Notes on Ash's Probability and Measure Theory

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1 Section 1.1: Some notes on set theory

Definition 1.0.1. If $A_1 \subset A_2 \subset ...$ and $\cup_{n=1}^{\infty} A_n = A$, we say that the A_n form a **increasing** sequence of sets with limit A or that the A_n increase to A; we write $A_n \uparrow A$. If $A_1 \supset A_2 \supset ...$ and $\cap_{n=1}^{\infty} A_n = A$, we say that the A_n form a **decreasing** sequence of sets with limit A or that the A_n decrease to A; we write $A_n \downarrow A$. $\triangle$

Now some remarks on representing unions as disjoint unions.

Remark 1.0.1. If $A_1, A_2, ...$ are subsets of some set Ω, then

$$\bigcup_{n=1}^{\infty} A_n = \dot{\bigcup_{n=1}^{\infty}} \left(A_n \cap A_{n-1}^c \cap ... \cap A_1^c \right) \tag{1.0.1}$$

In other words, any union can be re-represented as a disjoint union. This is useful because measures are countably additive on disjoint sets, so we prefer to work with collections of disjoint sets. $\triangle$

Remark 1.0.2. If $A_n \uparrow A$, then (1.0.1) becomes

$$\bigcup_{n=1}^{\infty} A_n = \dot{\bigcup_{n=1}^{\infty}} \left(A_n - A_{n-1} \right) \tag{1.0.2}$$

This is because $A_{n-1} \subset A_n$, so $A_{n-1}^c \supset A_n^c$ by contraposition. $\triangle$

2 Section 1.2: Fields, σ-fields, measures

2.1 Sec 1.2.1-1.2.2: Fields and σ-fields

Fields and σ-fields are important because they are the domain of measures. Here are some definitions.

Definition 2.1.1. Let $\mathcal{F}$ be a collection of subsets of a set Ω. Then $\mathcal{F}$ is called a **field** (or *algebra*) if

a) $\Omega \in \mathcal{F}$

b) If $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$.

c) If $A_1, ...A_n \in \mathcal{F}$ then $\cup_{i=1}^{n} A_i \in \mathcal{F}$.

that is, if $\Omega \in \mathcal{F}$ and $\mathcal{F}$ is closed under complementation and finite unions. $\triangle$

Definition 2.1.2. Let $\mathcal{F}$ be a collection of subsets of a set Ω. Then $\mathcal{F}$ is called a **sigma-field** (or *sigma-algebra*) if it satisfies Definition 2.1.1 after replacing condition c) with

c') If $A_1, A_2, ... \in \mathcal{F}$ then $\cup_{i=1}^{\infty} A_i \in \mathcal{F}$.

that is, if $\Omega \in \mathcal{F}$ and $\mathcal{F}$ is closed under complementation and *countable* unions. $\triangle$

Example 2.1.1. $\mathcal{F} = \{\emptyset, \Omega\}$ is the smallest σ-field on Ω. $\triangle$

Example 2.1.2. $\mathcal{F} = 2^{\Omega}$, i.e. the set of all subsets of Ω, is the largest σ-field on Ω. $\triangle$

Example 2.1.3. If $A \in \Omega$ is non-empty, then $\mathcal{F} = \{\emptyset, A, A^c, \Omega\}$ is the smallest σ-field containing A. $\triangle$

Notation 2.1.1. If $\mathcal{C}$ is a class of sets, the smallest σ-field containing the sets of $\mathcal{C}$ is written as $\sigma(\mathcal{C})$. This is sometimes called the *minimal σ-field over C* or the *σ-field generated by C*. $\triangle$

Question 2.1.1. Let $A_1, ..., A_n$ be subsets of Ω. Describe $\sigma(\{A_1, ..., A_n\})$, the smallest σ-field containing $A_1, ..., A_n$. Also describe the number of sets in $\mathcal{F}$. *This is Ash's Problem 1.2.8. For answer, see GoodNotes.* $\triangle$

Example 2.1.4. What is an example of a collection that is a *field*, but not a *σ-field*?

