Notes on Ash's Probability and Measure Theory

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1 Section 1.2: Fields, σ-fields, measures

1.1 Sec 1.2.1-1.2.2: Fields and σ-fields

Fields and σ-fields are important because they are the domain of measures. Here are some definitions.

Definition 1.1.1. Let $\mathcal{F}$ be a collection of subsets of a set Ω. Then $\mathcal{F}$ is called a **field** (or *algebra*) if

a) $\Omega \in \mathcal{F}$

b) If $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$.

c) If $A_1, ...A_n \in \mathcal{F}$ then $\cup_{i=1}^n A_i \in \mathcal{F}$.

that is, if $\Omega \in \mathcal{F}$ and $\mathcal{F}$ is closed under complementation and finite unions. $\triangle$

Definition 1.1.2. Let $\mathcal{F}$ be a collection of subsets of a set Ω. Then $\mathcal{F}$ is called a **sigma-field** (or *sigma-algebra*) if it satisfies Definition 1.1.1 after replacing condition c) with

c') If $A_1, A_2, ... \in \mathcal{F}$ then $\cup_{i=1}^\infty A_i \in \mathcal{F}$.

that is, if $\Omega \in \mathcal{F}$ and $\mathcal{F}$ is closed under complementation and *countable* unions. $\triangle$

Example 1.1.1. $\mathcal{F} = \{\emptyset, \Omega\}$ is the smallest σ-field on Ω. $\triangle$

Example 1.1.2. $\mathcal{F} = 2^\Omega$, i.e. the set of all subsets of Ω, is the largest σ-field on Ω. $\triangle$

Example 1.1.3. If $A \in \Omega$ is non-empty, then $\mathcal{F} = \{\emptyset, A, A^c, \Omega\}$ is the smallest σ-field containing A. $\triangle$

Notation 1.1.1. If $\mathcal{C}$ is a class of sets, the smallest σ-field containing the sets of $\mathcal{C}$ is written as $\sigma(\mathcal{C})$. This is sometimes called the *minimal σ-field over C* or the *σ-field generated by C*. $\triangle$

Question 1.1.1. Let $A_1, ..., A_n$ be subsets of Ω. Describe $\sigma(\{A_1, ..., A_n\})$, the smallest σ-field containing $A_1, ..., A_n$. Also describe the number of sets in $\mathcal{F}$. *This is Ash's Problem 1.2.8. For answer, see GoodNotes.* $\triangle$

Example 1.1.4. What is an example of a collection that is a *field*, but not a *σ-field*?

Let $\Omega = \mathbb{R}$ and $\mathcal{F}_0 = \{\text{finite disjoint unions of } [a, b), a \neq b\}$. Then $\mathcal{F}_0$ is a field, as can be easily verified. But $\mathcal{F}_0$ is <u>not</u> a σ-field. Note that if $A_n = [0, \frac{1}{n})$, then $\bigcap_{n=1}^\infty A_n = \{0\} \notin \mathcal{F}_0$.

$\triangle$

1.1.1 "Good sets" strategy

Ash says that there is a type of reasoning that occurs so often in problems involving σ-fieldsthat it deserves explicit mention. It is called the *good sets strategy*. Suppose you want to show that all members of a σ-algebra $\mathcal{F}$ have some property P. Define "good sets" as those that satisfy the property

$$\mathcal{G} := \{G \in \mathcal{F} : G \text{ has property } P\}$$

The strategy is then to simply

1. Show $\mathcal{G}$ is a σ-algebra
2. Show $\mathcal{G}$ contains some class $\mathcal{C}$ such that $\mathcal{F} = \sigma(\mathcal{C})$

Then you're done!

Why does this work?

$$
\begin{aligned}
& \mathcal{C} \subset \mathcal{G} && \text{by 2} \\
& \implies \sigma(\mathcal{C}) \subset \sigma(\mathcal{G}) && \\
& \implies \mathcal{F} \subset \mathcal{G} && \text{by 1,2}
\end{aligned}
$$

Yet $\mathcal{G} \subset \mathcal{F}$ by definition of $\mathcal{G}$.

So $\mathcal{G} = \mathcal{F}$.

So all sets in $\mathcal{F}$ are good.

In the text, Ash uses this strategy to show that if $\mathcal{C}$ is a class of subsets of Ω, and $A \in \Omega$, then

$$\underbrace{\sigma_\Omega(\mathcal{C}) \cap A}_{\text{take minimal sigma field first, then intersect}} = \underbrace{\sigma_A(\mathcal{C} \cap A)}_{\text{intersect first, then take minimal sigma-field}}$$

For another application, see handwritten homework exercises.

1.2 Sec 1.2.3-1.2.8: Measures, related set functions, and their properties

Definition 1.2.1. A **measure** on a σ-field $\mathcal{F}$ is a non-negative, extended real-valued function μ on $\mathcal{F}$ such that whenever $A_1, A_2, ...$ form a finite or countably infinite collection of disjoint sets in $\mathcal{F}$, we have countable additivity; that is,

$$\mu\left(\dot{\bigcup_n} A_n\right) = \sum_n \mu(A_n)$$

$\triangle$

Definition 1.2.2. A **probability measure** is a measure (Definition 1.2.1) where $\mu(\Omega) = 1$. $\triangle$

Remark 1.2.1. (*Measure-like functions on fields*) A measure-like set function can be defined on *fields* as well as *sigma-fields* if the countable additivity condition is taken to hold whenever a countable unions *does* happen to still be in the field. $\triangle$

Remark 1.2.2. Ash additionally assumes that a measure does not take $\mu(A) = \infty$ or $\mu(A) = -\infty$ for all $A \in \mathcal{F}$. From this, we automatically obtain $\mu(\emptyset) = 0$. For $\mu(A) < \infty$ for some A, and by considering the sequence $A, \emptyset, \emptyset, ...$, we have that $\mu(\emptyset) = 0$ by countable additivity. $\triangle$