Friday 08/25/2025: Theorems

CSCI 246: Discrete Structures

Textbook reference: Sec. 4, Scheinerman

Logistical Matters

- **Announcements** - Do you get the Canvas announcements? Preference for Canvas vs. Email?
- **Participation grade clarification** - 10% of your final grade is just showing up to class and doing group exercises. I check off who's here.
- **On group exercises** – We may not finish the group exercises in class, but can at least get started on them. You can try the rest as homework. Make sure you understand the solutions before the Friday quizzes.
- **Special circumstances for Friday 08/29** – Office hours from 9am-12pm. Guest instructor: Kunal Das, Ph.D.

Today's Agenda

- Logistics/practice quiz ($\approx$ 10 mins)
- Mini lecture ($\approx$ 15 mins)
- Group exercises ($\approx$ 15 mins) and review ($\approx$ 10 mins)

Practice Quiz

Replace each ? with a checkmark ✓ if the combination of truth values for propositions A and B is *possible* under the given logical connective. Replace it with a ✗ if the combination is *impossible*.

Propositions		**Logical Connectives**		
A	B	if A then B	if B then A	A if and only if B
T	T	?	?	?
T	F	?	?	?
F	T	?	?	?
F	F	?	?	?

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The goal of today's slides is to clarify the solution.

Outline for Mini-Lecture:

- If-then statements
- If-and-only-if statements
- Quiz solution
- The big picture: Propositional logic

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If-then: Truth Table

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Summary. If A is True, B must be True.

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Remark. If A is False, then no restrictions are placed on B.

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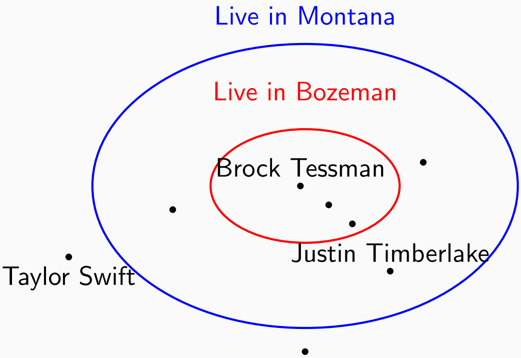
Remark. If A is False, then no restrictions are placed on B.

Example. Let $A =$ `This person lives in Bozeman`, $B =$ `This person lives in Montana`. Here, $A \implies B$. (Check this against the truth table.)

If-then: Set Theoretic Perspective

Let A = `This person lives in Bozeman` and B = `This person lives in Montana`. Then $A \implies B$!



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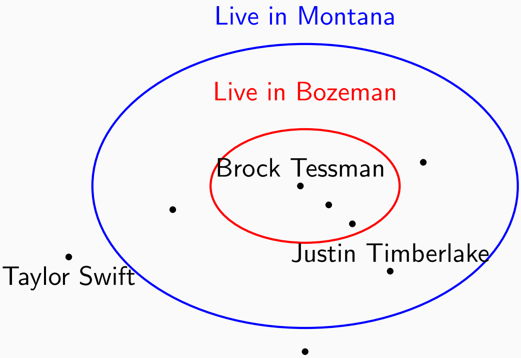

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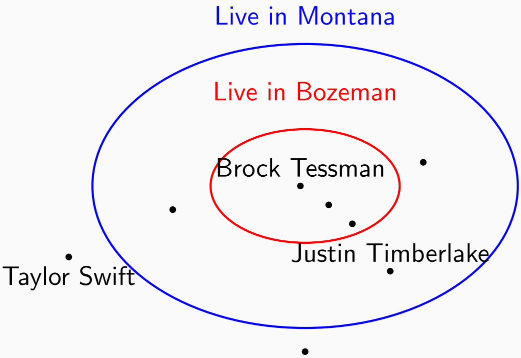

Set theoretic perspective. You can think about $A \implies B$ as "the set of things satisfying property A is contained within the set of things satisfying property B."

Poll: True or False? Every elephant flying with polka-dotted balloons above Bozeman is smoking a cigar.