Let $\Omega = \mathbb{R}$ and $\mathcal{F}_0 = \{\text{finite disjoint unions of right semi-closed intervals } (a, b], a \neq b\}$. Then $\mathcal{F}_0$ is a field, as can be easily verified.[1] But $\mathcal{F}_0$ is <u>not</u> a σ-field. Note that if $A_n = (-\frac{1}{n}, 0]$, then $\bigcap_{n=1}^{\infty} A_n = \{0\} \notin \mathcal{F}_0$. $\triangle$

Remark 2.1.1. A σ-field can also be described as a field that is closed under limits of increasing sequences. For if $A_n \in \mathcal{F}$ and $A_n \uparrow A$, then A is a countable union of sets in $\mathcal{F}$ by definition. Conversely, if $A = \cup_{n=1}^{\infty} A_n$, then set $B_N := \cup_{n=1}^{N} A_n$ and $B_N \uparrow A$. $\triangle$

2.1.1 "Good sets" strategy

Ash says that there is a type of reasoning that occurs so often in problems involving σ-fieldsthat it deserves explicit mention. It is called the *good sets strategy*. Suppose you want to show that all members of a σ-algebra $\mathcal{F}$ have some property P. Define "good sets" as those that satisfy the property

$$\mathcal{G} := \{G \in \mathcal{F} : G \text{ has property } P\}$$

The strategy is then to simply

1. Show $\mathcal{G}$ is a σ-algebra
2. Show $\mathcal{G}$ contains some class $\mathcal{C}$ such that $\mathcal{F} = \sigma(\mathcal{C})$

Then you're done!

Why does this work?

$$\begin{aligned} & \mathcal{C} \subset \mathcal{G} && \text{by 2} \\ \implies & \sigma(\mathcal{C}) \subset \sigma(\mathcal{G}) && \\ \implies & \mathcal{F} \subset \mathcal{G} && \text{by 1,2} \end{aligned}$$

Yet $\mathcal{G} \subset \mathcal{F}$ by definition of $\mathcal{G}$.

So $\mathcal{G} = \mathcal{F}$.

So all sets in $\mathcal{F}$ are good.

In the text, Ash uses this strategy to show that if $\mathcal{C}$ is a class of subsets of Ω, and $A \in \Omega$, then

$$\underbrace{\sigma_\Omega(\mathcal{C}) \cap A}_{\text{take minimal sigma field first, then intersect}} = \underbrace{\sigma_A(\mathcal{C} \cap A)}_{\text{intersect first, then take minimal sigma-field}}$$

For another application, see handwritten homework exercises.

[1] By convention, we also count (a, ∞) as right semi-closed for $-\infty \leq a < \infty$, which is necessary for the σ-field to be closed under complements.

2.2 Sec 1.2.3-1.2.4: Measures

Definition 2.2.1. A **measure** on a σ-field $\mathcal{F}$ is a non-negative, extended real-valued function μ on $\mathcal{F}$ such that whenever $A_1, A_2, ...$ form a finite or countably infinite collection of disjoint sets in $\mathcal{F}$, we have countable additivity; that is,

$$\mu\left(\dot{\bigcup_n} A_n\right) = \sum_n \mu(A_n)$$

$\triangle$

Definition 2.2.2. A **probability measure** is a measure (Definition 2.2.1) where $\mu(\Omega) = 1$. $\triangle$

Remark 2.2.1. Ash additionally assumes that a measure does not take $\mu(A) = \infty$ or $\mu(A) = -\infty$ for all $A \in \mathcal{F}$. From this, we automatically obtain $\mu(\emptyset) = 0$. For $\mu(A) < \infty$ for some A, and by considering the sequence $A, \emptyset, \emptyset, ...$, we have that $\mu(\emptyset) = 0$ by countable additivity. $\triangle$

Example 2.2.1. Let $\Omega = \{x_1, x_2, ...\}$ be a finite or countably infinite set. Let $p_1, p_2, ...$ be non-negative reals. Let $\mathcal{F} = 2^\Omega$. Define

$$\mu(A) = \sum_{x_i \in A} p_i \quad \text{for all } A \in \mathcal{F}$$

Then μ is a measure on $\mathcal{F}$. We might call it the “point weighting" measure.

- If $p_i \equiv 1 \; \forall\, i$, then μ is called the **counting measure**.
- If $\sum_i p_i = 1$, then μ is a probability measure.

$\triangle$

Example 2.2.2. (*Lesbesgue measure*) Define μ such that

$$\mu(a, b] = b - a \quad \forall\, a, b \in \mathbb{R} : b > a$$

As we will see in Section 3, this requirement determines μ on a large collection of sets, the Borel Sets $\mathcal{B}(\mathbb{R})$, defined as the smallest σ-field of subsets of $\mathbb{R}$ containing all intervals $(a, b] \subset \mathbb{R}$.

