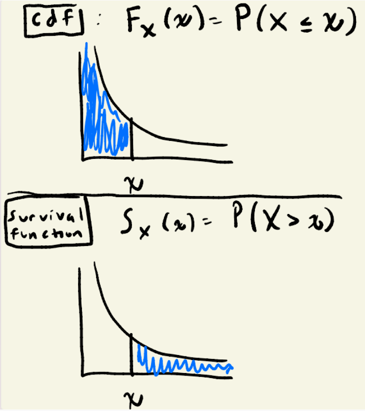
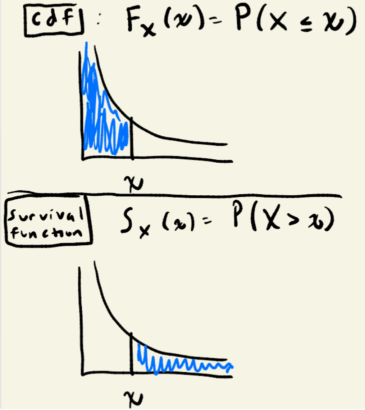
Integrating survival functions

September 9, 2022

Survival functions



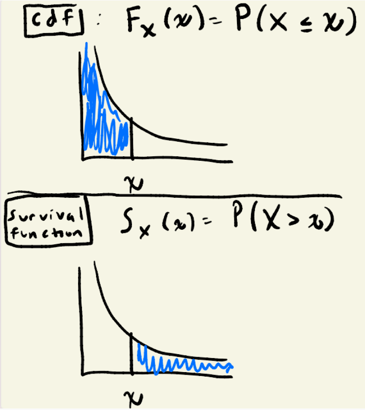
Survival functions



Question: What is the meaning of

$$\int_0^\infty S_X(x)\,dx = \int_0^\infty P(X > x)\,dx$$

Survival functions



Answer: If X is non-negative,

$$\int_0^\infty P(X > x)\, dx = \mathbb{E}[X] \quad !!!$$

Why?

We will use measure theory to make the relationship intuitive.

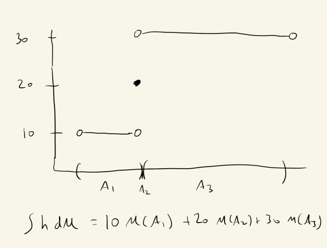
Review: General integration

Integrals of simple functions

Definition

Let h be simple, say $h = \sum_{i=1}^{r} y_i I_{A_i}$ where the A_i are disjoint sets in $\mathcal{F}$. Then

$$\int_\Omega h \, d\mu \; := \sum_{i=1}^{r} y_i \, \mu(A_i). \tag{1}$$



Note: The integral of a simple function exists whenever ∞ and $-\infty$ do not both appear in the sum.

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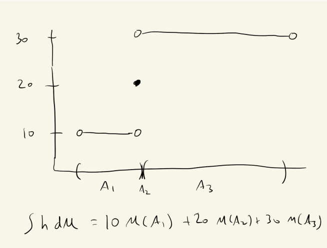

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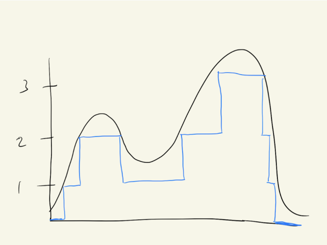
Note: An example of when we'd want to use some μ other than Lebesgue measure will come up shortly!

Integrals of non-negative Borel measurable functions

Definition

If h is non-negative Borel measurable, we define

$$\int_\Omega h \, d\mu = \sup\left\{ \int_\Omega s \, d\mu \ : s \quad \text{simple}, \quad 0 \leq s \leq h \right\}$$



The integral of a non-negative Borel measurable function *always* exists (although it may take on the value $+\infty$).

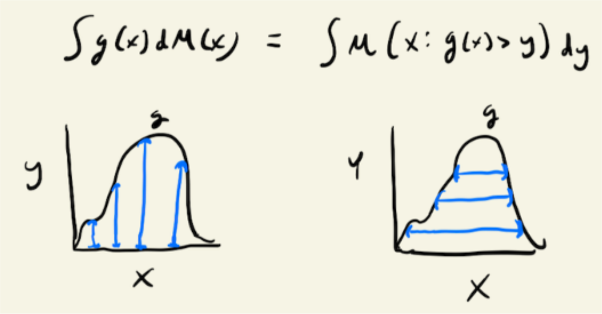
Resolution

Different ways to measure "area under the curve"

Example 13.5.3. *(Area under the curve can be obtained by either horizontal or vertical sections.)* [Durrett, 2010, Exercise 1.7.2] Let $g \geq 0$ be a measurable function on a sigma-finite measure space $(\Omega, \mathcal{F}, \mu)$. Use the classical Fubini-Tonelli Theorem (Thm. 13.4.1) to conclude that

$$\int_\Omega g \, d\mu = (\mu \times \lambda)\{(x,y) : 0 \leq y < g(x)\} = \int_0^\infty \mu(\{x : g(x) > y\}) \, dy$$

for some choice of measure λ.



Resolution

Taking $g(x) = x$ and $\mu = P_X$ to be a probability measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, we obtain

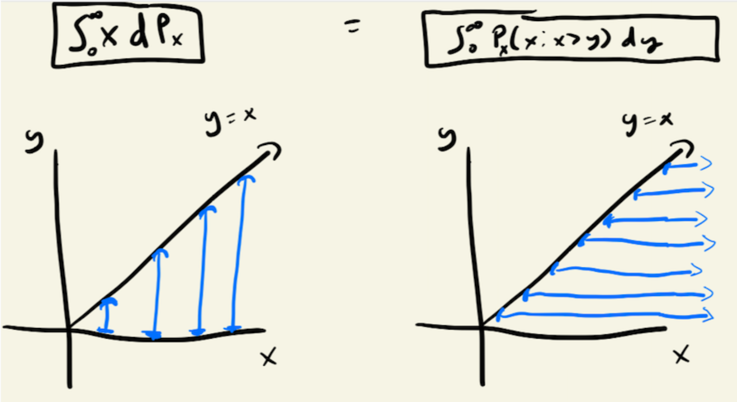


Figure 1: *The expected value of a non-negative random variable equals the integral of its survival function.* Note from the left that the expected value can be seen as the area under $y = x$ if the x-axis is measured with P_X.