Constructing measures

...on spaces with unit, finite, and infinite dimensionality

May 31, 2022

Goal: Construct measures on interesting spaces

Table of contents

Background

Measures are defined on σ-fields

Definition

Let $\mathcal{F}$ be a collection of subsets of a set Ω. Then $\mathcal{F}$ is called a **sigma-field** if it satisfies

1. $\Omega \in \mathcal{F}$
2. *(Closed under complementation.)* If $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$.
3. *(Closed under countable unions.)* If $A_1, A_2, ... \in \mathcal{F}$ then $\cup_{i=1}^{\infty} A_i \in \mathcal{F}$.

Measures are defined on σ-fields

Definition

Let $\mathcal{F}$ be a collection of subsets of a set Ω. Then $\mathcal{F}$ is called a **sigma-field** if it satisfies

1. $\Omega \in \mathcal{F}$
2. *(Closed under complementation.)* If $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$.
3. *(Closed under countable unions.)* If $A_1, A_2, ... \in \mathcal{F}$ then $\cup_{i=1}^{\infty} A_i \in \mathcal{F}$.

Remark

The definition implies that sigma-field are also closed under countable *intersections*.

Measures are defined on σ-fields

Definition

Let $\mathcal{F}$ be a collection of subsets of a set Ω. Then $\mathcal{F}$ is called a **sigma-field** if it satisfies

1. $\Omega \in \mathcal{F}$
2. *(Closed under complementation.)* If $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$.
3. *(Closed under countable unions.)* If $A_1, A_2, \ldots \in \mathcal{F}$ then $\cup_{i=1}^{\infty} A_i \in \mathcal{F}$.

Remark

The definition implies that sigma-field are also closed under countable *intersections*.

Example

If $A \in \Omega$ is non-empty, then $\mathcal{F} = \{\emptyset, A, A^c, \Omega\}$ is the smallest σ-field containing A.

Measures are defined on σ-fields

Definition

Let $\mathcal{F}$ be a collection of subsets of a set Ω. Then $\mathcal{F}$ is called a **sigma-field** if it satisfies

1. $\Omega \in \mathcal{F}$
2. *(Closed under complementation.)* If $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$.
3. *(Closed under countable unions.)* If $A_1, A_2, \ldots \in \mathcal{F}$ then $\cup_{i=1}^{\infty} A_i \in \mathcal{F}$.

Remark

The definition implies that sigma-field are also closed under countable *intersections*.

Example

If $A \in \Omega$ is non-empty, then $\mathcal{F} = \{\emptyset, A, A^c, \Omega\}$ is the smallest σ-field containing A.

Definitions

A set $A \in \mathcal{F}$ is called a **measurable set**.

$(\Omega, \mathcal{F})$ is called a **measurable space**.

Measures

Definition

A **measure** on a σ-field $\mathcal{F}$ is a non-negative, extended real-valued function μ on $\mathcal{F}$ such that whenever $A_1, A_2, \ldots$ form a finite or countably infinite collection of disjoint sets in $\mathcal{F}$, we have countable additivity; that is,

$$\mu\left(\dot{\bigcup_n} A_n\right) = \sum_n \mu(A_n)$$

Measures

Definition

A **measure** on a σ-field $\mathcal{F}$ is a non-negative, extended real-valued function μ on $\mathcal{F}$ such that whenever $A_1, A_2, \ldots$ form a finite or countably infinite collection of disjoint sets in $\mathcal{F}$, we have countable additivity; that is,

$$\mu\left(\dot{\bigcup_n} A_n\right) = \sum_n \mu(A_n)$$

Example

A **probability measure** is a measure where $\mu(\Omega) = 1$.

One dimension: Lebesgue measure

Elementary Families

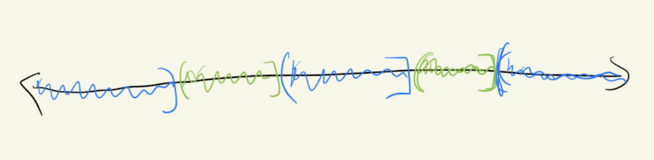
Definition

An **elementary family** is a collection $\mathcal{E}$ of subsets of Ω such that

1. $\emptyset \in \mathcal{E}$
2. if $E, F \in \mathcal{E}$ then $\mathcal{E} \cap \mathcal{F} \in \mathcal{E}$
3. if $E \in \mathcal{E}$, then E^c is a finite disjoint union of members of $\mathcal{E}$.