We may alternately characterize $\mathcal{B}(\mathbb{R})$ as the smallest σ-field containing

- all intervals $[a, b],\ a, b \in \mathbb{R}$.
- all intervals $(a, b),\ a, b \in \mathbb{R}$
- all intervals $(a, \infty),\ a \in \mathbb{R}$.
- all intervals $[a, \infty),\ a \in \mathbb{R}$.
- all intervals $(-\infty, b),\ b \in \mathbb{R}$.
- all intervals $(-\infty, b],\ b \in \mathbb{R}$.
- all intervals $[a, b],\ a, b \in \mathbb{R}$.
- all open sets of $\mathbb{R}$.[2]
- all closed sets of $\mathbb{R}$.[3]

[2] Recall that an open set is a countable union of open intervals.
[3] Recall that a set is open iff its complement is closed.

To illustrate these equivalences, let us equate the first two conditions. That is, let us show that a σ-field contains all open intervals (a, b) iff it contains all right semi-closed intervals $(a, b]$. To see this, simply note

$$(a, b] = \bigcap_{n=1}^{\infty} \left(a, b + \frac{1}{n}\right)$$

and

$$(a, b) = \bigcup_{n=1}^{\infty} \left(a, b - \frac{1}{n}\right]$$

$\triangle$

Question 2.2.1. The text gives another description of the Borel sets $\mathcal{B}(\mathbb{R})$ as the smallest σ-field containing $\mathcal{F}_0$, the field of disjoint unions of right semi-closed intervals $(a, b]$. Can we make the same statement about the field of finite disjoint unions of left semi-closed intervals? $\triangle$

2.3 Sec 1.2.5-1.2.6: Generalizations of measures, and their properties

The text considers some generalizations of measures that can be obtained by restricting the domain to a field, by assuming merely finite additivity, or by allowing the range to be extended reals ($\bar{\mathbb{R}}$) instead of non-negative extended reals ($\bar{\mathbb{R}}_{\geq 0}$).

Remark 2.3.1. The first two relaxations above often go together. However, a countably additive function can be defined on a *field* (rather than σ-field) if the condition is taken to hold whenever a countable union *does* happen to still be in the field. In my notes, I will simply things by assuming that countably additive functions are always defined on σ-fields. $\triangle$

	Range	
	non-negative	**signed**
finitely additive	μ_0	$\tilde{\mu}_0$
countably additive	μ measure	$\tilde{\mu}$ signed measure

Table 1: Notation for generalizations of measure (For assumed domain in each case, see Remark 2.3.1.)

In Table 1, we introduce some notation to try to clarify more immediately when results hold. Note the relations[4]

$$\{\mu\} \subset \{\mu_0\}, \{\tilde{\mu}\} \subset \{\tilde{\mu}_0\}.$$

Using the notation in Table 1, we rewrite Theorem 1.2.5 of the text:

Theorem 2.3.1. *Let $\tilde{\mu}_0$ be a finitely additive set function on the field $\mathcal{F}_0$. Then*

a) $\tilde{\mu}_0(\emptyset) = 0$

b) $\tilde{\mu}_0(A \cup B) + \tilde{\mu}_0(A \cap B) = \tilde{\mu}_0(A) + \tilde{\mu}_0(B)$ *for all* $A, B \in \mathcal{F}_0$.

c) If $A, B \in \mathcal{F}_0$ *and* $B \subset A$*, then*

$$\tilde{\mu}_0(A) = \tilde{\mu}_0(B) + \tilde{\mu}_0(A - B) \quad \textit{(piece-to-whole)}$$

Moreover, under non-negativity,

$$\mu_0(A) \leq \mu_0(B) \quad \textit{(monotonicity)}$$

[4]So, for example, if something holds for $\tilde{\mu}_0$, it holds for μ. A simple mnemonic is that adding stuff to the notation generalizes the function.

d) *Subadditivity holds, i.e.*

$$\mu_0(\cup_{i=1}^n A_i) \le \sum_{i=1}^n \mu_0(A_i)$$
$$\mu(\cup_{i=1}^\infty A_i) \le \sum_{i=1}^\infty \mu(A_i)$$

Proof. We prove Theorem 2.3.1 (b). The rest is an exercise for the reader (or see the text).
First, we break things into disjoint pieces

$$A = \Big(A \cap B\Big) \dot\bigcup \Big(A \cap B^c\Big) \qquad \Longrightarrow \tilde{\mu}_0(A) = \tilde{\mu}_0(A \cap B) + \tilde{\mu}_0(A \cap B^c) \tag{1}$$
$$B = \Big(A \cap B\Big) \dot\bigcup \Big(A^c \cap B\Big) \qquad \Longrightarrow \tilde{\mu}_0(B) = \tilde{\mu}_0(A \cap B) + \tilde{\mu}_0(A^c \cap B) \tag{2}$$
$$A \cup B = \Big(A \cap B\Big) \dot\bigcup \Big(A \cap B^c\Big) \dot\bigcup \Big(A^c \cap B\Big) \qquad \Longrightarrow \tilde{\mu}_0(A \cup B) = \tilde{\mu}_0(A \cap B) + \tilde{\mu}_0(A \cap B^c) + \tilde{\mu}_0(A^c \cap B) \tag{3}$$

