# Independence Properties of Directed Probabilistic Graphical Models

November 12, 2020

# Table of contents

# Motivation

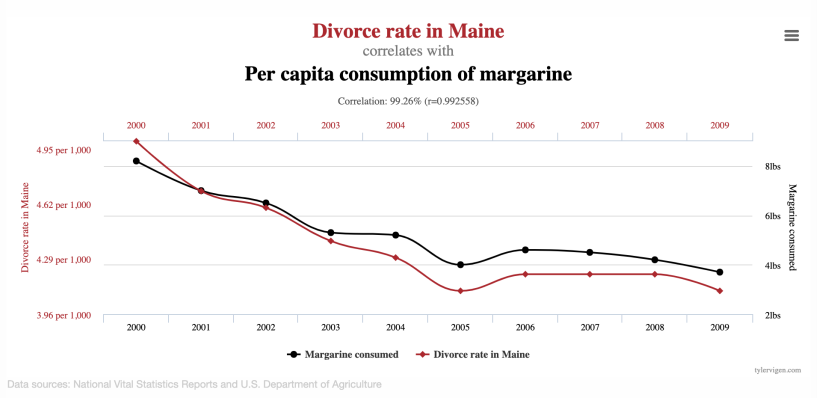
# Introductory Question

**Claim:** Buying margarine is unethical.

# Introductory Question

**Claim:** Buying margarine is unethical.

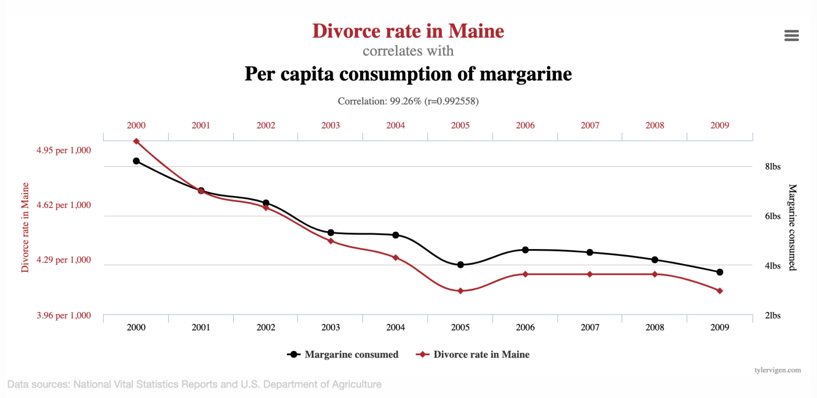
**Argument:**



# Introductory Question

**Claim:** Buying margarine is unethical.

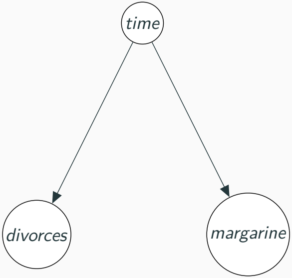
**Argument:**



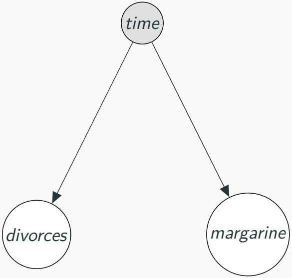

**Q:** What are the problems with this argument?

# Marginal vs. Conditional Independence

## The“third variable” problem



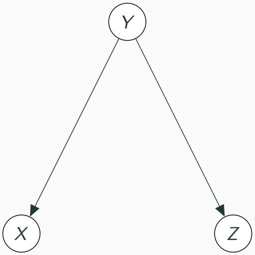

divorces $\not\perp\!\!\!\perp$ margarine




divorces $\perp\!\!\!\perp$ margarine $|$ time

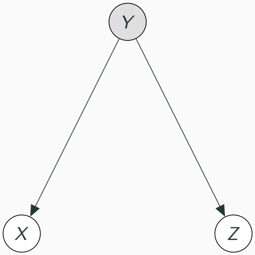
# Marginal vs. Conditional Independence

**Common parent**




$Z \not\perp\!\!\!\perp X$

**Common parent**




$Z \perp\!\!\!\perp X \mid Y$

# Motivation

Consider a Bayesian Hidden Markov Model
You may see statements like: *The future is independent of the past given the current hidden state.*

# Motivation

Consider a Bayesian Hidden Markov Model

You may see statements like: *The future is independent of the past given the current hidden state.*

- How can we know this?

