02/19/2026: Brownian Motion

CSCI 546: Diffusion Models

Textbook reference: Chang Sec 5.1-5.6, GS Sec 8.1, 8.5

Announcement (Sign-in Sheet)

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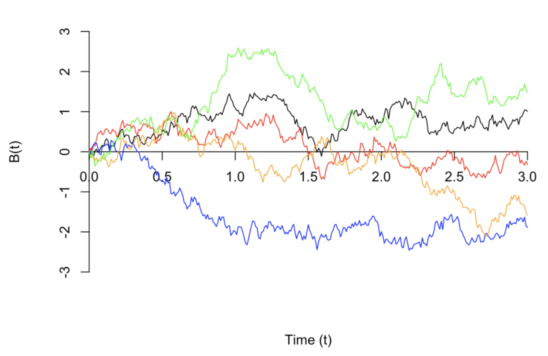
Personal Note

Review Problem Set #10

Concepts for Problem Set #11

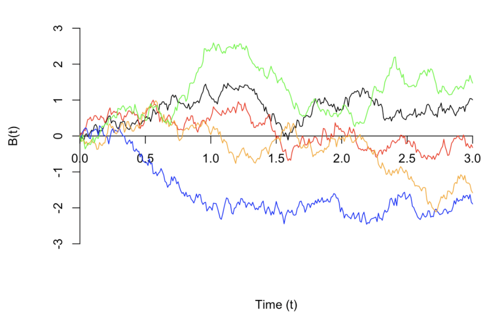
Stochastic Process



Definition. A **stochastic process** $\{X(t) : t \in \mathbb{R}\}$ is a function $X : \mathbb{R} \times \Omega \to \mathbb{R}$. Thus, we can view $X = X(t, \omega)$ in two ways:

- *As a collection of random variables.* For each fixed t, $X(t, \cdot)$ is a random variable.
- *As a random function.* For each fixed ω, $X(\cdot, \omega)$ is a real-valued function (or "path").

Brownian Motion



Definition. A **standard Brownian motion** $\{B(t) : t \geq 0\}$ is a stochastic process having

(i) continuous paths,

(ii) stationary, independent increments, and

(iii) $B(t) \sim N(0, t)$ for all $t \geq 0$.

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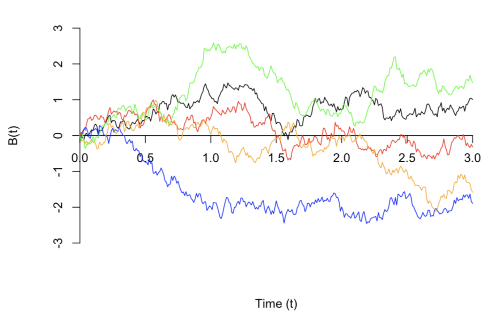

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Poll. What does "stationary, independent increments" mean?

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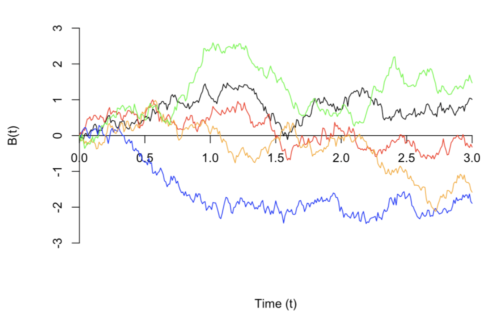

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Remark. A general (μ, σ^2) Brownian motion has mean and variance increasing at rate μ and σ^2 per unit time:

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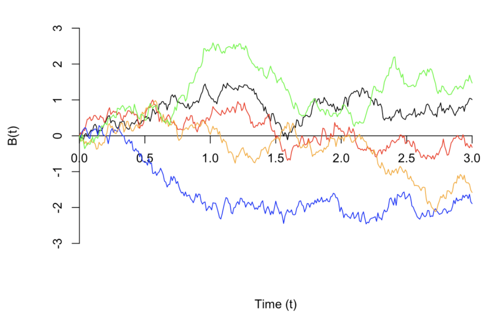

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Remark. A general (μ, σ^2) Brownian motion has mean and variance increasing at rate μ and σ^2 per unit time: $X(t) = X(0) + \mu t + \sigma B(t)$.

Is Brownian Motion a Gaussian Process?

Definition. A **Gaussian process** $\{X(t) : t \geq 0\}$ is a stochastic process such that for all numbers $n = 1, 2, \ldots$ and all times $t_1, t_2, \ldots t_n$, the random vector $\big(X(t_1), X(t_2), \ldots, X(t_n)\big)$ has a joint normal distribution.