Vacuous Truths

Solution to poll

Let A = `this is an elephant flying with polka-dotted balloons above Bozeman` and B = `this is smoking a cigar`. Then A is always False. So $A \implies B$ is TRUE!!! This is called a **vacuous truth.** Per Scheinerman, these statements are true "because they have no exceptions."

Vacuous Truths

Solution to poll

Let A = `this is an elephant flying with polka-dotted balloons above Bozeman` and B = `this is smoking a cigar`. Then A is always False. So $A \implies B$ is TRUE!!! This is called a **vacuous truth.** Per Scheinerman, these statements are true "because they have no exceptions."

Reference material

Given two propositions A and B, the statement "If A, then B" ($A \implies B$) holds whenever following "truth table" applies:

A	B	$\overbrace{A \implies B}^{\text{If A, then B}}$
T	T	T
T	F	F
F	T	T
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If-then: Other jargon

Some other terminology which means the same thing:

- A implies B
- A is sufficient for B
- B is necessary for A (A "only if" B)

If-then: Valid argument and logical fallacy

Modus Ponens (A valid argument) ✓

Structure	**Example**
If A, then B.	If it is raining, the ground is wet.
A.	It is raining.
Therefore B.	Therefore, the ground is wet.

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Structure	**Example**
If A, then B.	If it is raining, the ground is wet.
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Affirming the Consequent (A logical fallacy) ✗

Structure	**Example**
If A, then B.	If it is raining, the ground is wet.
B.	The ground is wet.
Therefore A.	Therefore, it is raining.

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- If-and-only-if statements
- Quiz solution
- The big picture: Propositional logic

If and only if: Definition and shorthand

If and only if statements: Definition

$$\underbrace{A \iff B}_{\text{A if and only if B}} \quad \text{means} \quad \underbrace{(A \implies B)}_{\text{A only if B}} \texttt{ and } \underbrace{(A \impliedby B)}_{\text{A if B}}$$

If and only if: Definition and shorthand

If and only if statements: Definition

$$\underbrace{A \iff B}_{\text{A if and only if B}} \quad \text{means} \quad \underbrace{(A \implies B)}_{\text{A only if B}} \ \texttt{and} \ \underbrace{(A \impliedby B)}_{\text{A if B}}$$

Shorthand. For convenience, we can abbreviate `A if and only if B` as `A iff B`.

If and only if: Truth Table

The truth table for the "`and`" operator (also written $\wedge$) is given by

Original propositions		New proposition
X	Y	$X \wedge Y$
T	T	T
T	F	F
F	T	F
F	F	F

If and only if: Truth Table

The truth table for the "and" operator (also written $\wedge$) is given by

Original propositions		**New proposition**
X	Y	$X \wedge Y$
T	T	T
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F	T	F
F	F	F

Now we apply the $\wedge$ operator to the results of the $\implies$ and $\impliedby$ operators.

Orig. props.		**New props.**		
A	B	$\overbrace{A \implies B}^{X}$	$\overbrace{B \implies A}^{Y}$	$\overbrace{(A \implies B) \wedge (B \implies A)}^{X \wedge Y}$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
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Given two propositions A and B, the statement "A if and only if B" (that is, the property $A \iff B$) holds whenever following "truth table" applies:

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Summary. A and B are <u>both</u> True or <u>both</u> False.

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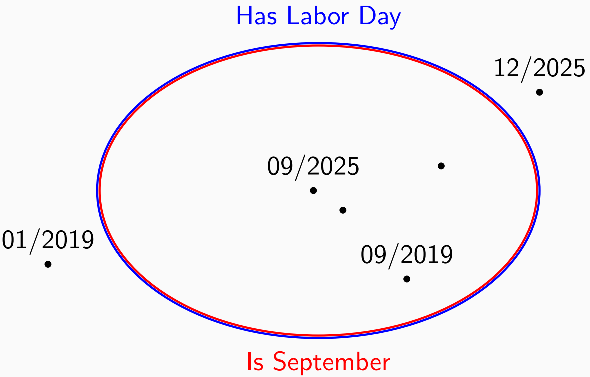
Summary. A and B are both True or both False.

