

# Notes on Ash's Probability and Measure Theory

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## Contents

<b>1</b>	<b>Section 1.2: Fields, <math>\sigma</math>-fields, measures</b>	<b>2</b>
1.1	Sec 1.2.1-1.2.2: Fields and $\sigma$ -fields . . . . .	2
1.2	Sec 1.2.3-1.2.8: Measures, related set functions, and their properties . . . . .	2

## 1 Section 1.2: Fields, $\sigma$ -fields, measures

### 1.1 Sec 1.2.1-1.2.2: Fields and $\sigma$ -fields

Fields and  $\sigma$ -fields are important because they are the domain of measures. Here are some definitions.

**Definition 1.1.1.** Let  $\mathcal{F}$  be a collection of subsets of a set  $\Omega$ . Then  $\mathcal{F}$  is called a **field** if

- a)  $\Omega \in \mathcal{F}$
- b) If  $A \in \mathcal{F}$ , then  $A^c \in \mathcal{F}$ .
- c) If  $A_1, \dots, A_n \in \mathcal{F}$  then  $\cup_{i=1}^n A_i \in \mathcal{F}$ .

that is, if  $\Omega \in \mathcal{F}$  and  $\mathcal{F}$  is closed under complementation and finite unions.  $\triangle$

**Definition 1.1.2.** Let  $\mathcal{F}$  be a collection of subsets of a set  $\Omega$ . Then  $\mathcal{F}$  is called a **sigma-field** if it satisfies Definition 1.1.1 after replacing condition c) with

- c') If  $A_1, A_2, \dots \in \mathcal{F}$  then  $\cup_{i=1}^{\infty} A_i \in \mathcal{F}$ .

that is, if  $\Omega \in \mathcal{F}$  and  $\mathcal{F}$  is closed under complementation and *countable* unions.  $\triangle$

**Example 1.1.1.**  $\mathcal{F} = \{\emptyset, \Omega\}$  is the smallest  $\sigma$ -field on  $\Omega$ .  $\triangle$

**Example 1.1.2.**  $\mathcal{F} = 2^\Omega$ , i.e. the set of all subsets of  $\Omega$ , is the largest  $\sigma$ -field on  $\Omega$ .  $\triangle$

**Example 1.1.3.** If  $A \in \Omega$  is non-empty, then  $\mathcal{F} = \{\emptyset, A, A^c, \Omega\}$  is the smallest  $\sigma$ -field containing  $A$ .  $\triangle$

**Question 1.1.1.** Let  $A_1, \dots, A_n$  be subsets of  $\Omega$ . Describe  $\sigma(\{A_1, \dots, A_n\})$ , the smallest  $\sigma$ -field containing  $A_1, \dots, A_n$ . Also describe the number of sets in  $\mathcal{F}$ . *This is Ash's Problem 1.2.8. For answer, see GoodNotes.*  $\triangle$

**Example 1.1.4.** What is an example of a collection that is a *field*, but not a  $\sigma$ -field?

Let  $\Omega = \mathbb{R}$  and  $\mathcal{F}_0 = \{\text{finite disjoint unions of } [a, b), a \neq b\}$ . Then  $\mathcal{F}_0$  is a field, as can be easily verified. But  $\mathcal{F}_0$  is not a  $\sigma$ -field. Note that if  $A_n = [0, \frac{1}{n})$ , then  $\bigcap_{n=1}^{\infty} A_n = \{0\} \notin \mathcal{F}_0$   $\triangle$

**Remark 1.1.1.** Ash says that there is a type of reasoning that occurs so often in problems involving  $\sigma$ -fields that it deserves explicit mention. It is called the *good sets* principle. See GoodNotes.  $\triangle$

### 1.2 Sec 1.2.3-1.2.8: Measures, related set functions, and their properties

**Definition 1.2.1.** A **measure** on a  $\sigma$ -field  $\mathcal{F}$  is a non-negative, extended real-valued function  $\mu$  on  $\mathcal{F}$  such that whenever  $A_1, A_2, \dots$  form a finite or countably infinite collection of disjoint sets in  $\mathcal{F}$ , we have countable additivity; that is,

$$\mu\left(\bigcup_n A_n\right) = \sum_n \mu(A_n)$$

$\triangle$

**Definition 1.2.2.** A **probability measure** is a measure (Definition 1.2.1) where  $\mu(\Omega) = 1$ .  $\triangle$

**Remark 1.2.1.** (*Measure-like functions on fields*) A measure-like set function can be defined on *fields* as well as *sigma-fields* if the countable additivity condition is taken to hold whenever a countable unions *does* happen to still be in the field.  $\triangle$

**Remark 1.2.2.** Ash additionally assumes that a measure does not take  $\mu(A) = \infty$  or  $\mu(A) = -\infty$  for all  $A \in \mathcal{F}$ . From this, we automatically obtain  $\mu(\emptyset) = 0$ . For  $\mu(A) < \infty$  for some  $A$ , and by considering the sequence  $A, \emptyset, \emptyset, \dots$ , we have that  $\mu(\emptyset) = 0$  by countable additivity.  $\triangle$

## References