02/10/2026: Markov Chains (Part 1)

CSCI 546: Diffusion Models

Textbook reference: Sec 6.1-6.2

Announcement (Sign-in Sheet)

Please sign the sign-in sheet.

Review Problem Set #7

Concepts for Problem Set #8

Markov Chains

Suppose we observe a random sequence $(x_i)_{i=1}^{\infty}$.

By chain rule, we can always write the joint density over the first n observations as:

$$p(x_1, x_2, \ldots, x_n) = p(x_1)p(x_2 \mid x_1)p(x_3 \mid x_1, x_2) \cdots p(x_n \mid x_1, x_2, \ldots x_{n-2}, x_{n-1})$$

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Markov chains are special sequences whose joint density simplifies to:

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That is, Markov chains satisfy

$$p(x_{n+1} \mid x_{1:n}) = p(x_{n+1} \mid x_n)$$

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In words:

The future is conditionally independent of the past given the present.

Example: Simple random walk

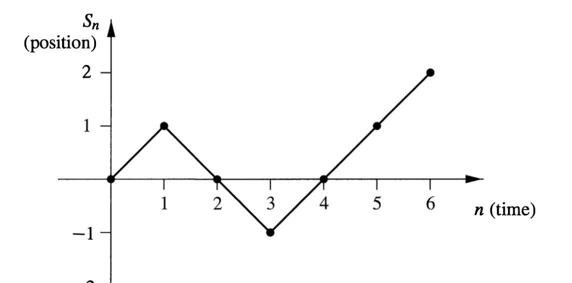
For timesteps $n = 1, 2, 3, \ldots$, let

$$X_n = \begin{cases} 1, & \text{with probability } p \\ -1, & \text{with probability } q = 1 - p \end{cases}$$

Then a simple random walk (in one dimension) is given by

$$S_n = S_0 + \sum_{i=1}^{n} X_i,$$

where S_0 is the starting position.



Random Groups

Aubrey Williams: 6
Austin Barton : 6
Blake Sigmundstad: 3
Diego Moylan: 2
Dillon Shaffer: 1
Ismoiljon Muzaffarov: 7
Jacob Tanner: 3

Josh Stoneback: 3
Joshua Bowen: 1
Joshua Culwell: 5
Laura Banaszewski: 5
Lina Hammel: 4
Logan Racz: 1

Matt Hall: 4
Micah Miller: 8
Mike Kadoshnikov: 8
Owen Cool: 2
Racquel Bowen: 2
Samuel Mocabee: 4
Tatiana Kirillova: 7

Group exercises - Problem Set 8

1. (6.1.2) A die is rolled repeatedly. Which of the following are Markov chains? For those that are, supply the transition matrix
   (a) The largest number X_n shown up to the n-th roll.
   (b) The number N_n of sixes in n rolls.
   (c) At time r, the time C_r since the most recent six.
   (d) At time r, the time B_r until the next six.
2. (Sec 6.2) Show that a simple random walk is periodic with period 2. Then show that a simple random walk is recurrent if $p = \frac{1}{2}$ but transient if $p \neq \frac{1}{2}$.

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   - For additional help, see slide 22 here.