Example. Let $A = \texttt{This month contains Labor Day}$, $B = \texttt{This month is September}$. Here, $A \iff B$. (Check this against the truth table.)

If and only if: Set theoretic perspective

Let A = `This month contains Labor Day` and B = `This month is September`. Then $A \iff B$!



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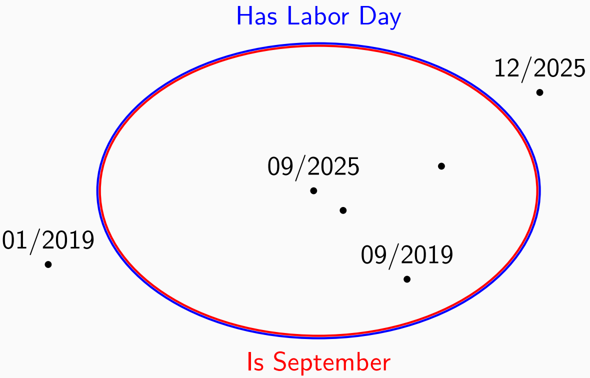

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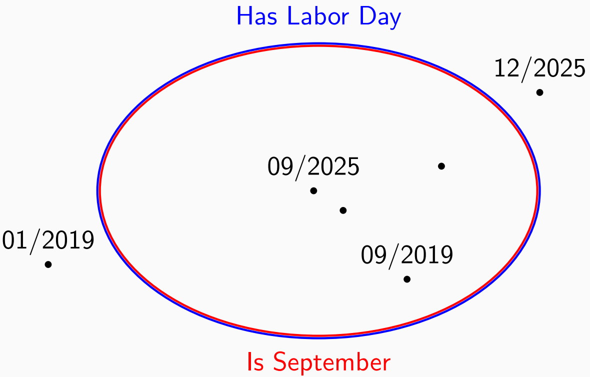

Set theoretic perspective. You can think about $A \iff B$ as "the set of things satisfying property A is identical to the set of things satisfying property B."

If and only if: Big picture

$A \iff B$ means that A and B are just two different descriptions of the same thing!

If and only if: Why should computer scientists care?

Consider the following theorem.

Theorem (Graph Theory)

A connected undirected graph with n vertices has a cycle if and only if it has at least n edges.

