

Notes on Ash's Probability and Measure Theory

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1 Section 1.1: Some set theory practice

Remark 1.0.1. If A_1, A_2, \dots are subsets of some set Ω , then

$$\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} \left(A_i \cap A_{i-1}^c \cap \dots \cap A_1^c \right)$$

In other words, any union can be re-represented as a disjoint union. This is useful because measures are countably additive on disjoint sets, so we prefer to work with collections of disjoint sets. \triangle

2 Section 1.2: Fields, σ -fields, measures

2.1 Sec 1.2.1-1.2.2: Fields and σ -fields

Fields and σ -fields are important because they are the domain of measures. Here are some definitions.

Definition 2.1.1. Let \mathcal{F} be a collection of subsets of a set Ω . Then \mathcal{F} is called a **field** (or *algebra*) if

- a) $\Omega \in \mathcal{F}$
- b) If $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$.
- c) If $A_1, \dots, A_n \in \mathcal{F}$ then $\bigcup_{i=1}^n A_i \in \mathcal{F}$.

that is, if $\Omega \in \mathcal{F}$ and \mathcal{F} is closed under complementation and finite unions. \triangle

Definition 2.1.2. Let \mathcal{F} be a collection of subsets of a set Ω . Then \mathcal{F} is called a **sigma-field** (or *sigma-algebra*) if it satisfies Definition 2.1.1 after replacing condition c) with

- c') If $A_1, A_2, \dots \in \mathcal{F}$ then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$.

that is, if $\Omega \in \mathcal{F}$ and \mathcal{F} is closed under complementation and *countable* unions. \triangle

Example 2.1.1. $\mathcal{F} = \{\emptyset, \Omega\}$ is the smallest σ -field on Ω . \triangle

Example 2.1.2. $\mathcal{F} = 2^{\Omega}$, i.e. the set of all subsets of Ω , is the largest σ -field on Ω . \triangle

Example 2.1.3. If $A \in \Omega$ is non-empty, then $\mathcal{F} = \{\emptyset, A, A^c, \Omega\}$ is the smallest σ -field containing A . \triangle

Notation 2.1.1. If \mathcal{C} is a class of sets, the smallest σ -field containing the sets of \mathcal{C} is written as $\sigma(\mathcal{C})$. This is sometimes called the *minimal σ -field over \mathcal{C}* or the *σ -field generated by \mathcal{C}* . \triangle

Question 2.1.1. Let A_1, \dots, A_n be subsets of Ω . Describe $\sigma(\{A_1, \dots, A_n\})$, the smallest σ -field containing A_1, \dots, A_n . Also describe the number of sets in \mathcal{F} . *This is Ash's Problem 1.2.8. For answer, see GoodNotes.* \triangle

Example 2.1.4. What is an example of a collection that is a *field*, but not a σ -field?

Let $\Omega = \mathbb{R}$ and $\mathcal{F}_0 = \{\text{finite disjoint unions of left semi-closed intervals } [a, b), a \neq b\}$. Then \mathcal{F}_0 is a field, as can be easily verified.¹ But \mathcal{F}_0 is not a σ -field. Note that if $A_n = [0, \frac{1}{n})$, then $\bigcap_{n=1}^{\infty} A_n = \{0\} \notin \mathcal{F}_0$. \triangle

¹By convention, we also count $(-\infty, b)$ as left semi-closed for $-\infty < b \leq \infty$, which is necessary for the σ -field to be closed under complements.

2.1.1 “Good sets” strategy

Ash says that there is a type of reasoning that occurs so often in problems involving σ -fields that it deserves explicit mention. It is called the *good sets strategy*. Suppose you want to show that all members of a σ -algebra \mathcal{F} have some property P . Define “good sets” as those that satisfy the property

$$\mathcal{G} := \{G \in \mathcal{F} : G \text{ has property } P\}$$

The strategy is then to simply

1. Show \mathcal{G} is a σ -algebra
2. Show \mathcal{G} contains some class \mathcal{C} such that $\mathcal{F} = \sigma(\mathcal{C})$

Then you’re done!

