01/20/2026: Random Variables and their Distributions (Part 1)

CSCI 546: Diffusion Models

Textbook reference: Sec 2.1-2.3

Solutions to Group Exercises

The solutions manual for all exercises in the textbook for the math module have been posted to Canvas.

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Announcement

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Review Problem Set #1

Monty Hall Problem

There are three doors on the set for a game show. Behind one door is a car and behind the other two doors are goats.

You get to pick a door to open. The host of the show then opens one of the other doors and reveals a goat. What should you do if you want to maximize your chance of winning the car: stay with your original door or switch – or would the likelihood of winning be the same either way?

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Try it! `https://www.mathwarehouse.com/monty-hall-simulation-online/`

Mini-lecture

(5) Example. A random variable which is neither continuous nor discrete. A coin is tossed, and a head turns up with probability $p(= 1-q)$. If a head turns up then a rod is flung on the ground and the angle measured as in Example (4). Then $\Omega = \{T\} \cup \{(H, x) : 0 \le x < 2\pi\}$, in the obvious notation. Let $X : \Omega \to \mathbb{R}$ be given by

$$X(T) = -1, \qquad X((H, x)) = x.$$

The random variable X takes values in $\{-1\} \cup [0, 2\pi)$ (see Figure 2.3 for a sketch of its distribution function). We say that X is continuous with the exception of a 'point mass (or *atom*) at -1'. ●

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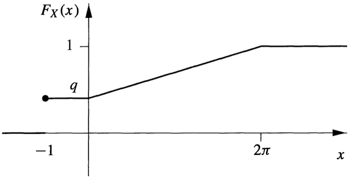

Figure 2.3. The distribution function F_X of the random variable X in Example (5).

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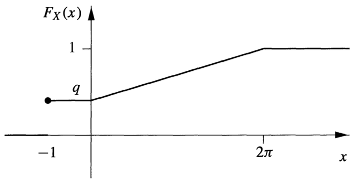

Figure 2.3. The distribution function F_X of the random variable X in Example (5).

(6) Lemma. *A distribution function F has the following properties:*

(a) $\lim_{x\to-\infty} F(x) = 0, \quad \lim_{x\to\infty} F(x) = 1,$

(b) *if $x < y$ then $F(x) \le F(y)$,*

(c) *F is right-continuous, that is, $F(x + h) \to F(x)$ as $h \downarrow 0$.*

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Discrete and continuous RVs

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(2) Definition. The random variable X is called **continuous** if its distribution function can be expressed as

$$F(x) = \int_{-\infty}^{x} f(u)\, du \qquad x \in \mathbb{R},$$

for some integrable function $f : \mathbb{R} \to [0, \infty)$ called the (**probability**) **density function** of X.

Terminology Note

Given a random variable X, I use the conventional terminology

1. **Cumulative distribution function** to refer to $F_X \triangleq P(X \leq x)$
2. **Probability mass function** to refer to $f_X \triangleq P(X = x)$ when X is discrete.
3. **Probability density function** to refer to f_X such that $P(X \leq x) = \int_{-\infty}^{x} f_X(t)\, dt$ when X is continuous.

And I use the term **probability distribution** to refer broadly to any of these, since each is sufficient to fully characterize the behavior of the random variable.

Group exercises - Problem Set 2

1. (2.1.6) **Uniform Distribution.** A random variable that is equally likely to take any value in a finite set S is said to to have the *uniform distribution* on S. If U is such a random variable and $\emptyset \neq R \subseteq S$, show that the distribution of U conditional on $\{U \in R\}$ is uniform in R.
2. (2.2.3) Let $\{X_r : r \geq 1\}$ be observations which are independent and identically distributed with unknown distribution function F. Describe and justify a method for estimating $F(x)$.
3. (2.3.4) Show that, if f and g are density functions, and $0 \leq \lambda \leq 1$, then $\lambda f + (1 - \lambda)g$ is a density. Is the product fg a density function?