Lebesgue Integration

January 8, 2025

Preface

Goal

In this section, we will introduce the general theory of integration of a function with respect to a general measure, as introduced by Lebesgue. This is referred to as *integration*, *abstract integration*, or *Lesbesgue integration*.

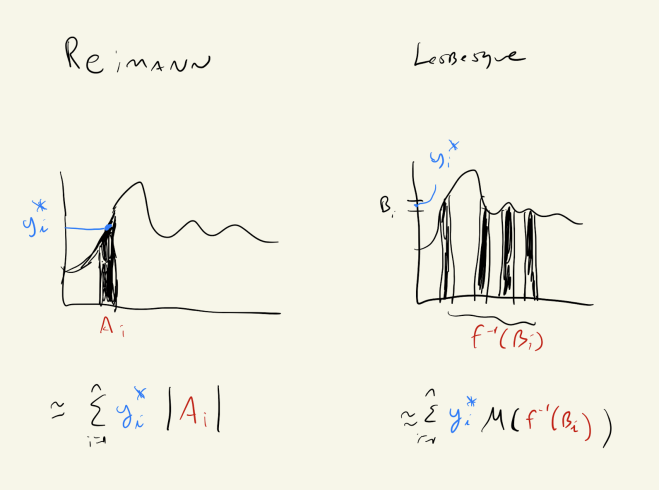
Integration

Intuition

Folland summarizes the difference between the Riemann and Lebesgue approaches thus: "to compute the Riemann integral of f, one partitions the domain [...] into subintervals", while in the Lebesgue integral, "one is in effect partitioning the range of f " (*folland1999real*).

Intuition

The figure below compares how the Riemann and Lesbesgue approaches would approximate the area under the curve of a function $f : \mathbb{R} \to \mathbb{R}$.



What differences do you see?

One Difference – Grouping values adaptively

Since Lebesgue partitions the range and not the domain, it can *group values adaptively* when computing the area under the curve as the sum over n contributions.

The Lebesgue definition makes it possible to calculate integrals for a broader class of functions.

For example, consider the *Dirichlet function*, which is 0 where its argument is irrational and 1 otherwise. The Reimann integral is undefined, because the upper sum and lower sum don't converge as the partition gets finer.

One Difference – Grouping values adaptively

Lebesgue summarized his approach to integration in a letter to Paul Montel:

> *I have to pay a certain sum, which I have collected in my pocket. I take the bills and coins out of my pocket and give them to the creditor in the order I find them until I have reached the total sum. This is the Riemann integral. But I can proceed differently. After I have taken all the money out of my pocket I order the bills and coins according to identical values and then I pay the several heaps one after the other to the creditor. This is my integral.*

The insight is that one should be able to rearrange the values of a function freely, while preserving the value of the integral. This process of rearrangement can convert a very pathological function into one that is "nice" from the point of view of integration

A second difference - Liberation from intervals

We can integrate over arbitrary regions, that aren't necessarily intervals.

Consider:

$$\int_{x\,:\,\text{some condition on x holds}} f$$

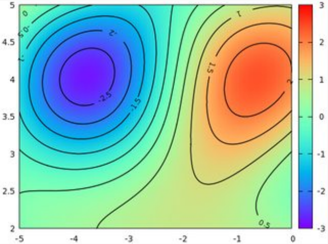
A third difference - arbitrary measure

The Reimann approach implicitly assumes that sets in the domain have sizes that are given by Lesbesgue measure ($\mu(A) = |A|$), whereas the Lesbesgue approach allows sets in the domain to have sizes given by any arbitrary measure μ.

Two-dimensional example

Suppose we want to find a mountain's volume (above sea level).

- **The Riemann approach**: Divide the base of the mountain into a grid of 1 meter squares. Measure the altitude of the mountain at the center of each square.
- **The Lebesgue approach**: Draw a contour map of the mountain, where adjacent contours are 1 meter of altitude apart.




Riemann integral


Lebesgue integral

Notation

Let $(\Omega, \mathcal{F})$ be a measurable space, fixed throughout the discussion.

In this section, we define integral of a measurable function h on $(\Omega, \mathcal{F})$ against arbitrary measure μ. The integral can be written as:

$$\int_\Omega h \, d\mu \,, \qquad \int_\Omega h(\omega) \, d\mu(\omega) \,, \quad \text{or} \quad \int_\Omega h(\omega)\mu(d\,\omega)$$

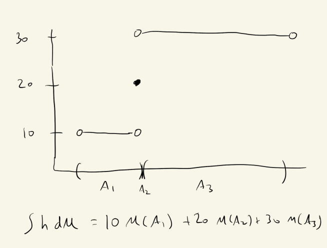
h is measurable if the inverse image of every measurable set is measurable.

Integrals of simple functions

Definition

Let h be simple, say $h = \sum_{i=1}^{r} y_i I_{A_i}$ where the A_i are disjoint sets in $\mathcal{F}$. Then

$$\int_\Omega h \, d\mu := \sum_{i=1}^{r} y_i \, \mu(A_i). \tag{1}$$



Note: The integral of a simple function exists whenever ∞ and $-\infty$ do not both appear in the sum.

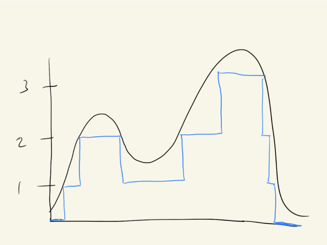
Example: Integrating the Dirichlet function (see notes).

Integrals of non-negative Borel measurable functions

Definition

If h is non-negative Borel measurable, we define

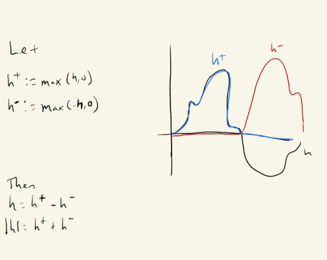
$$\int_\Omega h \, d\mu = \sup\left\{ \int_\Omega s \, d\mu \, : s \quad \text{simple}, \quad 0 \le s \le h \right\}$$



The integral of a non-negative Borel measurable function *always* exists (although it may take on the value $+\infty$).

Integrals of arbitrary Borel measurable functions

Let h be an arbitrary Borel measurable function. We will express an arbitrary Borel measurable function as as the difference of two non-negative Borel measurable functions.



We can define the integral of h by

$$\int_\Omega h\, d\mu = \int_\Omega h^+\, d\mu - \int_\Omega h^-\, d\mu$$

The integral of an arbitrary non-negative Borel function exists so long as it does not take the form $+\infty - \infty$.