Independence Properties of Directed Probabilistic Graphical Models

November 11, 2020

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Motivation

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Consider, for example, a Bayesian Hidden Markov Model.

$$\underset{\textit{joint}}{p(\pi, \theta, X, Y)} = \underbrace{p(\pi)\, p(\theta)}_{\text{prior}} \quad \underbrace{p(X_0) \prod_{t=1}^{T} p(X_t \mid X_{t-1})\, p(Y_t \mid X_t)}_{\text{(complete data) likelihood}}$$

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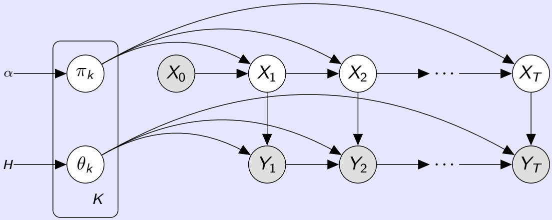
Representation as a ***probabilistic graphical model***



Directed Probabilistic Graphical Models

Joint distributions

The starting point for a directed probabilistic graphical model is a particular factorization of a joint density:

$$p(X_1, ..., X_n) = \prod_{i=1}^{n} p(X_i \mid \pi_i) \tag{2.1}$$

where the conditioning set π_i is referred to as the parents of variable i.

(2.1) simplifies the factorizations which are *always* true, by the chain rule of probability:

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Consider, e.g., the structure imposed in Bayesian models by independent priors or conditionally i.i.d likelihoods.

Directed probabilistic graphical models

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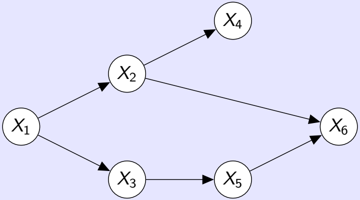

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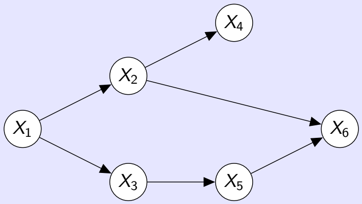

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$$p(X) = p(X_1)\, p(X_2 \mid X_1)\, p(X_3 \mid X_1)\, p(X_4 \mid X_2)\, p(X_5 \mid X_3)\, p(X_6 \mid X_5, X_2)$$

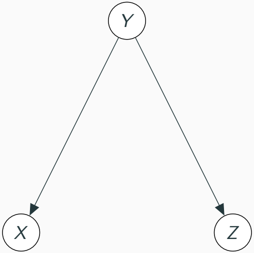
Independence in Canonical Graphs

Three canonical graphs

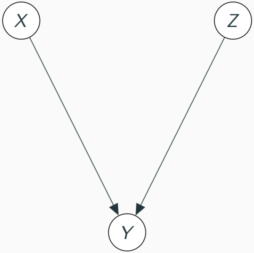
Three canonical graphs

Cascade

Common parent



v-structure

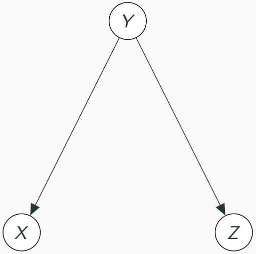


Three canonical graphs : Marginal Independence

Cascade

$Z \not\perp\!\!\!\perp X$

Common parent



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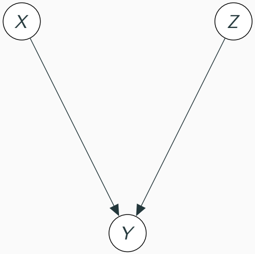
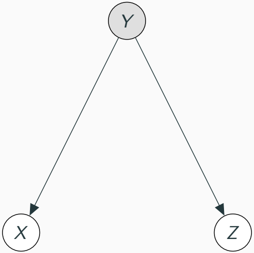


$Z \perp\!\!\!\perp X$

Cascade

$Z \perp\!\!\!\perp X \mid Y$

Common parent



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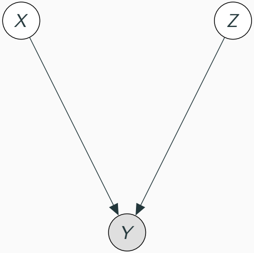

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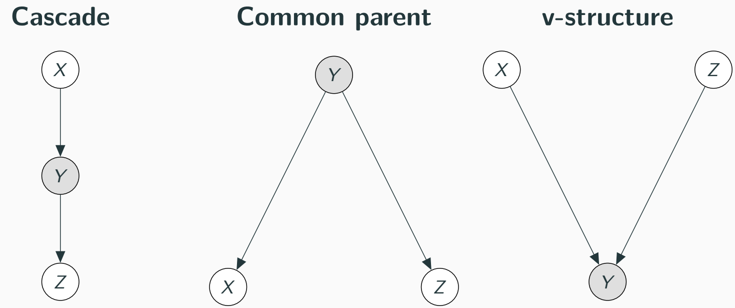
Three canonical graphs : Take Home

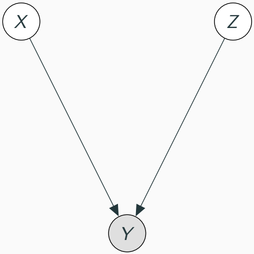


Knowing Y decouples X and Z

Knowing Y couples X and Z

Competing explanations

v-structure



The independence properties of the v-structure is commonly understood through a competing explanations paradigm.

Suppose your house has a twitchy burglar alarm that is also sometimes triggered by earthquakes.