# Motivation

Consider a Bayesian Hidden Markov Model

You may see statements like: *The future is independent of the past given the current hidden state.*

- How can we know this?
- More generally: How can we easily answer queries about (conditional *or* marginal) independence ?

# Motivation

Consider a Bayesian Hidden Markov Model

You may see statements like: *The future is independent of the past given the current hidden state.*

- How can we know this?
- More generally: How can we easily answer queries about (conditional *or* marginal) independence ?

## Joint Distribution

Notating transition matrix $\pi$, emissions parameters $\theta$. hidden states $X$, and observations $Y$, and suppressing hyperparameters, we have

$$\underbrace{p(\pi, \theta, X, Y)}_{\textit{joint density}} = \underbrace{p(\pi)\, p(\theta)}_{\text{prior}} \quad \underbrace{p(X_0) \prod_{t=1}^{T} p_\pi(X_t \mid X_{t-1})\; p_\theta(Y_t \mid X_t)}_{\text{(complete data) likelihood}}$$

# Motivation

Consider a Bayesian Hidden Markov Model

You may see statements like: *The future is independent of the past given the current hidden state.*

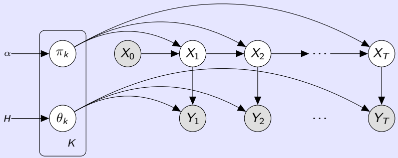
- How can we know this?
- More generally: How can we easily answer queries about (conditional *or* marginal) independence ?

## Joint Distribution

Notating transition matrix $\pi$, emissions parameters $\theta$. hidden states $X$, and observations $Y$, and suppressing hyperparameters, we have

$$\underbrace{p(\pi, \theta, X, Y)}_{\textit{joint density}} = \underbrace{p(\pi)\, p(\theta)}_{\text{prior}} \quad \underbrace{p(X_0) \prod_{t=1}^{T} p_\pi(X_t \mid X_{t-1})\, p_\theta(Y_t \mid X_t)}_{\text{(complete data) likelihood}}$$

## Representation as a *probabilistic graphical model*



# Directed Probabilistic Graphical Models

# Joint distributions

The starting point for a directed probabilistic graphical model is a particular factorization of a joint density:

$$p(X_1, ..., X_n) = \prod_{i=1}^{n} p(X_i \mid \pi_i) \tag{2.1}$$

where the conditioning set $\pi_i$ is referred to as the parents of variable $i$.

( In the intro example, who are the parents of margarine?

# Joint distributions

The starting point for a directed probabilistic graphical model is a particular factorization of a joint density:

$$p(X_1, ..., X_n) = \prod_{i=1}^{n} p(X_i \mid \pi_i) \tag{2.1}$$

where the conditioning set $\pi_i$ is referred to as the parents of variable $i$.

( In the intro example, who are the parents of margarine? In the HMM example, who are the parents of the hidden state $x_t$? Of the observation $y_t$? )

# Joint distributions

The starting point for a directed probabilistic graphical model is a particular factorization of a joint density:

$$p(X_1, ..., X_n) = \prod_{i=1}^{n} p(X_i \mid \pi_i) \tag{2.1}$$

where the conditioning set $\pi_i$ is referred to as the parents of variable $i$.

