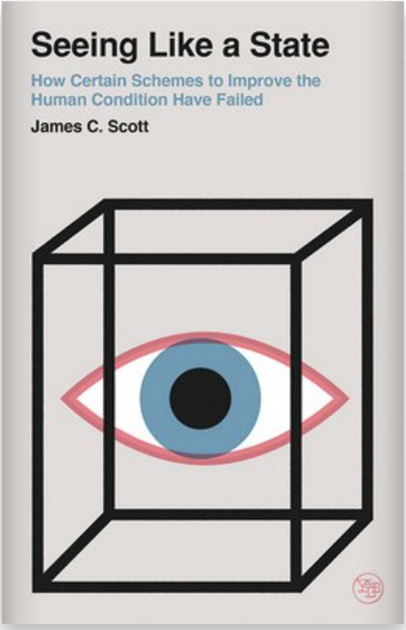
Seeing like a sigma-field

February 3, 2023


Seeing Like a State
How Certain Schemes to Improve the
Human Condition Have Failed
James C. Scott

σ-field

We want to define $\mathbb{E}[Y \mid \mathcal{F}]$ the conditional expectation of a random variable Y with respect to a σ-field $\mathcal{F}$.

Definition

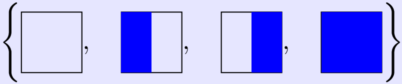
Let $\mathcal{F}$ be a collection of subsets of a set Ω. Then $\mathcal{F}$ is called a **sigma-field** if it satisfies

1. $\Omega \in \mathcal{F}$
2. If $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$.
3. If $A_1, A_2, ... \in \mathcal{F}$ then $\cup_{i=1}^{\infty} A_i \in \mathcal{F}$.

that is, if $\Omega \in \mathcal{F}$ and $\mathcal{F}$ is closed under complementation and countable unions.

Example

Let Ω be the unit square, and

$\mathcal{F} =$ 

Conditional expectation w.r.t. a σ-field

There are two conditions for a function h on $(\Omega, \mathcal{F})$ to be the conditional expectation $\mathbb{E}[Y \mid \mathcal{F}]$:

1. [*"Average matching".*] $\int_F Y \, dP = \int_F h \, dP$ for all $F \in \mathcal{F}$.

2. [*"Measurability".*] h is measurable with respect to $\mathcal{F}$.
   (That is, we must have $h^{-1}(B) \in \mathcal{F}$ for all Borel sets $B \in \mathcal{B}(\overline{\mathbb{R}})$).

Example

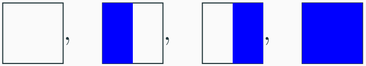
Consider the probability space given by

$$\Omega = [0,1]^2 \qquad \text{(the unit square)}$$
$$\mathcal{F} = \mathcal{B}([0,1]^2)$$
$$P = \texttt{Uniform}$$

and the random variable Y on $(\Omega, \mathcal{F}, P)$ given by

3	1
5	3

Now consider the sub σ-field $\mathcal{G} \subset \mathcal{F}$ given by

$\mathcal{G} = \Big\{$  $\Big\}$

We want to think about the conditional expectation $\mathbb{E}[Y \mid \mathcal{G}]$ given a σ-field. In particular, we want to use this example to illuminate the two conditions for the conditional expectation.

We ask, can $\mathbb{E}[Y \mid \mathcal{G}]$ be ...

9	20

? ✗ This violates the *average matching* condition of conditional expectation. (

3	1
5	3

? ✗ This violates the the *measurability* condition of conditional expectation.[91]

4	2

? ✓ This perfectly meets both conditions.

Seeing like a σ-field

This example highlights two fundamental points about conditional expectations:

1. **The true primitive for (the conditioning set of) a conditional expectation is a σ-field, <u>not</u> a random variable.** No additional random variable is mentioned in this example! Indeed, the conditional expectation with respect to a random variable $E[Y \mid X]$ is a special case of the conditional expectation with respect to a σ-field $E[Y \mid \mathcal{G}]$. In case of $E[Y \mid X]$, the additional random variable X simply plays the intermediary role of *inducing* a particular kind of σ-field over the sample space Ω.
2. **The notion of measurability is <u>not</u> needed simply to avoid odd pathological sets** (like Lebesgue non-measurable sets); it is in fact fundamental to the notion of conditional expectation, even when dealing with very simple collections of sets.