Example

Consider the **right semi-closed (r.s.c) intervals**:



From elementary families to fields

Definition

Let $\mathcal{F}_0$ be a collection of subsets of a set Ω. Then $\mathcal{F}_0$ is called a **field** if it satisfies

1. $\Omega \in \mathcal{F}_0$
2. *(Closed under complementation.)* If $A \in \mathcal{F}_0$, then $A^c \in \mathcal{F}_0$.
3. *(Closed under finite unions.)* If $A_1, A_2, ... \in \mathcal{F}_0$ then $\cup_{i=1}^n A_i \in \mathcal{F}_0$.

From elementary families to fields

Definition

Let $\mathcal{F}_0$ be a collection of subsets of a set Ω. Then $\mathcal{F}_0$ is called a **field** if it satisfies

1. $\Omega \in \mathcal{F}_0$
2. *(Closed under complementation.)* If $A \in \mathcal{F}_0$, then $A^c \in \mathcal{F}_0$.
3. *(Closed under finite unions.)* If $A_1, A_2, ... \in \mathcal{F}_0$ then $\cup_{i=1}^n A_i \in \mathcal{F}_0$.

Proposition

If $\mathcal{E}$ is an elementary family then the collection

$$\mathcal{F}_0 := \{\textit{finite disjoint unions of members of } \mathcal{E}\}$$

is a field.

From elementary families to fields

Definition

Let $\mathcal{F}_0$ be a collection of subsets of a set Ω. Then $\mathcal{F}_0$ is called a **field** if it satisfies

1. $\Omega \in \mathcal{F}_0$
2. *(Closed under complementation.)* If $A \in \mathcal{F}_0$, then $A^c \in \mathcal{F}_0$.
3. *(Closed under finite unions.)* If $A_1, A_2, ... \in \mathcal{F}_0$ then $\cup_{i=1}^n A_i \in \mathcal{F}_0$.

Proposition

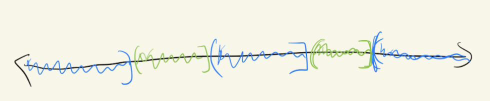
If $\mathcal{E}$ is an elementary family then the collection

$$\mathcal{F}_0 := \{\textit{finite disjoint unions of members of } \mathcal{E}\}$$

is a field.

Example

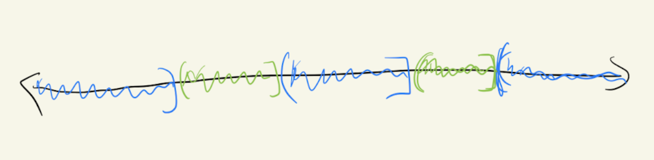
The collection of **finite disjoint unions of r.s.c intervals** forms a field:



Pre-measure on a field

Example: Lebesgue pre-measure

To measure a finite disjoint union of r.s.c intervals



Set

$$\mu\left(\dot{\bigcup_{i=1}^{n}}(a_i, b_i]\right) = \sum_{i=1}^{n} b_i - a_i$$

Carathéodory Extension Theorem

Theorem

Let μ be a pre-measure on a field $\mathcal{F}_0$ of subsets of Ω, and assume that μ is σ-finite on $\mathcal{F}_0$. Then μ has a unique extension to a measure on $\mathcal{F} := \sigma(\mathcal{F}_0)$, the minimal σ-field over $\mathcal{F}_0$.

