02/17/2026: Markov Chains (Part 3)

CSCI 546: Diffusion Models

Textbook reference: Sec 6.5, 6.6, 6.9

Announcement (Sign-in Sheet)

Please sign the sign-in sheet.

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Intro Tutorial to Tempest

By Jess Kunke, Asst. Professor of Statistics

Wednesday, February 18th 4:10-5:00 PM

Wilson 1-154

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A look at the syllabus

- I've added some additional readings to the diffusion section which give some nice motivation and intuition.
- I'm considering splitting the final topic (stochastic calculus) up across 2 days. Thoughts?

Review Problem Set #9

Concepts for Problem Set #10

Outline for today's material

- Reversibility
- Chains with finitely many states
- Continuous-time Markov chains

Stationarity as “global balance”

Stationarity

Recall that $\boldsymbol{\pi}$ is a stationary distribution for a Markov chain if

$$\boldsymbol{\pi} = \boldsymbol{\pi}\boldsymbol{P},$$

where $\boldsymbol{\pi}$ is a row vector and $\boldsymbol{P}$ is the transition probability matrix of the Markov chain.

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The **most common interpretation** of stationarity: $\boldsymbol{\pi}$ gives the long-run frequencies of inhabiting each state. However, we can also give another interpretation.

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$$\begin{aligned} \boldsymbol{\pi} &= \boldsymbol{\pi}\boldsymbol{P} \\ \implies \pi_j &= \sum_i \pi_i P_{ij} && \text{(matrix multiplication)} \\ \implies \sum_i \pi_j P_{ji} &= \sum_i \pi_i P_{ij} && \text{(probabilities sum to 1)} \end{aligned}$$

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Definition

A Markov chain (X_n) is called (time) **reversible** if for each n

$$(X_0, X_1, \ldots, X_n) \stackrel{\mathcal{D}}{=} (X_n, X_{n-1}, \ldots X_0).$$

That is, the joint distribution of $(X_0, X_1, \ldots, X_n)$ is the same as the joint distribution of $(X_n, X_{n-1}, \ldots X_0)$.

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Proposition. An irreducible Markov chain is reversible if and only the distribution π of X_0 is stationary and satisfies local balance.

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Remark. Notice how local balance allowed the π factor to propagate through the product from the left end to the right, reversing the direction of all of the transitions along the way.

Ehrenfest model of diffusion

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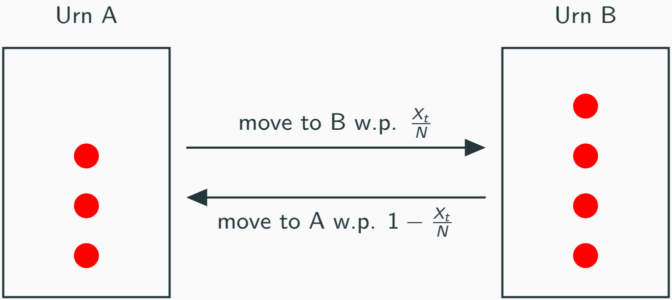

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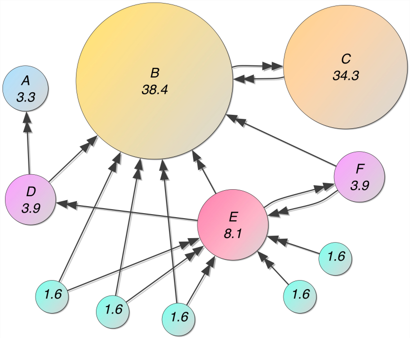
In particular, we have:

Theorem (Perron-Frobenius)(Partial)

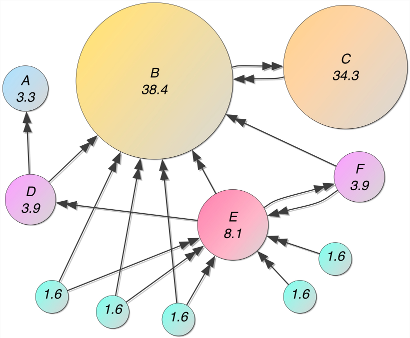
Any finite irreducible Markov chain has a unique stationary distribution $\boldsymbol{\pi}$. Furthermore, all components of $\boldsymbol{\pi}$ are strictly positive.

The 'Page Rank' Markov Chain



How does Google's patented 'Page Rank' algorithm describe the relative popularities of webpages?

The ‘Page Rank’ Markov Chain



Webpages form a directed graph with n vertices (representing webpages). A link from page i to page j is designated by an directed edge.

