

# **01/20/2026: Random Variables and their Distributions (Part 1)**

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CSCI 546: Diffusion Models

Textbook reference: Sec 2.1-2.3

## Solutions to Group Exercises

The solutions manual for all exercises in the textbook for the math module have been posted to Canvas.

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## Announcement

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# **Review Problem Set #1**

# Monty Hall Problem



There are three doors on the set for a game show. Behind one door is a car and behind the other two doors are goats.

You get to pick a door to open. The host of the show then opens one of the other doors and reveals a goat. What should you do if you want to maximize your chance of winning the car: stay with your original door or switch – or would the likelihood of winning be the same either way?

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Try it!

<https://www.mathwarehouse.com/monty-hall-simulation-online/>

# **Mini-lecture**

# Distribution Functions

**(5) Example.** A random variable which is neither continuous nor discrete. A coin is tossed, and a head turns up with probability  $p (= 1 - q)$ . If a head turns up then a rod is flung on the ground and the angle measured as in Example (4). Then  $\Omega = \{T\} \cup \{(H, x) : 0 \leq x < 2\pi\}$ , in the obvious notation. Let  $X : \Omega \rightarrow \mathbb{R}$  be given by

$$X(T) = -1, \quad X((H, x)) = x.$$

The random variable  $X$  takes values in  $\{-1\} \cup [0, 2\pi]$  (see Figure 2.3 for a sketch of its distribution function). We say that  $X$  is continuous with the exception of a ‘point mass (or atom) at  $-1$ ’.



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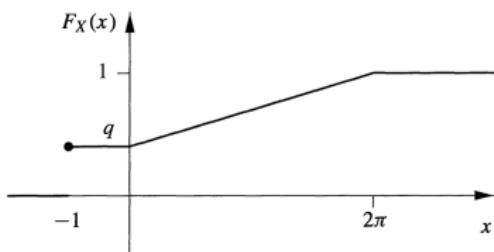


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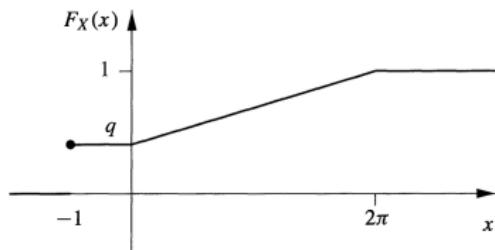


Figure 2.3. The distribution function  $F_X$  of the random variable  $X$  in Example (5).

(6) **Lemma.** A distribution function  $F$  has the following properties:

- $\lim_{x \rightarrow -\infty} F(x) = 0, \quad \lim_{x \rightarrow \infty} F(x) = 1,$
- if  $x < y$  then  $F(x) \leq F(y),$
- $F$  is right-continuous, that is,  $F(x + h) \rightarrow F(x)$  as  $h \downarrow 0$ .

# Discrete and continuous RVs

**(1) Definition.** The random variable  $X$  is called **discrete** if it takes values in some countable subset  $\{x_1, x_2, \dots\}$ , only, of  $\mathbb{R}$ . The discrete random variable  $X$  has **(probability) mass function**  $f : \mathbb{R} \rightarrow [0, 1]$  given by  $f(x) = \mathbb{P}(X = x)$ .

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**(2) Definition.** The random variable  $X$  is called **continuous** if its distribution function can be expressed as

$$F(x) = \int_{-\infty}^x f(u) du \quad x \in \mathbb{R},$$

for some integrable function  $f : \mathbb{R} \rightarrow [0, \infty)$  called the **(probability) density function** of  $X$ .

## Group exercises - Problem Set 2

1. (2.1.6) **Uniform Distribution.** A random variable that is equally likely to take any value in a finite set  $S$  is said to have the *uniform distribution* on  $S$ . If  $U$  is such a random variable and  $\emptyset \neq R \subseteq S$ , show that the distribution of  $U$  conditional on  $\{U \in R\}$  is uniform in  $R$ .
2. (2.2.3) Let  $\{X_r : r \geq 1\}$  be observations which are independent and identically distributed with unknown distribution function  $F$ . Describe and justify a method for estimating  $F(x)$ .
3. (2.3.4) Show that, if  $f$  and  $g$  are density functions, and  $0 \leq \lambda \leq 1$ , then  $\lambda f + (1 - \lambda)g$  is a density. Is the product  $fg$  a density function?