Filtrations and Martingales

March 3, 2023

Review: σ-field

Martingales depend on filtrations, which depend on σ-fields.

Definition

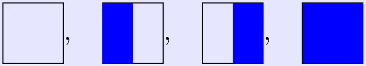
Let $\mathcal{F}$ be a collection of subsets of a set Ω. Then $\mathcal{F}$ is called a **sigma-field** if it satisfies

1. $\Omega \in \mathcal{F}$
2. If $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$.
3. If $A_1, A_2, \ldots \in \mathcal{F}$ then $\cup_{i=1}^{\infty} A_i \in \mathcal{F}$.

that is, if $\Omega \in \mathcal{F}$ and $\mathcal{F}$ is closed under complementation and countable unions.

Example

Let Ω be the unit square, and

$\mathcal{F} = \Bigg\{$  $\Bigg\}$

Filtrations

Definition

Let $(\Omega, \mathcal{F}, P)$ be a probability space.

Then a **filtration** $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots$ is an increasing sequence of sub σ-fields of $\mathcal{F}$.

A simple strategy for constructing a filtration

A simple method for constructing a filtration is as follows.

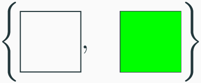
1. Construct a sequence $\{\Omega_n\}$ of increasingly refined partitions of Ω.
2. Define a filtration by setting $\mathcal{F}_n = \sigma(\Omega_n)$, i.e. each σ-field $\mathcal{F}_n$ consists of the sets that can be formed by taking unions of some subset of the cells in the partition Ω_n.

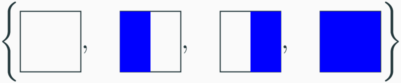
Some examples of filtrations formed by this strategy include:

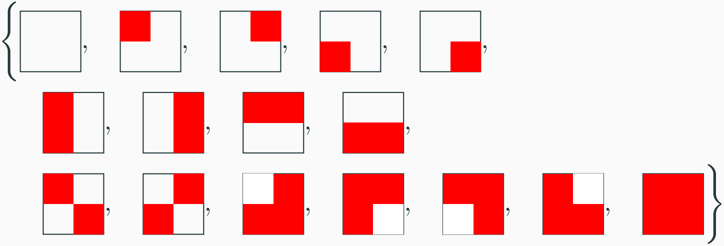
1. Take Ω to be the unit square. Form increasingly refined partitions by splitting cells of Ω_{n-1} in half, vertically if n odd and horizontally if n even.
2. Let Ω be the space of binary-valued sequences. Form increasingly refined partitions by having Ω_n group together all sequences whose values match along the first n coordinates.

Example of a filtration

Let Ω be the unit square. Define $\{\mathcal{F}_n\}$ by

$\mathcal{F}_0 =$ 

$\mathcal{F}_1 =$ 

$\mathcal{F}_2 =$ 

$\vdots \qquad \vdots$

Then $\{\mathcal{F}_n\}$ is a filtration.

Martingales

Definition

Let $\{X_n\}$ be a sequence of random variables and $\{\mathcal{F}_n\}$ be a filtration. If

1. The sequence $\{X_n\}$ is adapted to the filtration $\{\mathcal{F}_n\}$.
2. Each X_n is integrable.
3. $\mathbb{E}[X_{n+1} \mid \mathcal{F}_n] = X_n \quad$ for all n.

Then we say that $\{X_n\}$ is a **martingale** relative to $\{\mathcal{F}_n\}$.

If, in the last definition, $=$ is replaced by $\leq$ or $\geq$, then $\{X_n\}$ is said to be a **supermartingale** or **submartingale**, respectively.

Examples of Martingales

1. Random walks.
2. Polya Urn process.
3. Increasing information process. (See next slides.)

Concrete example of increasing information process

We define a probability space as follows:

$$\Omega = [0,1]^2 \quad \text{(the unit square)}$$
$$\mathcal{F} = \mathcal{B}([0,1]^2)$$
$$P = \text{Uniform distribution}$$

To construct a filtration, we begin by constructing a sequence $\{\Omega_n\}$ of increasingly refined partitions of Ω. We set

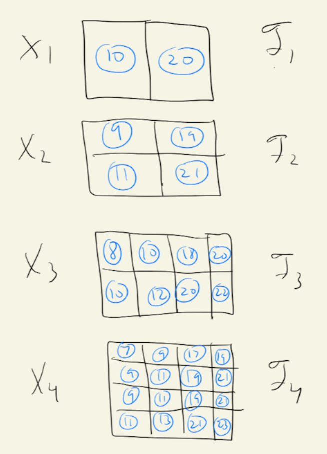
$$\Omega_0 = \left\{\emptyset, \Omega\right\}$$

Ω_n as the partition of Ω formed by splitting cells of Ω_{n-1} in half vertically if n odd and horizontally if n even

We then define a filtration by setting $\mathcal{F}_n = \sigma(\Omega_n)$, i.e. each σ-field $\mathcal{F}_n$ consists of the sets that can be formed by taking unions of some subset of the cells in the partition Ω_n.

Concrete example of increasing information process

Now let Y be an integrable random variable on $(\Omega, \mathcal{F}, P)$, and define $X_n = \mathbb{E}[Y \mid \mathcal{F}_n]$. For example:



For the first few elements in the sequence ($n = 1, \ldots, 4$), we show the values of the random variable $X_n \triangleq \mathbb{E}[Y \mid \mathcal{F}_n]$ over each element of a partition Ω_n from which the σ-field $\mathcal{F}_n$ is generated.

The Figure illustrates the martingale property: $X_n = \mathbb{E}[X_{n+1} \mid \mathcal{F}_n]$. In particular, we can see that both conditions for conditional expectation are satisfied:

- **average matching**: for any set in $\mathcal{F}_n$ (which is a rectangle or union of rectangles in the partition Ω_n), the value of X_n is the arithmetic average of the values of the corresponding subrectangles in $\mathcal{F}_{n+1}$.
- **measurability**: inverse images of any realization $X_n = x_n$ or set of realizations $X_n \in \{x_n^{(1)}, \ldots x_n^{(k)}\}$ exist in $\mathcal{F}_n$.

The Figure also illustrates what we mean by an "increasing information" process:

- As n increases, X_n gives more information about the values of Y on the square Ω.

  (In particular, X_n gives us the average value of Y over a grid of sub-rectangles that is a refinement of the corresponding grid over which X_{n-1} gave averages. Every time we split a rectangle in half, the original value a splits into two values b and c such that $a = \frac{b+c}{2}$. There are of course infinitely many possible choices for b and c given a, and we don't know what those values are until we observe the next random variable in the sequence.)

The increasing information process is a martingale

Proposition

Let Y be an integrable random variable and $\mathcal{F}_n$ be a filtration on $(\Omega, \mathcal{F}, P)$. Define

$$X_n \triangleq \mathbb{E}[Y \mid \mathcal{F}_n]$$

Then X_n is a martingale with respect to $\mathcal{F}_n$.

Proof.

The first two conditions of the definition follow immediately. For the third condition, note that

$$\begin{aligned}
\mathbb{E}[X_{n+1} \mid \mathcal{F}_n] &= \mathbb{E}\big[\mathbb{E}[Y \mid \mathcal{F}_{n+1}] \mid \mathcal{F}_n\big] && \text{def. } \{X_n\} \\
&= \mathbb{E}[Y \mid \mathcal{F}_n] && \text{“Smaller } \sigma\text{-field always wins”, a.k.a. the tower property} \\
&= X_n && \text{def. } \{X_n\}
\end{aligned}$$

$\square$