Thanks to the "iff", you know that edge count alone is enough to decide whether a cycle exists. Instead of running a full DFS or Union-Find cycle detection algorithm, you can just check counts:

```
def has_cycle(num_vertices, edges):
    return len(edges) >= num_vertices
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Take home. Why would computer scientists care about theorems of the form "A iff B"? It can sometimes happen that what we really want is A, but that's very hard to code up. Thanks to the theorem, we can just code up B instead, because it turns out (perhaps surprisingly!) that it's the exact same thing.

If and only if: How is it related to $==$?

Student question. How does iff relate to $==$ (the comparison operator in many programming languages)?

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Case Study

Consider propositions A, B as binary-valued functions of x, which are elements of some set $\mathcal{X}$. For instance, perhaps

$$\begin{aligned} \mathcal{X} &= \{\text{months since Labor Day was created (9/1882) until now}\} \\ A(x) &= \text{month } x \text{ has Labor Day in it} \\ B(x) &= \text{month } x \text{ is September} \end{aligned}$$

Then `A iff B` means

$$\text{for all } x \in \mathcal{X}, \qquad A(x) \quad \underbrace{=}_{\text{written } == \text{ in code}} \quad B(x)$$

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Then `A iff B` means

$$\text{for all } x \in \mathcal{X}, \qquad A(x) \underbrace{=}_{\text{written } == \text{ in code}} B(x)$$

Summary. iff means that $==$ returns True for <u>all</u> cases of two propositions.

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- If-then statements
- If-and-only-if statements
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Solution to practice quiz

A	B	if A then B	if B then A	A if and only if B
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T	F	F	T	F
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- The column headings show 3 new propositions, formed from the original propositions by **logical connectives**.

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- Besides `if-then` and `if-and-only-if`, there are other logical connectives (`and`, `or`, `xor`, etc.), some of which were discussed in the text.

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- Besides `if-then` and `if-and-only-if`, there are other logical connectives (`and, or, xor`, etc.), some of which were discussed in the text.
- The study of how to combine and change propositions under logical connectives to form more complex propositions is called **propositional logic**.

Propositional logic

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Terminology. The first two columns combined with one remaining column gives the **truth table** for that logical connective.

Group exercises

Random group assignments

Aaron Christensen: 1
Aidan Sinclair: 6
Brendan Kelly: 9
Buggy Garza: 6
Cedric Jefferson: 3
Conner Brost: 7
Connor Graville: 4
David Knauert: 2
David Oswald: 16
Elias Martin: 9
Ericson O'Guinn: 3
Erik Halverson: 15
Francis Bush: 6
Garrett Miller: 14
George Cutler: 7
Georgia Franks: 10
Gregor Schmidt: 16

Hakyla Riggs: 8
Izayah Abayomi: 1
Jacob Ketola: 11
Jacob Ruiz: 13
Jaden Hampton: 5
Jeremy Ness: 7
Jonah Day: 8
Karter Gress: 16
Kyle Hoerner: 17
Landry Clarke: 8
Leon BirdHat: 4
Lillian Ziegler: 17
Matthew Rau: 12
Micah Miller: 15
Michael Pitman: 1
Nathan Campbell: 2

Nathan Hooley: 10
Nicholas Rugani: 2
Noah Andersson: 13
Olivia Greuter: 4
Peter Van Vleet: 14
Pierce Dotson: 11
Quinn Carlson: 10
Ridley Christoferson: 12
Riley Smith: 5
Sierra Holleman: 11
Tanner Gramps: 9
Timothy True: 15
Titus Sykes: 5
Trey Randall: 12
William Grant: 13
William Sheldon: 14
Zachary Reller: 3

Note: If your name is not here, please come see me so I can get you on the roster.

Group exercises

1. It is a common mistake to confuse the following two statements (i) If A, then B and (ii) If B, then A. Find two conditions A and B such that statement (i) is true but statement (ii) is false. Then find two conditions A and B such that both statements are true.
2. Consider these two statements: (i) If A, then B, (ii) If (not B), then (not A). Under what circumstances are these statements true? When are they false? Explain whether these statements are identical or not. [Note: (ii) is called the **contrapositive** of (i).]
3. (Challenge problem, from philosopher Norman Swartz.) Is the following statement true or false, and why? *A's-being-a-necessary-condition-for-B is both a necessary and sufficient condition for B's-being-a-sufficient-condition-for-A.*

Question 1: Solution

$A \implies B$ but $B \not\Rightarrow A$:
A = I lived in Los Angeles
B = I lived in California.

$A \iff B$:
A = Valentine's Day is this month
B = This month is February.

Question 2: Solution

Recall from Slide 4 that the truth table for the $\implies$ operator is given by

X	Y	$\overbrace{X \implies Y}^{\text{If X, then Y}}$
T	T	T
T	F	F
F	T	T
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Now we apply the $\implies$ operator to the results of the "`not`" operator (also written $\neg$).

Orig. props.		**New props.**		
A	B	$\overbrace{\neg A}^{Y}$	$\overbrace{\neg B}^{X}$	$\overbrace{\neg B \implies \neg A}^{X \implies Y}$
T	T	F	F	T
T	F	F	T	F
F	T	T	F	T
F	F	T	T	T

Note that $\neg B \implies \neg A$ gives the same results as $A \implies B$ as on Slide 4.

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Note that $\neg B \implies \neg A$ gives the same results as $A \implies B$ as on Slide 4.

Remark: We've shown that a proposition is logically equivalent to its contrapositive. So what? Sometimes it's easier to verify the contrapositive version.

Question 3: Solution

The simplest way to see this is as follows:

- A's-being-a-necessary-condition-for-B can be expressed as $B \implies A$.
- B's-being-a-sufficient-condition-for-A can be expressed as $B \implies A$.
- In other words, both propositions are the same: $B \implies A$. And a proposition is always necessary and sufficient for itself.