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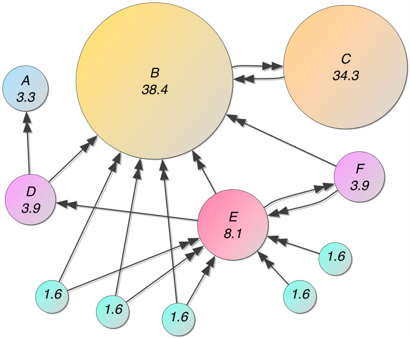

The behavior of a swiftly bored web surfer can be modeled by a random walk on the directed graph of webpages.

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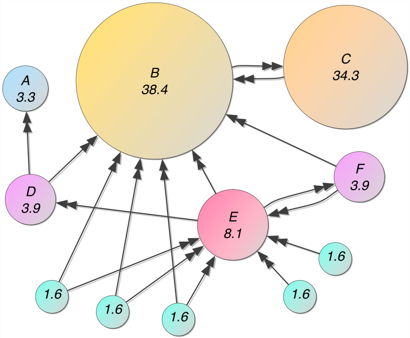

With probability b, the random walk moves to a randomly linked page from the current page (or to a random page in the graph, if a webpage has no outgoing links). With probability $1 - b$, the random walk moves to a random page in the graph.

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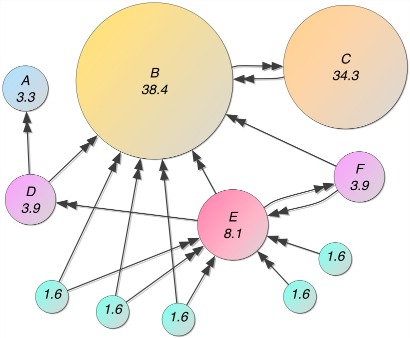
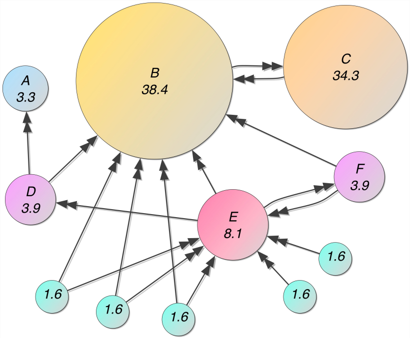


Let d_i be the out-degree of vertex i.

The 'Page Rank' Markov Chain



Then the transition matrix $\boldsymbol{P} = b\boldsymbol{Q} + (1-b)\boldsymbol{u}^\top \boldsymbol{e}$, where $\boldsymbol{Q} = (q_{ij})$ with

$$q_{ij} = \begin{cases} 1/d_i, & \text{if } i \to j \\ 1/n, & \text{if } i \text{ dangles } (d_i = 0) \\ 0, & \text{otherwise} \end{cases}$$

where $\boldsymbol{u}$ is a row vector with $u_i = 1/n$ and $\boldsymbol{e}$ is a row vector with entries 1.

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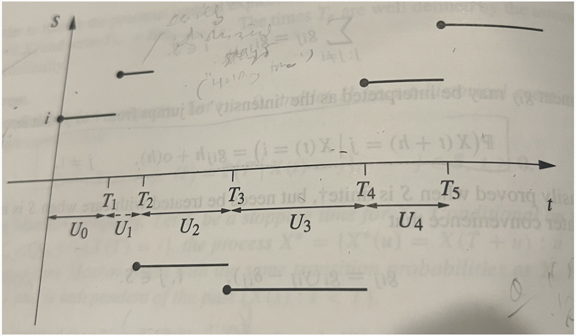
Continuous Time Markov Chains



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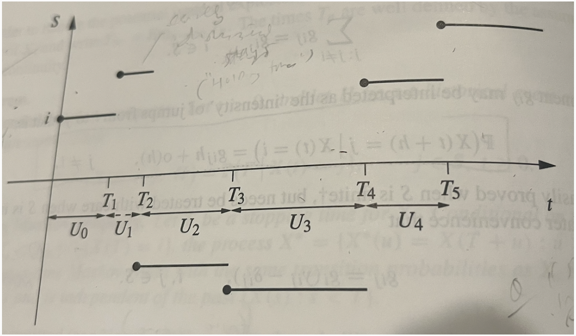

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Examples. Traffic light cycles, inventory systems with bulk orders, popcorn popping, customer queues, website traffic,

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Let $X = \{X(t) : t \geq 0\}$ be a family of random variables taking values in a countable state space S and indexed by the half line $[0, \infty)$.

