Proofs)

- Know how to prove Props 5.2, 5.3, 5.5, and 5.6.

Group exercises (2 questions)

Know how to do all of the group exercises from these sections, except for the Bonus problem from Sec 4.

Outline for today's material

- Practice quiz
- Two proof templates
- Group exercises

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Practice Quiz

Practice Quiz Question

Prove that the the sum of two even integers is even.

Use the appropriate proof template from the textbook.

Scheinerman Definition 3.1 (Even)

An integer is called *even* provided it is divisible by two.

Scheinerman Definition 3.2 (Divisible)

Let a and b be integers. We say that a is *divisible* by b provided there is an integer c such that $bc = a$. We also say that b *divides* a, or b is a *factor* of a, or b is a *divisor* of a. The notation for this is $b|a$.

Solution Sketch

Proposition. The sum of two even integers is even.

Proof.

Annotation	**Main Text**
Convert Prop. to "if-then" form	We show that if x and y are even integers, then $x + y$ is even.
State "if"	Let x and y be even integers
Unravel defs.	Then by Defs. 3.1 and 3.2, there exist integers a, b such that $x = 2a$ and $y = 2b$.
*** The glue ***	What goes here?!?!
Unravel defs.	So there is an integer c such that $x + y = 2c$.
State "then"	Hence, $x + y$ is even.

Solution

Proposition. The sum of two even integers is even.

Proof.

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Convert Prop. to "if-then" form	We show that if x and y are even integers, then $x + y$ is even.
State "if"	Let x and y be even integers
Unravel defs.	Then by Defs. 3.1 and 3.2, there exist integers a, b such that $x = 2a$ and $y = 2b$.
*** The glue ***	Hence, $x + y = 2a + 2b = 2(a + b)$.
Unravel defs.	So there is an integer $c = a + b$ such that $x + y = 2c$.
State "then"	Hence, $x + y$ is even.

Outline for today’s material

- Practice quiz
- **Two proof templates**
- Group exercises

Proof Template 1: Direct proof of an if-then theorem

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Proof Template 1: Direct proof of an if-then theorem

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4. Unravel the definitions, working forward from the beginning of the proof and backward from the end of the proof.

Proof Template 1: Direct proof of an if-then theorem

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2. At the beginning of the proof, write down the *antecedent* (the A in `if A then B`) as your assumption.
3. At the end of the proof, write down the *consequent* (the B in `if A then B`) as your conclusion.
4. Unravel the definitions, working forward from the beginning of the proof and backward from the end of the proof.
5. Forge a link between the two halves of the argument.

Proof Template 2: Direct proof of an if-and-only-if theorem

To prove a statement of the form “A iff B”:

- ($\Longrightarrow$) Prove “If A, then B”.
- ($\Longleftarrow$) Prove “If B, then A”.

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- Two proof templates
- Group exercises

Random group assignments

Aaron Christensen: 10
Aidan Sinclair: 17
Bennett Dijkstra: 6
Brendan Kelly: 7
Buggy Garza: 10
Cedric Jefferson: 12
Conner Brost: 11
Connor Graville: 3
David Knauert: 14
David Oswald: 5
Elias Martin: 4
Ericson O'Guinn: 13
Erik Halverson: 1
Francis Bush: 4
Garrett Miller: 4
George Cutler: 3
Georgia Franks: 13
Gregor Schmidt: 10

Hakyla Riggs: 16
Izayah Abayomi: 8
Jacob Ketola: 8
Jacob Ruiz: 3
Jaden Hampton: 6
Jeremy Ness: 14
Jonah Day: 1
Karter Gress: 7
Kyle Hoerner: 2
Landry Clarke: 9
Leon BirdHat: 5
Lillian Ziegler: 15
Matthew Rau: 6
Matyas Kari: 11
Micah Miller: 17
Michael Pitman: 1

Nathan Campbell: 13
Nathan Hooley: 8
Nicholas Rugani: 15
Noah Andersson: 16
Olivia Greuter: 12
Peter Van Vleet: 17
Pierce Dotson: 14
Quinn Carlson: 9
Ridley Christoferson: 7
Riley Smith: 9
Sierra Holleman: 5
Tanner Gramps: 16
Timothy True: 12
Titus Sykes: 2
Trey Randall: 18
William Grant: 11
William Sheldon: 15
Zachary Reller: 2

Group exercises

1. Prove that the square of an odd integer is odd.
2. Prove that the difference between consecutive perfect squares is odd.
3. Let x be an integer. Prove that $0|x$ if and only if $x = 0$.
4. Prove that an integer is odd if and only if it is the sum of two consecutive integers.

Group exercise #1: Solution

Proposition. The square of an odd integer is odd.

Proof.