Summing (1) and (2), we obtain

$$\tilde{\mu}_0(A) + \tilde{\mu}_0(B) = 2\tilde{\mu}_0(A \cap B) + \tilde{\mu}_0(A \cap B^c) + \tilde{\mu}_0(A^c \cap B).$$

We use (3) to simplify the RHS, and the result follows. □

Remark 2.3.2. In the proof of Theorem 2.3.1 (b), note that we use a common strategy – breaking sets into disjoint pieces so that we can apply the assumed (finite or countable) additivity of the set function. △

Remark 2.3.3. Being able to work with these generalizations will be important in Section 3 on extension of measures. In particular, it will help us show that we can construct the Lesbesgue measure on the Borel sets. △

2.4 Sec 1.2.7-1.2.8: Continuity of countably additive set functions

Countably additive set functions have a basic continuity property. Continuity of measure is a special case.

Theorem 2.4.1. *Let $\widetilde{\mu}$ be a countably additive set function on the σ-field $\mathcal{F}$. Then*

a) *(continuity from below) If $A_1, A_2, ... \in \mathcal{F}$ and $A_n \uparrow A$, then $\widetilde{\mu}(A_n) \to \widetilde{\mu}(A)$ as $n \to \infty$.*

b) *(continuity from above) If $A_1, A_2, ... \in \mathcal{F}$, $A_n \downarrow A$, and $\widetilde{\mu}(A_n)$ is finite, then $\widetilde{\mu}(A_n) \to \widetilde{\mu}(A)$ as $n \to \infty$.*

Proof. We prove continuity from below, and leave continuity from above as an exercise to the reader (or see text). First let us assume that all $\widetilde{\mu}(A_n)$ are finite (*). Then

$$\begin{aligned}
A &= A_1 \dot\cup A_2 - A_1 \dot\cup A_3 - A_2 \dot\cup ... && \text{by (1.0.2)}\\
\Longrightarrow \widetilde{\mu}(A) &= \widetilde{\mu}(A_1) + \widetilde{\mu}(A_2 - A_1) + \widetilde{\mu}(A_3 - A_2) + ... && \text{(countable additivity)}\\
&= \widetilde{\mu}(A_1) + \widetilde{\mu}(A_2) - \widetilde{\mu}(A_1) + \widetilde{\mu}(A_3) - \widetilde{\mu}(A_2) + ... && \text{(Theorem 2.3.1 c), (*)}\\
&= \lim_{n\to\infty} \widetilde{\mu}(A_n)
\end{aligned}$$

Now suppose $\widetilde{\mu}(A_n) = \infty$ for some n. So write

$$\begin{aligned}
A &= A_n \cup A - A_n && \text{(increasing sequence)} \\
\implies \widetilde{\mu}(A) &= \widetilde{\mu}(A_n) + \widetilde{\mu}(A - A_n) && \text{(countable additivity)} \\
&= \infty + \widetilde{\mu}(A - A_n)
\end{aligned}$$

So $\widetilde{\mu}(A) = \infty$.[5] Replace A by A_k for any $k \geq n$ to also find $\widetilde{\mu}(A_k) = \infty$ for all $k \geq n$ and the result follows.

Finally suppose $\widetilde{\mu}(A_n) = -\infty$ for some n. Then the result follows in the same way as for $\widetilde{\mu}(A_n) = \infty$.

□

Remark 2.4.1. The logic of the proof of Theorem 2.4.1 under the finiteness assumption is as follows. First, we re-represent the union as a disjoint union (the form is particularly simple since the sets are increasing). This allows us to apply countable additivity. Then we apply the part-to-whole decomposition (and the subtraction is defined under the finiteness assumption). △

Remark 2.4.2. In proving Theorem 2.4.1 for the case where $\mu(A_n) = \infty$ for some n, it is tempting to make the simpler argument

$$\begin{aligned}
\mu(A) &\geq \mu(A_n) && \text{(monotonicity)} \\
\mu(A_k) &\geq \mu(A_n) && \text{(monotonicity)}
\end{aligned}$$

for $k \geq n$. But recall from Theorem 2.3.1 that monotonicity only holds under non-negativity, and the theorem statement is more general, applying to *signed* set functions as well. △

3 Section 1.3: Extension of measures

[5] Note that we cannot have $\widetilde{\mu}(A - A_n) = -\infty$, because that would violate additivity.