Let

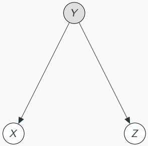
$$
\begin{aligned}
X &= \{\text{your house got robbed}\} \\
Z &= \{\text{an earthquake occurred nearby}\} \\
Y &= \{\text{your burglar alarm goes off}\}
\end{aligned}
$$

Then it is (perhaps) intuitive that

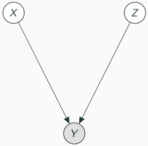
$$
\begin{aligned}
&Z \perp\!\!\!\perp X \\
&Z \not\!\perp\!\!\!\perp X \mid Y
\end{aligned}
$$

Relevance to real models

Common parent



v-structure



In real models ...

- the common parent structure shows up with conditional i.i.d data models. (So imagine Y is a parameter and X and Z are two observations.) The observations are conditionally independent, but integrating out the random parameter induces dependencies in the observations. Note in particular that the observations are, in general, *dependent* in the predictive posterior.
- the v-structure shows up with independent priors. (So imagine X and Z are model parameters given independent priors and Y is an observation.) Then the parameters are independent when generating data (i.e. in the prior), but they become dependent when doing inference (i.e. in the posterior).

Independence in Directed PGM's

d-separation

If variables X and Z are **d-separated** given a **set** of variables E
Then X and Z are **conditionally independent** given the **set** E

Definition #2:
Variables X and Z are d-separated given a **set** of evidence variables E iff there does **not** exist a path in the **undirected ancestral moral** graph **with E removed**.

1. **Ancestral graph**: keep only X, Z, E and their ancestors
2. **Moral graph**: add undirected edge between all pairs of each node's parents
3. **Undirected graph**: convert all directed edges to undirected
4. Givens Removed: delete any nodes in E

Example Query: $A \perp\!\!\!\perp B \mid \{D, E\}$

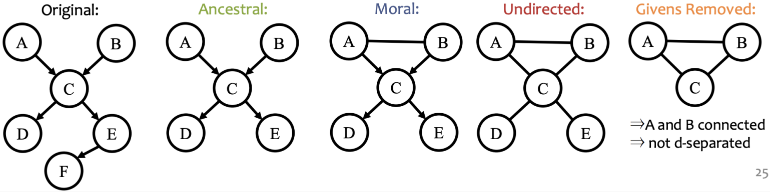


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Image Credit: Matt Gormley (CMU).

Worksheet for practice

Fundamental property of Bayes networks

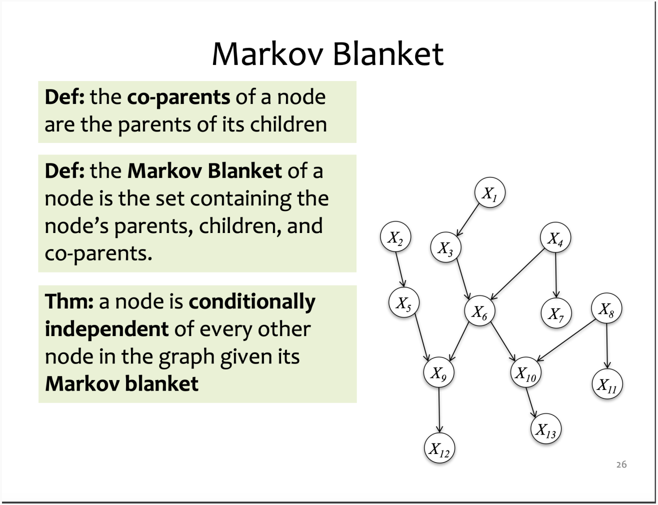
An oft-stated fact is:

A node is independent of its non-descendants given its parents.

This can easily be proven via d-separation.

- The first step ("ancestral graph") will remove all of X's children.
- The fourth step ("remove givens") will remove X's parents.
- Thus, X will be disconnected from the rest of the graph.

Markov blankets



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Markov blankets

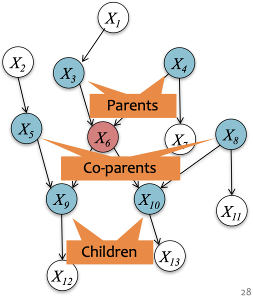
Markov Blanket

Def: the **co-parents** of a node are the parents of its children

Def: the **Markov Blanket** of a node is the set containing the node's parents, children, and co-parents.

Thm: a node is **conditionally independent** of every other node in the graph given its **Markov blanket**

Example: The Markov Blanket of X_6 is $\{X_3, X_4, X_5, X_8, X_9, X_{10}\}$



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Markov Blankets: Why *co*-parents?

Why is it not sufficient for the Markov Blanket to only include the parents and children of X_i?

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Why is it not sufficient for the Markov Blanket to only include the parents and children of X_i?

The phenomenon of explaining away means that the observations of child nodes will not block paths to the co-parents.

This is why step 2 of the d-separation algorithm ("moralization") connects parents.

In the previous graph, the transformed graph would still have paths from X_6 to, for example, X_8 (and to X_{11}).

Proof of Markov Blanket statement

Let us consider the conditional distribution of some variable X_i given the factorization in (2.1):

$$p(X_i \mid X_{-i}) = \frac{p(X_1, \ldots, X_n)}{\int p(X_1, \ldots, X_n) d\, X_i}$$

$$= \frac{\prod_{k=1}^{n} p(X_k \mid \pi_k)}{\int \prod_{k=1}^{n} p(X_k \mid \pi_k)\ d\, X_i}$$

All terms will cancel in the numerator and denominator except for terms of the form

1. $p(X_i \mid \pi_i)$, i.e. terms where i is the node itself
2. $\{p(X_k \mid \pi_k) : i \in \pi_k\}$, i.e. terms where i is one of the parents.

Terms of type (1) will depend on X_i's parents, and terms of type (2) will depend on X_i's children and co-parents.