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Definition

A continuous time chain X satisfies the **Markov property** if

$$P\big(X(t_n) = i_n \mid X(t_1) = i_1, X(t_1) = i_2, \ldots, X(t_{n-1}) = i_{n-1}\big)$$
$$= P\big(X(t_n) = i_n \mid X(t_{n-1}) = i_{n-1}\big)$$

for any increasing sequence of times $t_1 < t_2 < \ldots < t_n$ and states $i_1, i_2, \ldots i_n$.

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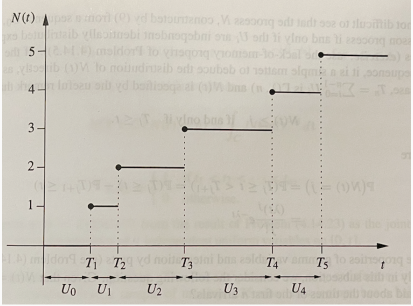
for any increasing sequence of times $t_1 < t_2 < \ldots < t_n$ and states $i_1, i_2, \ldots i_n$.

Remark. For technical reasons, the text restricts consideration to *right continuous* chains – that is those which have *stepwise* sample paths.

Example: Poisson Process

A Poisson process $N(t)$ can be used to model the number of 'arrivals' or 'occurrences' or 'events' by time t.



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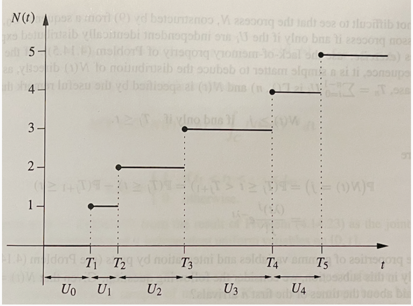

Example. Emission of particles by a radioactive source.

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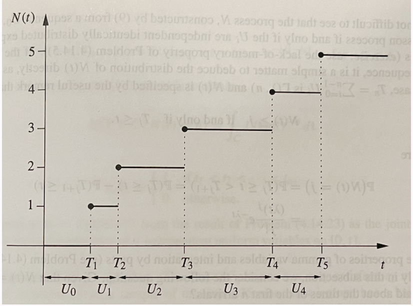

A Poisson process with intensity λ has pmf

$$P\big(N(t) = j\big) = \frac{(\lambda t)^j}{j!} e^{-\lambda t}, \qquad j = 0, 1, 2, \ldots$$

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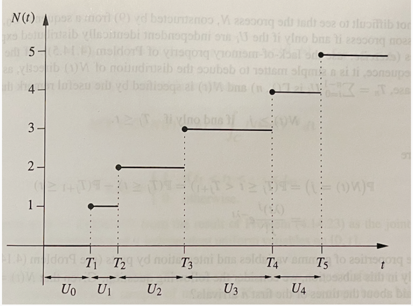

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$$P\big(N(t) = j\big) = \frac{(\lambda t)^j}{j!} e^{-\lambda t}, \qquad j = 0, 1, 2, \ldots$$

Hence, the expected value is $E[N(t)] = \lambda t$.

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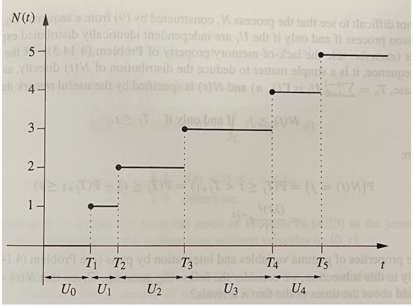

Theorem. The **interarrival times** $U_n = T_{n+1} - T_n$ are independent and exponentially distributed with parameter λ.

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2. **Finite population.** The number of kernels is limited, so there is not an unbounded number of possible pops.
3. **Dependent events.** Although rare, a pop from one kernel could trigger another through physical collision.

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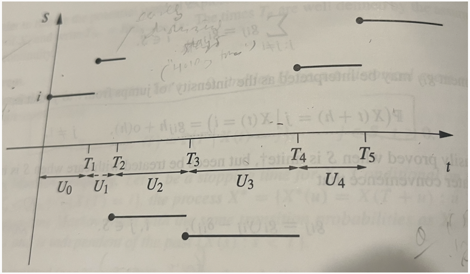

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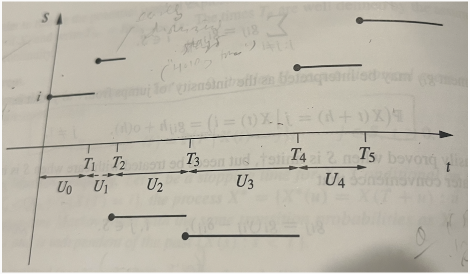