Why does this work?

$$\mathcal{C} \subset \mathcal{G} \quad \text{by 2}$$

$$\implies \sigma(\mathcal{C}) \subset \sigma(\mathcal{G})$$

$$\implies \mathcal{F} \subset \mathcal{G} \quad \text{by 1,2}$$

Yet $\mathcal{G} \subset \mathcal{F}$ by definition of \mathcal{G} .

So $\mathcal{G} = \mathcal{F}$.

So all sets in \mathcal{F} are good.

In the text, Ash uses this strategy to show that if \mathcal{C} is a class of subsets of Ω , and $A \in \Omega$, then

$$\underbrace{\sigma_{\Omega}(\mathcal{C}) \cap A}_{\text{take minimal sigma field first, then intersect}} = \underbrace{\sigma_A(\mathcal{C} \cap A)}_{\text{intersect first, then take minimal sigma-field}}$$

For another application, see handwritten homework exercises.

2.2 Sec 1.2.3-1.2.4: Measures

Definition 2.2.1. A **measure** on a σ -field \mathcal{F} is a non-negative, extended real-valued function μ on \mathcal{F} such that whenever A_1, A_2, \dots form a finite or countably infinite collection of disjoint sets in \mathcal{F} , we have countable additivity; that is,

$$\mu\left(\bigcup_n A_n\right) = \sum_n \mu(A_n)$$

△

Definition 2.2.2. A **probability measure** is a measure (Definition 2.2.1) where $\mu(\Omega) = 1$.

△

Remark 2.2.1. Ash additionally assumes that a measure does not take $\mu(A) = \infty$ or $\mu(A) = -\infty$ for all $A \in \mathcal{F}$. From this, we automatically obtain $\mu(\emptyset) = 0$. For $\mu(A) < \infty$ for some A , and by considering the sequence $A, \emptyset, \emptyset, \dots$, we have that $\mu(\emptyset) = 0$ by countable additivity.

△

Example 2.2.1. Let $\Omega = \{x_1, x_2, \dots\}$ be a finite or countably infinite set. Let p_1, p_2, \dots be non-negative reals. Let $\mathcal{F} = 2^{\Omega}$. Define

$$\mu(A) = \sum_{x_i \in A} p_i \quad \text{for all } A \in \mathcal{F}$$

Then μ is a measure on \mathcal{F} . We might call it the “point weighting” measure.

- If $p_i \equiv 1 \forall i$, then μ is called the **counting measure**.
- If $\sum_i p_i = 1$, then μ is a probability measure.

△

Example 2.2.2. (*Lebesgue measure*) Define μ such that

$$\mu(a, b] = b - a \quad \forall a, b \in \mathbb{R} : b > a$$

This requirement determines μ on a large collection of sets, the Borel Sets $\mathcal{B}(\mathbb{R})$, defined as the smallest σ -field of subsets of \mathbb{R} containing all intervals $(a, b] \subset \mathbb{R}$.

We may alternately characterize $\mathcal{B}(\mathbb{R})$ as the smallest σ -field containing

- all intervals $[a, b]$, $a, b \in \mathbb{R}$.
- all intervals (a, b) , $a, b \in \mathbb{R}$
- all intervals (a, ∞) , $a \in \mathbb{R}$.
- all intervals $[a, \infty)$, $a \in \mathbb{R}$.
- all intervals $(-\infty, b)$, $b \in \mathbb{R}$.
- all intervals $(-\infty, b]$, $b \in \mathbb{R}$.
- all intervals $[a, b]$, $a, b \in \mathbb{R}$.
- all open sets of \mathbb{R} .²
- all closed sets of \mathbb{R} .³

To illustrate these equivalences, let us equate the first two conditions. That is, let us show that a σ -field contains all open intervals (a, b) iff it contains all right semi-closed intervals $(a, b]$. To see this, simply note

$$(a, b] = \bigcap_{n=1}^{\infty} \left(a, b + \frac{1}{n} \right)$$

and

$$(a, b) = \bigcup_{n=1}^{\infty} \left(a, b - \frac{1}{n} \right]$$

△

Question 2.2.1. The text gives another description of the Borel sets $\mathcal{B}(\mathbb{R})$ as the smallest σ -field containing \mathcal{F}_0 , the field of disjoint unions of right semi-closed intervals $(a, b]$. Can we make the same statement about the field of finite disjoint unions of left semi-closed intervals (as presented in Example 2.1.4)?

