Review of measures

March 23, 2022

Definition

Definition

Let $\mathcal{F}$ be a collection of subsets of a set Ω. Then $\mathcal{F}$ is called a **sigma-field** (or *sigma-algebra*) if it satisfies

1. $\Omega \in \mathcal{F}$
2. If $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$.
3. If $A_1, A_2, \ldots \in \mathcal{F}$ then $\cup_{i=1}^{\infty} A_i \in \mathcal{F}$.

that is, if $\Omega \in \mathcal{F}$ and $\mathcal{F}$ is closed under complementation and countable unions.

Definition

Definition

Let $\mathcal{F}$ be a collection of subsets of a set Ω. Then $\mathcal{F}$ is called a **sigma-field** (or *sigma-algebra*) if it satisfies

1. $\Omega \in \mathcal{F}$
2. If $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$.
3. If $A_1, A_2, \ldots \in \mathcal{F}$ then $\cup_{i=1}^{\infty} A_i \in \mathcal{F}$.

that is, if $\Omega \in \mathcal{F}$ and $\mathcal{F}$ is closed under complementation and countable unions.

Definition

A **measure** on a σ-field $\mathcal{F}$ is a non-negative, extended real-valued function μ on $\mathcal{F}$ such that whenever $A_1, A_2, \ldots$ form a finite or countably infinite collection of disjoint sets in $\mathcal{F}$, we have countable additivity; that is,

$$\mu\left(\dot{\bigcup_n} A_n\right) = \sum_n \mu(A_n)$$

Examples

Example

Let Ω be any set. Fix $x_0 \in \Omega$. For any $A \in \mathcal{F}$ define $\mu(A) = 1$ if $x_0 \in A$ and $\mu(A) = 0$ if $x_0 \notin A$. Then μ may be called the **unit mass** concentrated at x_0.

Examples

Example

Let Ω be any set. Fix $x_0 \in \Omega$. For any $A \in \mathcal{F}$ define $\mu(A) = 1$ if $x_0 \in A$ and $\mu(A) = 0$ if $x_0 \notin A$. Then μ may be called the **unit mass** concentrated at x_0.

Example

Let $\Omega = \{x_1, x_2, ...\}$ be a finite or countably infinite set. Let $p_1, p_2, ...$ be non-negative reals. Define

$$\mu(A) = \sum_{x_i \in A} p_i \quad \text{for all } A \in \mathcal{F}$$

Then μ is a measure on $\mathcal{F}$. We might call it the "point weighting" measure.

- If $p_i \equiv 1 \; \forall \, i$, then μ is called the **counting measure**.
- If $\sum_i p_i = 1$, then μ is a **discrete probability measure**.

More examples

Example

Define μ such that

$$\mu(a, b] = b - a \quad \forall\, a, b \in \mathbb{R} : b > a$$

This requirement determines μ on a large collection of sets, a sigma-field called the Borel Sets $\mathcal{B}(\mathbb{R})$, defined as the smallest σ-field of subsets of $\mathbb{R}$ containing all intervals. The measure is called **Lesbesgue measure**.

More examples

Example

Define μ such that

$$\mu(a, b] = b - a \quad \forall\, a, b \in \mathbb{R} : b > a$$

This requirement determines μ on a large collection of sets, a sigma-field called the Borel Sets $\mathcal{B}(\mathbb{R})$, defined as the smallest σ-field of subsets of $\mathbb{R}$ containing all intervals. The measure is called **Lesbesgue measure**.

Example

Let F be a *distribution function* on $\mathbb{R}$; that is, $F : \mathbb{R} \to \mathbb{R}$ is a map which is increasing and right continuous.

Define

$$\mu(a, b] = F(b) - F(a)$$

This is called a **Lesbesgue-Stieltjes measure**.