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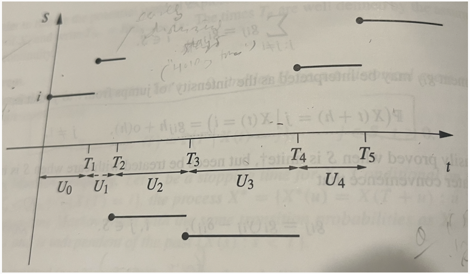

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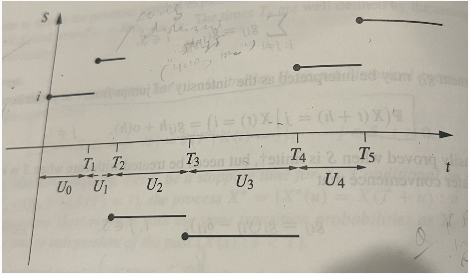

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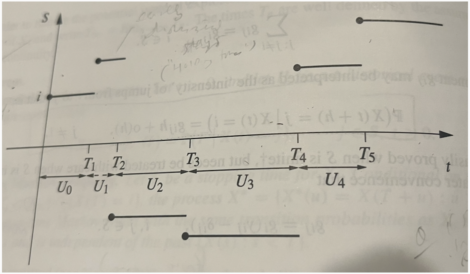

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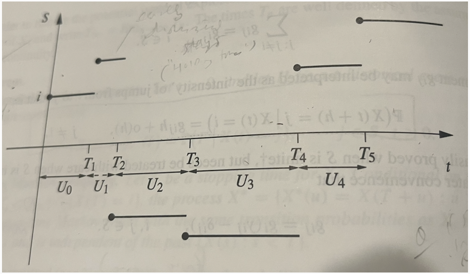

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Fact from Problem 4.14.5a

Let $g : [0, \infty) \to [0, \infty)$ be such that $g(s+t) = g(s)g(t)$ for $s, t \geq 0$. If g is monotone, then $g(s) = e^{\mu s}$ for some μ.

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which in turn implies:

[title

The **exponential distribution** is the only continuous distribution with the 'lack of memory' property

$$P(X > s + t \mid X > s) = P(X > t).$$

Random Groups

Aubrey Williams: 5
Austin Barton : 3
Blake Sigmundstad: 7
Diego Moylan: 6
Dillon Shaffer: 1
Ismoiljon Muzaffarov: 2
Jacob Tanner: 8

Josh Stoneback: 4
Joshua Bowen: 1
Joshua Culwell: 3
Laura Banaszewski: 1
Lina Hammel: 6
Logan Racz: 2

Matt Hall: 5
Micah Miller: 8
Mike Kadoshnikov: 4
Owen Cool: 2
Racquel Bowen: 4
Samuel Mocabee: 7
Tatiana Kirillova: 3

Group exercises - Problem Set 10

1. (6.5) **The Ehrenfest model.** Find the stationary distribution for the Ehrenfest model of diffusion. (Hint: Rather than solve the equation $\boldsymbol{\pi} = \boldsymbol{\pi}\boldsymbol{P}$, consider that the diffusion model might be reversible in equilibrium. That is, look for solutions to the detailed balance equation $\pi_i p_{ij} = \pi_j p_{ji}$.)
2. (6.6.9) **The 'Page Rank' Markov chain.** Recall the construction of a random walk on the world wide web, whose transition matrix is $\boldsymbol{P} = b\boldsymbol{Q} + (1-b)\boldsymbol{u}^\top \boldsymbol{e}$, where $\boldsymbol{Q} = (q_{ij})$ with

$$q_{ij} = \begin{cases} 1/d_i, & \text{if } i \to j \\ 1/n, & \text{if } i \text{ dangles } (d_i = 0) \\ 0, & \text{otherwise} \end{cases}$$

   where n is the number of webpages, d_i is the outgoing degree of the i-th webpage, $\boldsymbol{u}$ is a row vector with $u_i = 1/n$, and $\boldsymbol{e}$ is a row vector of 1's.
   a) Deduce that the stationary distribution $\boldsymbol{\pi}$ is given by $\boldsymbol{\pi} = \{(1-b)/n\}\boldsymbol{e}(\boldsymbol{I} - b\boldsymbol{Q})^{-1}$, where $\boldsymbol{I}$ is the identity matrix.
   b) Explain why the elements of $\boldsymbol{\pi}$, when re-arranged in decreasing order, supply a description of the relative popularities of webpages.
3. (6.9) **Holding times.** Let X be a continuous time Markov chain. Let $X(0) = i$, and $U_0 = \inf\{t : X(t) \neq i\}$ be the time of the first change in value. Show that U_0 has the exponential distribution.