△

²Recall that an open set is a countable union of open intervals.

³Recall that a set is open iff its complement is closed.

2.3 Sec 1.2.5-1.2.6: Generalizations of measures, and their properties

The text considers some generalizations of measures that can be obtained by relaxing the domain to a field, by assuming merely finite additivity, or by allowing the range to be extended reals ($\bar{\mathbb{R}}$) instead of non-negative extended reals ($\bar{\mathbb{R}}_{\geq 0}$).

Remark 2.3.1. The first two relaxations above often go together. However, a countably additive function can be defined on a *field* (rather than σ -field) if the condition is taken to hold whenever a countable union *does* happen to still be in the field. In my notes, I will simply things by assuming that countably additive functions are always defined on σ -fields. \triangle

	Range	
	non-negative	signed
finitely additive	μ_0	$\tilde{\mu}_0$
countably additive	μ measure	$\tilde{\mu}$ signed measure

Table 1: Notation for generalizations of measure (For assumed domain in each case, see Remark 2.3.1.)

In Table 1, we introduce some notation to try to clarify more immediately when results hold. Note the relations⁴

$$\{\mu\} \subset \{\mu_0\}, \{\tilde{\mu}\} \subset \{\tilde{\mu}_0\}.$$

Using the notation in Table 1, we rewrite Theorem 1.2.5 of the text:

Theorem 2.3.1. *Let $\tilde{\mu}_0$ be a finitely additive set function on the field \mathcal{F}_0 . Then*

- a) $\tilde{\mu}_0(\emptyset) = 0$
- b) $\tilde{\mu}_0(A \cup B) + \tilde{\mu}_0(A \cap B) = \tilde{\mu}_0(A) + \tilde{\mu}_0(B)$ for all $A, B \in \mathcal{F}_0$.
- c) If $A, B \in \mathcal{F}_0$ and $B \subset A$, then $\tilde{\mu}_0(A) = \tilde{\mu}_0(B) + \tilde{\mu}_0(A - B)$.
- d) Subadditivity holds, i.e.

$$\mu_0(\cup_{i=1}^n A_i) \leq \sum_{i=1}^n \mu_0(A_i)$$

$$\mu(\cup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} \mu(A_i)$$

Proof. We prove Theorem 2.3.1 (b). The rest is an exercise for the reader (or see the text).

First, we break things into disjoint pieces

$$A = (A \cap B) \cup (A \cap B^c) \implies \mu_0(A) = \mu_0(A \cap B) + \mu_0(A \cap B^c) \quad (1)$$

$$B = (A \cap B) \cup (A^c \cap B) \implies \mu_0(B) = \mu_0(A \cap B) + \mu_0(A^c \cap B) \quad (2)$$

$$A \cup B = (A \cap B) \cup (A \cap B^c) \cup (A^c \cap B) \implies \mu_0(A \cup B) = \mu_0(A \cap B) + \mu_0(A \cap B^c) + \mu_0(A^c \cap B) \quad (3)$$

⁴So, for example, if something holds for $\tilde{\mu}_0$, it holds for μ . A simple mnemonic is that adding stuff to the notation generalizes the function.

Summing (1) and (2), we obtain

$$\tilde{\mu}_0(A) + \tilde{\mu}_0(B) = 2\tilde{\mu}_0(A \cap B) + \tilde{\mu}_0(A \cap B^c) + \tilde{\mu}_0(A^c \cap B).$$

We use (3) to simplify the RHS, and the result follows. \square

Remark 2.3.2. In the proof of Theorem 2.3.1 (b), note that we use a common strategy – breaking sets into disjoint pieces so that we can apply the assumed (finite or countable) additivity of the set function. \triangle