( In the intro example, who are the parents of margarine? In the HMM example, who are the parents of the hidden state $x_t$? Of the observation $y_t$? )

(2.1) simplifies the factorizations which are *always* true, by the chain rule of probability:

$$p(X_1, ..., X_n) = \prod_{i=1}^{n} p(X_i \mid X_1, .., X_{i-1})$$

# Directed probabilistic graphical models

Once we have specified our desired factorization via (2.1), we can identify it with a directed acyclic graph (DAG) $\mathcal{G} = (E, V)$ by:

- identifying each random variable with a node

# Directed probabilistic graphical models

Once we have specified our desired factorization via (2.1), we can identify it with a directed acyclic graph (DAG) $\mathcal{G} = (E, V)$ by:

- identifying each random variable with a node
- drawing a directed arc from $A$ to $B$ if $A$ is a parent of $B$

# Directed probabilistic graphical models

Once we have specified our desired factorization via (2.1), we can identify it with a directed acyclic graph (DAG) $\mathcal{G} = (E, V)$ by:

- identifying each random variable with a node
- drawing a directed arc from $A$ to $B$ if $A$ is a parent of $B$

We call this representation a directed probabilistic graphical model (or a Bayesian network) .

# Directed probabilistic graphical models

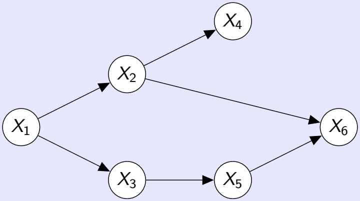
Once we have specified our desired factorization via (2.1), we can identify it with a directed acyclic graph (DAG) $\mathcal{G} = (E, V)$ by:

- identifying each random variable with a node
- drawing a directed arc from $A$ to $B$ if $A$ is a parent of $B$

We call this representation a directed probabilistic graphical model (or a Bayesian network) .

## Example

For example, the directed acyclic graph (DAG) below




corresponds to the factorization (Any guesses?)

# Directed probabilistic graphical models

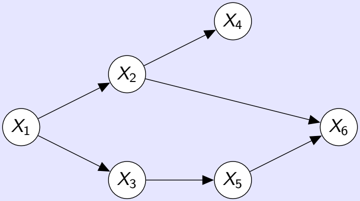
Once we have specified our desired factorization via (2.1), we can identify it with a directed acyclic graph (DAG) $\mathcal{G} = (E, V)$ by:

- identifying each random variable with a node
- drawing a directed arc from $A$ to $B$ if $A$ is a parent of $B$

We call this representation a directed probabilistic graphical model (or a Bayesian network) .

## Example

For example, the directed acyclic graph (DAG) below



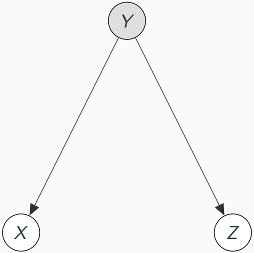

corresponds to the factorization (Any guesses?)

$$p(X) = p(X_1)\, p(X_2 \mid X_1)\, p(X_3 \mid X_1)\, p(X_4 \mid X_2)\, p(X_5 \mid X_3)\, p(X_6 \mid X_5, X_2)$$

# Exercise

Prove that $X \perp\!\!\!\perp Y \mid Z$ for the common parent structure.

