02/03/2026: Continuous Random Variables (Part 1)

CSCI 546: Diffusion Models

Textbook reference: Sec 4.1-4.5

Announcement (Sign-in Sheet)

Please sign the sign-in sheet.

Review Problem Set #5

Concepts for Problem Set #6

Continuous random variables

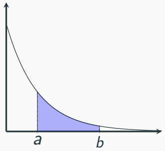
Definition (GS Sec 4.1)

A random variable X is called **continuous** if its distribution function $F(x) = P(X \leq x)$ can be written as

$$F(x) = \int_{-\infty}^{x} f(u)\, du$$

for some integrable $f : \mathbb{R} \to [0, \infty)$.

Remark. If X is a random variable with probability density f below



then

$$P(a \leq X \leq b) = \texttt{shaded area}$$

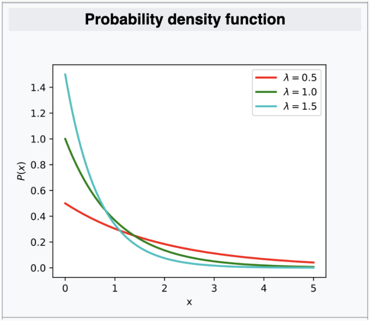
Example: Exponential Distribution

Definition

A continuous random variable whose probability density function is given, for some $\lambda > 0$, by

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & \text{if } x \geq 0 \\ 0, & \text{if } x < 0 \end{cases}$$

is said to be an **exponential** random variable with parameter λ.



Applications. An exponential distribution often arises as the amount of time until some specific event occurs. For example, the amount of time until...

- ... an earthquake occurs
- ... a new war breaks out
- ... you get a spam telephone call

Memorylessness

Definition

We say that a non-negative random variable X is **memoryless** if

$$P(X > s + t \mid X > t) = P(X > s), \qquad \text{for all } s, t \geq 0 \tag{1}$$

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Poll. Who can explain what the above equation is saying?

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Interpretation. If we think of X as waiting time for the next snowfall, Eq. (1) says:

> *If we're still waiting for snowfall at time t, the distribution of the remaining amount of time that we must wait is the same when we started waiting.*

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> *If we're still waiting for snowfall at time t, the distribution of the remaining amount of time that we must wait is the same when we started waiting.*

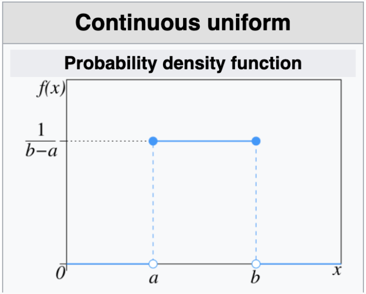
That is, it is as if Mother Nature does not "remember" that we've already been waiting for time t.

Example: Uniform Distribution

Definition

A continuous random variable X is said to be **uniform** on $[a, b]$ (or, equivalently, on (a, b)) if it has the probability density function

$$f(x) = \begin{cases} \frac{1}{b-a}, & \text{if } a < x < b \\ 0, & \text{otherwise} \end{cases}$$



Applications. Although simple, the uniform distribution is surprisingly useful. E.g.:

- Random number generation by a computer. (Non-uniform random numbers are often obtained by transforming uniform samples.)
- Random angles in physics.
- Monte carlo estimates of an integral $\int_a^b f(u)\, du$

Random Groups

Aubrey Williams: 7
Austin Barton : 4
Blake Sigmundstad: 3
Diego Moylan: 3
Dillon Shaffer: 2
Ismoiljon Muzaffarov: 5
Jacob Tanner: 8

Josh Stoneback: 1
Joshua Bowen: 2
Joshua Culwell: 6
Laura Banaszewski: 4
Lina Hammel: 1
Logan Racz: 8

Matt Hall: 4
Micah Miller: 2
Mike Kadoshnikov: 1
Owen Cool: 5
Racquel Bowen: 6
Samuel Mocabee: 7
Tatiana Kirillova: 3

Group exercises - Problem Set 6

1. (4.14.5b) Prove that the exponential distribution is memoryless. (Note: the exponential distribution is the *only* continuous distribution with this property!)
2. (4.5.5a) Let X and Y be independent continuous random variables. Show that
$$\mathbb{E}[g(X)h(Y)] = \mathbb{E}[g(X)]\mathbb{E}[h(Y)]$$
whenever these expectations exist. If X and Y have the exponential distribution with parameter 1, find $\mathbb{E}[e^{\frac{1}{2}(X+Y)}]$.
3. (4.5.4) Let X and Y be independent random variables each having the uniform distribution on $[0, 1]$. Let $U = \min\{X, Y\}$ and $V = \max\{X, Y\}$. Find $\mathbb{E}[U]$ and $\mathbb{E}[V]$, and then calculate $\mathrm{Cov}[U, V]$. Are U and V independent?