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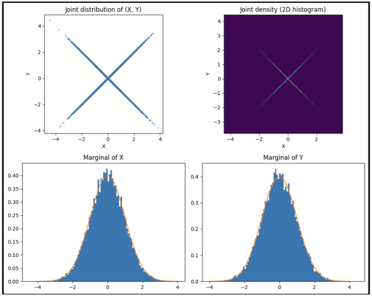
Warning. **Gaussian marginals do not imply a jointly Gaussian distribution.** For instance, let

- $Z \sim \mathcal{N}(0, 1)$,
- ε be independent of Z, with $P(\varepsilon = 1) = P(\varepsilon = -1) = \frac{1}{2}$,
- $X := Z, \qquad Y := \varepsilon Z.$



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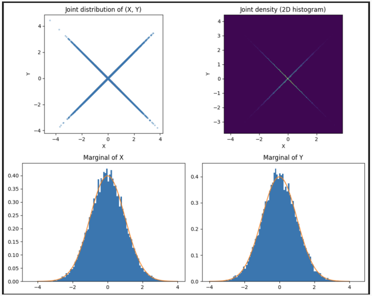

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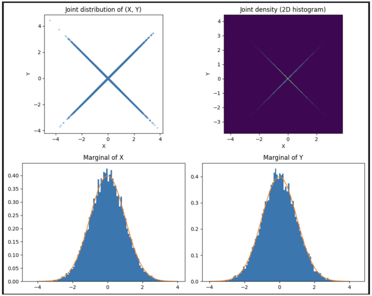

Both marginals are Gaussian:

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But the joint is not Gaussian:

- (X, Y) lives on the two lines $y = x$ and $y = -x$.

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- Since the Δ_i's are also independent, the vector of increments

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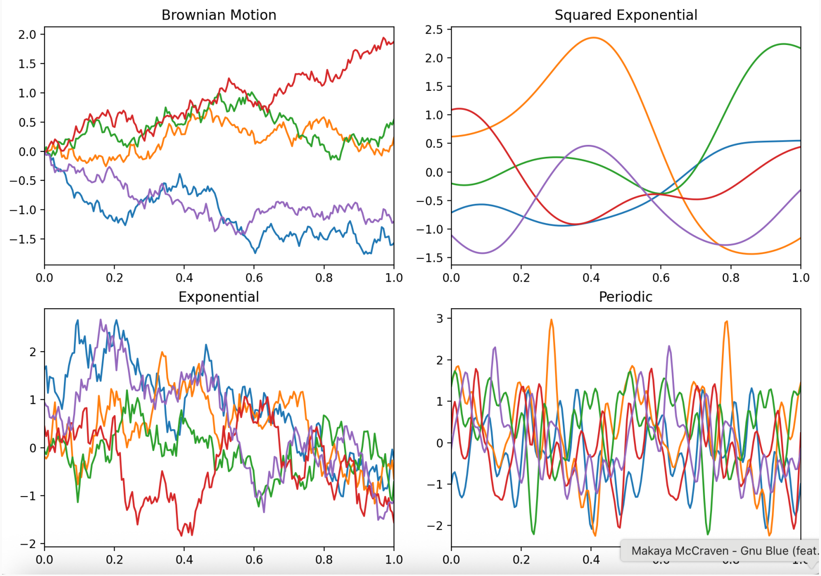
- We can write

$$(X(t_1), X(t_2), \ldots, X(t_n)) = A\Delta$$

  for some matrix A, i.e. as a linear transformation of Δ. Linear transformations of multivariate Gaussians are multivariate Gaussians.

Gaussian Processes



Different covariance functions give different types of GPs.

Brownian Motion as a Gaussian Process

Fact (from Chang). A Gaussian process having continuous paths, mean 0, and covariance function $r(s, t) = s \wedge t$ is a standard Brownian motion.

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Notation. $s \wedge t = \min(s, t)$.

Random Groups

Aubrey Williams: 6
Austin Barton : 2
Blake Sigmundstad: 4
Diego Moylan: 5
Dillon Shaffer: 3
Ismoiljon Muzaffarov: 3
Jacob Tanner: 1

Josh Stoneback: 1
Joshua Bowen: 1
Joshua Culwell: 5
Laura Banaszewski: 3
Lina Hammel: 4
Logan Racz: 6

Matt Hall: 2
Micah Miller: 4
Mike Kadoshnikov: 7
Owen Cool: 2
Racquel Bowen: 8
Samuel Mocabee: 8
Tatiana Kirillova: 7

Group exercises - Problem Set 11

1. (Chang Exercise 5.5) **Brownian Motion keeps "restarting", and is Markov.** Suppose that W is standard Brownian motion, and let $c > 0$. Define $X(t) = W(c+t) - W(c)$. Then $\{X(t) : t \geq 0\}$ is a standard Brownian motion that is independent of $\{W(t) : 0 \leq t \leq c\}$.
2. (Chang Sec 5.1) **Covariance of Brownian Motion.** Prove that for Brownian motion, $\mathrm{Cov}(W_s, W_t) = s \wedge t$. (Recall the notation that $s \wedge t := \min(s,t)$).
3. (GS 8.6.5) Let W be standard Brownian motion. Which of the following are also standard Brownian motions?

$$(a) -W(t), \qquad (b)\, \sqrt{t}W(1), \qquad (c)\, tW(1/t).$$