**Common parent**




$$Z \perp\!\!\!\perp X \mid Y$$

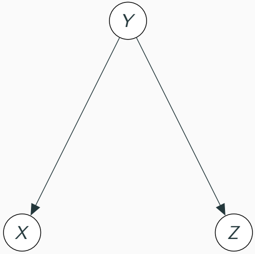
# Independence in Canonical Graphs

# Three canonical graphs

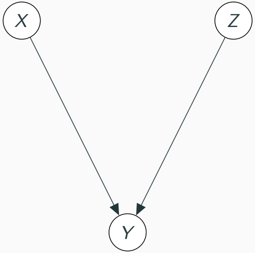
# Three canonical graphs

### Cascade


### Common parent




### v-structure

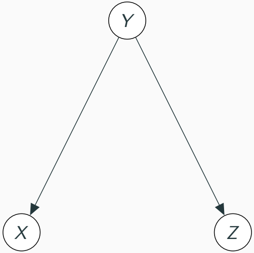


# Three canonical graphs : Marginal Independence

**Cascade**

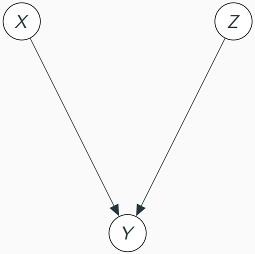

$Z \not\perp\!\!\!\perp X$

**Common parent**




$Z \not\perp\!\!\!\perp X$

**v-structure**



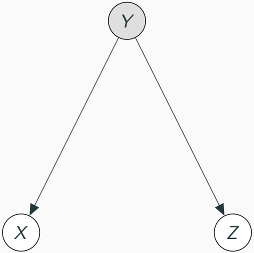

$Z \perp\!\!\!\perp X$

# Three canonical graphs : Conditional Independence

**Cascade**

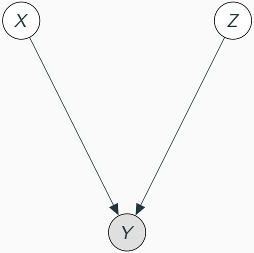

$$Z \perp\!\!\!\perp X \mid Y$$

**Common parent**




$$Z \perp\!\!\!\perp X \mid Y$$

**v-structure**

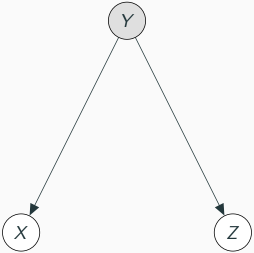



$$Z \not\!\perp\!\!\!\perp X \mid Y$$

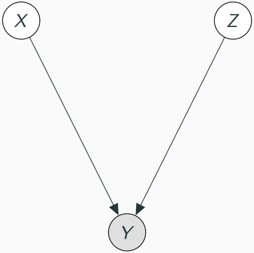
# Three canonical graphs : Take Home

**Cascade**


**Common parent**



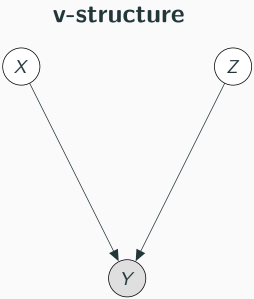

**v-structure**




Knowing $Y$ decouples $X$ and $Z$

Knowing $Y$ couples $X$ and $Z$

# Competing explanations




The independence properties of the v-structure is commonly understood through a competing explanations paradigm.

Suppose your house has a twitchy burglar alarm that is also sometimes triggered by earthquakes.

Let

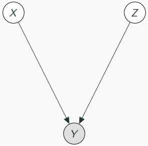
$$X = \{\text{your house got robbed}\}$$
$$Z = \{\text{an earthquake occurred nearby}\}$$
$$Y = \{\text{your burglar alarm goes off}\}$$

Then it is (perhaps) intuitive that

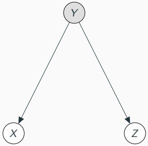
$$Z \perp\!\!\!\perp X$$
$$Z \not\perp\!\!\!\perp X \mid Y$$

# Relevance to real models

**v-structure**




**Common parent**

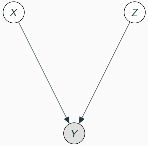



In real models ...

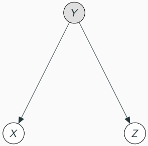
- the v-structure shows up with independent priors. ( So imagine $X$ and $Z$ are model parameters given independent priors and $Y$ is an observation.) Then the parameters are independent when generating data (i.e. in the prior), but they become dependent when doing inference (i.e. in the posterior).

# Relevance to real models

**v-structure**




**Common parent**




In real models ...

- the v-structure shows up with independent priors. ( So imagine $X$ and $Z$ are model parameters given independent priors and $Y$ is an observation.) Then the parameters are independent when generating data (i.e. in the prior), but they become dependent when doing inference (i.e. in the posterior).
- the common parent structure shows up with conditional i.i.d data models. (So imagine $Y$ is a parameter and $X$ and $Z$ are two observations.) The observations are conditionally independent, but integrating out the random parameter induces dependencies in the observations. (Imagine ollecting observations from a normal distribution with unknown $\mu$, $\Sigma$.)