Now we can measure all sorts of things:

$$[a, b] = \bigcap_{n=1}^{\infty} \left(a - \frac{1}{n}, b\right] \qquad \text{Closed intervals}$$

$$(a, b) = \bigcup_{n=1}^{\infty} \left(a, b - \frac{1}{n}\right] \qquad \text{Open intervals}$$

$$\texttt{Cantor set} = \bigcap_{i=1}^{\infty} \bigcap_{j=1}^{3^{i-1}-1} \left[0, \frac{3j+1}{3^i}\right] \cup \left[\frac{3j+2}{3^i}, 1\right] \qquad \text{Weird things}$$

$$\vdots \qquad \vdots \qquad\qquad\qquad \vdots$$

n dimensions: Product measure

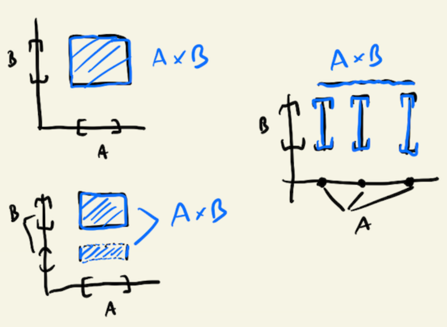
Measurable rectangles

Let $(X, \mathcal{M})$ and $(Y, \mathcal{N})$ be measurable spaces (i.e. sets and associated σ-fields).

Definition

A **measurable rectangle** is a subset $A \times B$ of $X \times Y$ where $A \in \mathcal{M}$ and $B \in \mathcal{N}$ are measurable subsets of X and Y, respectively.

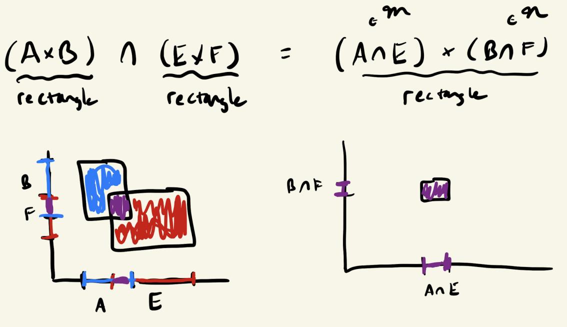
Examples



Note: The "sides" of a measurable rectangle $A \times B$ are not required to be intervals. For instance, if $\mathbb{R}$ is equipped with the Borel σ-field, then $\mathbb{Q} \times \mathbb{Q}$ is a measurable rectangle in $\mathbb{R} \times \mathbb{R}$.

Measurable rectangles are an elementary family.

For example, measurable rectangles are closed under intersection.



Construction of the product measure

Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be measure spaces.

Let $C \in F_0$ be a finite disjoint union of rectangles $A_1 \times B_1, A_2 \times B_2, \dots A_n \times B_n$.

Define the set function

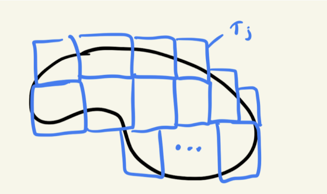
$$\pi_0(C) := \sum_{i=1}^{n} \mu(A_i)\nu(B_i)$$

Then π_0 is well-defined, and a premeasure on $\mathcal{F}_0$.

By the Carathéodory Extension Theorem, π_0 extends to a measure π on $\mathcal{F} := \sigma(\mathcal{F}_0)$.

Application

To find the product measure of D, an arbitrary measurable set in $X \times Y = \mathbb{R}^2$

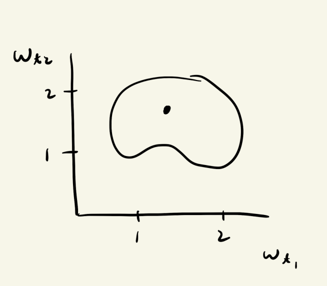


We take the least upper bound of the measure of countable rectangles which cover it.

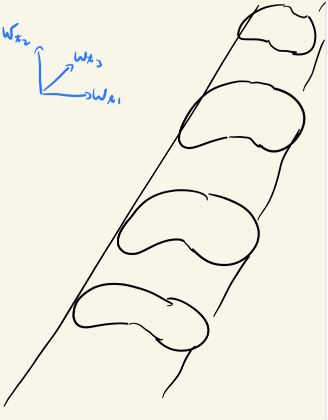
$$\mu \times \nu(D) := \inf \left\{ \sum_{n=1}^{\infty} \mu(A_n)\nu(B_n) : \cup_{n=1}^{\infty}(A_n \times B_n) \supseteq D,\ A_n, B_n \in \mathcal{B}(\mathbb{R}) \right\} \tag{0.1}$$

Infinite dimensions

Measurable cylinders – an elementary family



A base (in 2 coordinates) for a cylinder



The cylinder projected to the first three coordinates.