Annotation	Main Text
Convert Prop. to "if-then" form	We show that if x is an odd integer, then x^2 is odd.
State "if"	Let x be an odd integer.
Unravel defs.	Then by definition of *odd*, there is an integer a such that $x = 2a + 1$.
*** The glue ***	So $x^2 = (2a + 1)(2a + 1) = 4a^2 + 4a + 1 = 2(2a^2 + 2a) + 1$.
Unravel defs.	So there is an integer b (where $b = 2a^2 + 2a$) such that $x^2 = 2b + 1$.
State "then"	Hence, x^2 is odd.

Group exercise #1: Solution

Proposition. The square of an odd integer is odd.

Proof.

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Convert Prop. to "if-then" form	We show that if x is an odd integer, then x^2 is odd.
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Unravel defs.	So there is an integer b (where $b = 2a^2 + 2a$) such that $x^2 = 2b + 1$.
State "then"	Hence, x^2 is odd.

Remark. You do not need to provide the annotations or colors in your own proofs. I am using them here in the solution to highlight the formulaic structure of an if-then proof.

Group exercise #2: Solution

Proposition. The difference between consecutive perfect squares is odd.

Proof.

Annotation	Main Text
Convert Prop. to "if-then" form	We show that if x and y are consecutive perfect squares, then $x - y$ is odd.
State "if"	Let x and y be consecutive perfect squares
Unravel defs.	Then $x = (z+1)^2$ and $y = z^2$ where z is an integer.
*** The glue ***	So $x - y = (z+1)^2 - z^2 = (z^2+2z+1) - z^2 = 2z+1$.
Unravel defs.	So there is an integer b (where $b = z$) such that $x - y = 2b + 1$.
State "then"	Hence, $x - y$ is odd.

Group exercise #2: Solution

Proposition. The difference between consecutive perfect squares is odd.

Proof.

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Convert Prop. to "if-then" form	We show that if x and y are consecutive perfect squares, then $x - y$ is odd.
State "if"	Let x and y be consecutive perfect squares
Unravel defs.	Then $x = (z+1)^2$ and $y = z^2$ where z is an integer.
*** The glue ***	So $x - y = (z+1)^2 - z^2 = (z^2+2z+1) - z^2 = 2z+1$.
Unravel defs.	So there is an integer b (where $b = z$) such that $x - y = 2b + 1$.
State "then"	Hence, $x - y$ is odd.

Remark. You do not need to provide the annotations or colors in your own proofs. I am using them here in the solution to highlight the formulaic structure of an if-then proof.

Group exercise #3: Solution

Proposition. Let x be an integer. Prove that $0|x$ if and only if $x = 0$.

Proof. We decompose the *if-and-only-if* statement into two *if-then* statements.

(a) We show that if $0|x$, then $x = 0$.

Let x be an integer such that $0|x$.

Then by definition of *divisible*, there is an integer a such that $0 \cdot a = x$.

But $0 \cdot a = 0$.

Hence $x = 0$.

(b) We show that if $x = 0$, then $0|x$.

Let $x = 0$.

Let a be any integer. (For example, take $a = 7$.) Then $a \cdot 0 = 0$.

Hence, there is an integer a such that $0 \cdot a = x$.

Hence, $0|x$.

Group exercise #3: Solution

Proposition. Let x be an integer. Prove that $0|x$ if and only if $x = 0$.

Proof. We decompose the *if-and-only-if* statement into two *if-then* statements.

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Let $x = 0$.

Let a be any integer. (For example, take $a = 7$.) Then $a \cdot 0 = 0$.

Hence, there is an integer a such that $0 \cdot a = x$.

Hence, $0|x$.

Remark. An *if-and-only-if* proof consists of two *if-then* proofs. Each uses the same *if-then* template (and same color-scheme) as in Group Exercises #1 and #2. Note that some green rows were skipped (as there was no definition to unravel for $x = 0$).

Group exercise #4: Solution

Proposition. An integer is odd if and only if it is the sum of two consecutive integers.

Proof. We decompose the *if-and-only-if* statement into two *if-then* statements.

(a) We show that if x is the sum of two consecutive integers, then x is an odd integer.

Let x be the sum of two consecutive integers.

So there is an integer a such that $x = a + (a+1)$.

So $x = 2a + 1$

Hence, there is an integer a such that $x = 2a + 1$.

Hence, x is an odd integer.

(b) We show that if x is an odd integer, then x is the sum of two consecutive integers.

Let x be an odd integer.

Then by definition of *odd*, there is an integer a such that $x = 2a + 1$.

So we have $x = 2a + 1 = a + (a+1)$.

Hence x is the sum of two consecutive integers.

Group exercise #4: Solution

Proposition. An integer is odd if and only if it is the sum of two consecutive integers.

Proof. We decompose the *if-and-only-if* statement into two *if-then* statements.

(a) We show that if x is the sum of two consecutive integers, then x is an odd integer.

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Hence, there is an integer a such that $x = 2a + 1$.

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Hence x is the sum of two consecutive integers.

Remark. An *if-and-only-if* proof consists of two *if-then* proofs. Each uses the same *if-then* template (and same color-scheme) as in Group Exercises #1 and #2.