# Independence in Directed PGM's

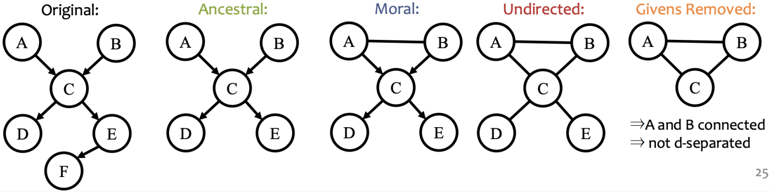
# d-separation

**If** variables X and Z are **d-separated** given a **set** of variables E
**Then** X and Z are **conditionally independent** given the **set** E

**Definition #2:**
Variables X and Z are d-separated given a **set** of evidence variables E iff there does **not** exist a path in the **undirected ancestral moral** graph **with E removed**.

1. **Ancestral graph**: keep only X, Z, E and their ancestors
2. **Moral graph**: add undirected edge between all pairs of each node's parents
3. **Undirected graph**: convert all directed edges to undirected
4. Givens Removed: delete any nodes in E

**Example Query:** $A \perp\!\!\!\perp B \mid \{D, E\}$




25

Image Credit: Matt Gormley (CMU).

# Worksheet for practice

# Revisiting the HMM statement

*The future is independent of the past given the current state*

Is this true?

1. $Y_2 \perp\!\!\!\perp Y_1 \mid X_2$ ? (Try it.)

# Revisiting the HMM statement

*The future is independent of the past given the current state*

Is this true?

1. $Y_2 \perp\!\!\!\perp Y_1 \mid X_2$ ? (Try it.) ✗

# Revisiting the HMM statement

*The future is independent of the past given the current state*

Is this true?

1. $Y_2 \perp\!\!\!\perp Y_1 \mid X_2$ ? (Try it.) ✗
2. $Y_2 \perp\!\!\!\perp Y_1 \mid X_2, \theta, \pi$ ?

# Revisiting the HMM statement

*The future is independent of the past given the current state*

Is this true?

1. $Y_2 \perp\!\!\!\perp Y_1 \mid X_2$ ? (Try it.) ✗
2. $Y_2 \perp\!\!\!\perp Y_1 \mid X_2, \theta, \pi$ ? ✓

# Revisiting the HMM statement

*The future is independent of the past given the current state*

Is this true?

1. $Y_2 \perp\!\!\!\perp Y_1 \mid X_2$ ? (Try it.) ✗
2. $Y_2 \perp\!\!\!\perp Y_1 \mid X_2, \theta, \pi$ ? ✓
3. (In fact, $Y_2 \perp\!\!\!\perp Y_1 \mid X_2, \theta$)

# Fundamental property of Bayes networks

Let us generalize this finding.
An oft-stated fact is:

> *A node is independent of its non-descendants given its parents.*

# Fundamental property of Bayes networks

Let us generalize this finding.
An oft-stated fact is:

*A node is independent of its non-descendants given its parents.*

(Look again at the HMM)

# Fundamental property of Bayes networks

Let us generalize this finding.
An oft-stated fact is:

*A node is independent of its non-descendants given its parents.*

(Look again at the HMM)

This can easily be proven via d-separation.

- The first step (“ancestral graph”) will remove all of $X$'s children.
- The fourth step (“remove givens”) will remove $X$'s parents.
- Thus, $X$ will be disconnected from the rest of the graph.

# Markov blankets

## Markov Blanket

**Def:** the **co-parents** of a node are the parents of its children

**Def:** the **Markov Blanket** of a node is the set containing the node's parents, children, and co-parents.

**Thm:** a node is **conditionally independent** of every other node in the graph given its **Markov blanket**

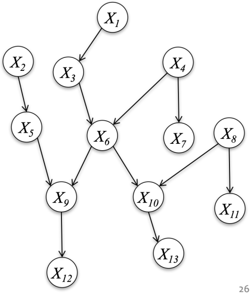





Image Credit: Matt Gormley (CMU).

