Counting Measure and Stationary Time Series

June 16, 2023

Measure theory is convenient in unifying various kinds of random variables.

In this lightning chat, I give a real example that came up when investigating time series.

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Measure spaces

A measure space $(\Omega, \mathcal{F}, \mu)$ consists of

1. Some set Ω.
2. A sigma-field $\mathcal{F}$, which specifies *measurable* subsets of Ω.
3. A measure μ.

Example measures

- *Lebesgue measure.* This gives the length of intervals

$$\mu(a, b] = b - a \quad \forall\, a, b \in \mathbb{R} : b > a$$

  but can be used to measure other kinds of sets.
- *Counting measure*. This gives the number of points a set.

Integrals of simple functions

Definition

Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. Then s is a **simple function** if we can write

$$s = \sum_{i=1}^{n} c_i 1_{A_i}$$

where 1 is the indicator function and the A_i are disjoint sets in $\mathcal{F}$.

Definition

Let $s = \sum_{i=1}^{n} c_i 1_{A_i}$ be a simple function. Then the **integral of a simple function** is defined as

$$\int_{\Omega} s \, d\mu \ := \sum_{i=1}^{n} c_i \ \mu(A_i). \tag{1}$$

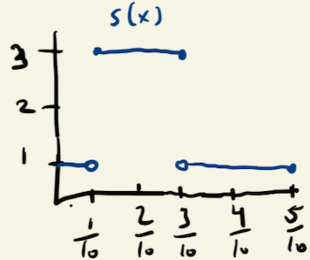
Examples of integrals of simple functions

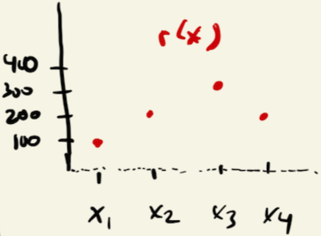
μ = Lesbesgue measure

ν = Counting measure

$$\int s\,d\mu = 1\left(\frac{3}{10}\right) + 3\left(\frac{2}{10}\right)$$

$$\int r\,d\nu = 100(1) + 200(2) + 300(1)$$





Series as integrals against counting measure

Let $\Omega = \{1, 2, 3, ...\}$, $\mathcal{F} = 2^\Omega$ (i.e. all subsets of Ω), and μ be the counting measure.

A real-valued function f on Ω can be written as a sequence of real numbers; we write $f = \{a_n\}, n = 1, 2, \ldots$.

It can be shown that an integral on this space is really a series:

$$\int_\Omega f \; d\mu \; = \sum_{n=1}^{\infty} a_n$$

Why do we care?

We can uniformly apply everything we know about integration regardless of whether our space is Euclidean, discrete, or something more exotic.

In the next section, we'll exploit this fact by applying:

Triangle inequality for integrals

If $\int_\Omega h\, d\mu$ exists, then $\left| \int_\Omega h\, d\mu \right| \leq \int_\Omega |h|\, d\mu$.

Tonelli's theorem

Suppose that $(X, \mathcal{F}, \mu)$ and $(Y, \mathcal{G}, \nu)$ are σ-finite measure spaces. If f is a non-negative measurable function on $X \times Y$ then

$$\int f\, d(\mu \times \nu) = \int \int f(x, y)\, d\nu(y)\; d\mu(x) = \int \int f(x, y)\, d\mu(x)\; d\nu(y)$$

Stationary time series

Stationary time series

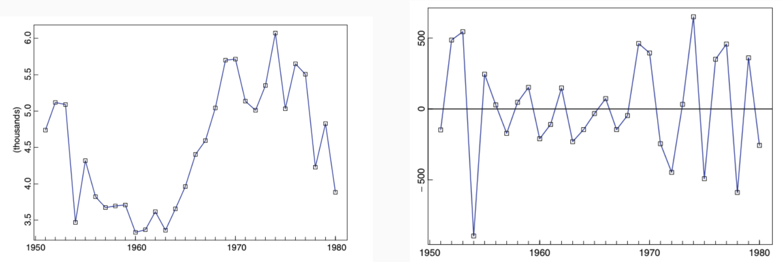
Loosely speaking, a time series $\{X_t, t = 0, \pm 1, \ldots\}$ is said to be stationary if it has statistical properties similar to those of the "time-shifted" series $\{X_{t+h}, t = 0, \pm 1, \ldots\}$ for each integer h.

Definition

A time series $\{X_t\}$ is said to be **covariance stationary** if it has finite variance and

$$\mathbb{E}[X_t] = c \qquad \forall t$$
$$\mathrm{Cov}[X_t, X_{t+h}] = \gamma(h) \qquad \forall t, h$$

Example of stationary time series



(a) Strikes in the U.S.A, 1951-1980 **(b)** Strikes in the U.S.A., after detrending

Figure 1: The time series on the right can be accurately modeled as a stationary time series. Image credits: Brockwell, Peter J., and Richard A. Davis (2016) *Introduction to time series and forecasting.*

Linear processes

A linear process is linearly filtered white noise.

Definition

A process $\{W_t\}$ is called **white noise**, denoted $\{W_t\} \sim \mathrm{WN}(0, \sigma_W^2)$, if it is a sequence of uncorrelated random variables with mean 0 and finite variance σ^2.

Definition

A time series X_t is said to be a **linear process** if it has the representation

$$X_t = \sum_{j=-\infty}^{\infty} \psi_j W_{t-j} \tag{2}$$

for all t, where $W_t \sim WN(0, \sigma_W^2)$, and $\{\psi_j\}$ is a sequence of constants which is absolutely summable ($\sum_{j=0}^{\infty} |\psi_j| < \infty$).

Why do we care?

Every covariance stationary process is either a linear process or can be transformed to one by subtracting a deterministic component.

This important result is known as Wold's decomposition.

Application of general integration theory

Linear processes almost surely converge

We show that the infinite sum in (2) converges almost surely:

$$
\begin{aligned}
\mathbb{E}|X_t| &= \mathbb{E}\left|\sum_{j=-\infty}^{j=\infty} \psi_j\, W_{t-j}\right| \\
&\leq \mathbb{E}\sum_{j=-\infty}^{j=\infty} |\psi_j\, W_{t-j}| && \text{by } \triangle \text{ inequality} \\
&= \sum_{j=-\infty}^{j=\infty} |\psi_j|\, \mathbb{E}|W_{t-j}| && \text{by Tonelli} \\
&\leq \left(\sum_{j=-\infty}^{j=\infty} |\psi_j|\right)\sigma_W \\
&< \infty.
\end{aligned}
$$

Thus, the series defining X_t is almost surely finite.