# Markov blankets

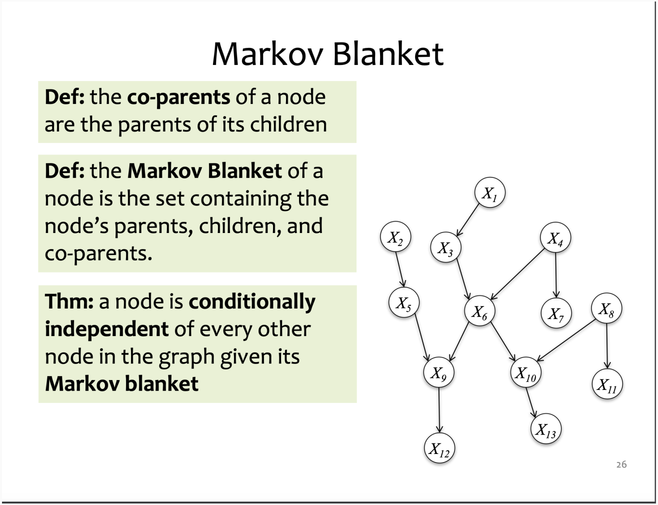



Image Credit: Matt Gormley (CMU).

**Q**: What is the Markov Blanket of $X_6$? Why?

# Markov blankets

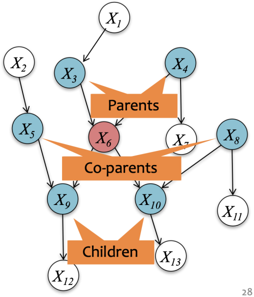
## Markov Blanket

**Def:** the **co-parents** of a node are the parents of its children

**Def:** the **Markov Blanket** of a node is the set containing the node's parents, children, and co-parents.

**Thm:** a node is **conditionally independent** of every other node in the graph given its **Markov blanket**

**Example:** The Markov Blanket of $X_6$ is $\{X_3, X_4, X_5, X_8, X_9, X_{10}\}$




28

Image Credit: Matt Gormley (CMU).

# Markov Blankets: Why *co*-parents?

Why is it not sufficient for the Markov Blanket to only include the parents and children of $X_i$?

# Markov Blankets: Why *co*-parents?

Why is it not sufficient for the Markov Blanket to only include the parents and children of $X_i$?

The phenomenon of explaining away means that the observations of child nodes will not block paths to the co-parents.

# Markov Blankets: Why *co*-parents?

Why is it not sufficient for the Markov Blanket to only include the parents and children of $X_i$?

The phenomenon of explaining away means that the observations of child nodes will not block paths to the co-parents.

This is why step 2 of the d-separation algorithm ("moralization") connects parents.

In the previous graph, the transformed graph would still have paths from $X_6$ to, for example, $X_8$ (and to $X_{11}$).

# Proof of Markov Blanket statement

Let us consider the conditional distribution of some variable $X_i$ given the factorization in (2.1):

$$
\begin{aligned}
p(X_i \mid X_{-i}) &= \frac{p(X_1, ..., X_n)}{\int p(X_1, ..., X_n) d\, X_i} \\
&= \frac{\prod_{k=1}^{n} p(X_k \mid \pi_k)}{\int \prod_{k=1}^{n} p(X_k \mid \pi_k)\; d\, X_i}
\end{aligned}
$$

All terms will cancel in the numerator and denominator except for terms of the form

1. $p(X_i \mid \pi_i)$, i.e. terms where $i$ is the node itself
2. $\{p(X_k \mid \pi_k) : i \in \pi_k\}$, i.e. terms where $i$ is one of the parents.

Terms of type (1) will depend on $X_i$'s parents, and terms of type (2) will depend on $X_i$'s children and co-parents.