

# Maximum Likelihood Approaches

---

November 9, 2020

# Table of contents

1. Maximum Likelihood
2. Maximum Likelihood Example
3. Anomaly Detection
4. Optimization
5. Exponential Family

# Acknowledgements

This slide deck borrows heavily from an excellent course on statistical ML by Peter Orbanz.

Other useful resources came from David Blei and Michael Jordan.

# Maximum Likelihood

---

# Parametric Models

## Models

A **model**  $\mathcal{P}$  is a set of probability distributions. We index each distribution with a parameter value  $\theta \in \Theta$ ; we can then write the model as

$$\mathcal{P} = \{P_\theta \mid \theta \in \Theta\}$$

The set  $\Theta$  is called the **parameter space** of the model.

## Parametric model

The model is called **parametric** if the number of parameters (i.e. the vector  $\theta$ ) is (1) finite and (2) independent of the number of data points. Intuitively, the complexity of a parametric model does not increase with sample size.

## Density representation

For parametric models, we can assume that  $\Theta \subset \mathbb{R}^d$  for some fixed dimension  $d$ . We usually represent each  $P_\theta$  via a density function  $p(x \mid \theta)$ .

# Maximum Likelihood Estimation

## Setting

- Given: Data  $x_1, \dots, x_n$ , parametric model  $\mathcal{P} = \{P_\theta \mid \theta \in \Theta\}$
- Objective: Find the distribution in  $\mathcal{P}$  which best explain the data.  
That means we have to choose a “best” parameter value  $\hat{\theta}$ .

## Maximum Likelihood approach

Maximum Likelihood assumes that the data is best explained by the distribution in  $\mathcal{P}$  under which it has the highest probability (or highest density value).

Hence, the **maximum likelihood estimator** is defined as

$$\hat{\theta}_{\text{ML}} := \operatorname{argmax}_{\theta \in \Theta} p(x_1, \dots, x_n \mid \theta)$$

the parameter which maximizes the joint density of the data.

# The i.i.d. assumption

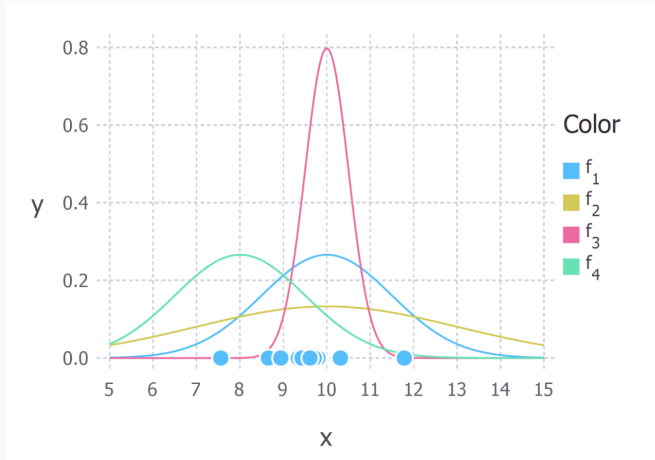
## The i.i.d. assumption

The standard assumption of ML methods is that the data is **independent and identically distributed (i.i.d.)**, that is, generated by independently sampling repeatedly from the same distribution  $\mathcal{P}$ .

If the density of  $\mathcal{P}$  is  $p(x \mid \theta)$ , that means the joint density decomposes as

$$p(x_1, \dots, x_n) = \prod_{i=1}^n p(x_i \mid \theta)$$

# Illustration



Ten data points and four possible Gaussians from which they were drawn:  
 $f_1 \sim \mathcal{N}(10, 2.25)$ ,  $f_2 \sim \mathcal{N}(10, 9)$ ,  $f_3 \sim \mathcal{N}(10, 0.25)$ ,  $f_4 \sim \mathcal{N}(8, 2.25)$ .

Image Credit: Jonny Brooks-Bartlett



## Maximum Likelihood equation

In practice, the criterion for a maximum likelihood estimator (under the i.i.d assumption) is

$$\nabla_{\theta} \left( \prod_{i=1}^n p(x_i | \theta) \right) = 0$$

We use the “logarithm trick” to avoid a huge product rule computation.

## Recall: Logarithms turn products into sums

$$\log \left( \prod_i f_i \right) = \sum_i \log(f_i)$$

## Logarithms and maxima

The logarithm is monotonically increasing on  $\mathbb{R}_+$ .

Consequence: Application of log does not change the *location* of a maximum or minimum:

$$\max_y \log(g(y)) \neq \max_y g(y) \quad \text{The *value* changes.}$$

$$\operatorname{argmax}_y \log(g(y)) = \operatorname{argmax}_y g(y) \quad \text{The *location* does not change.}$$

## Likelihood and logarithm trick

$$\hat{\theta}_{\text{ML}} = \operatorname{argmax}_{\theta} \prod_{i=1}^n p(x_i|\theta) = \operatorname{argmax}_{\theta} \log \left( \prod_{i=1}^n p(x_i|\theta) \right) = \operatorname{argmax}_{\theta} \sum_{i=1}^n \log p(x_i|\theta)$$

## Maximum Likelihood in practice (revisited)

$$0 = \sum_{i=1}^n \nabla_{\theta} \log p(x_i|\theta) = \sum_{i=1}^n \frac{\nabla_{\theta} p(x_i|\theta)}{p(x_i|\theta)}$$

Whether or not we can solve this analytically depends on the choice of model!

# Maximum Likelihood Example

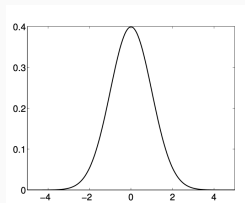
---

# Example: Gaussian Distribution

## Gaussian density in one dimension

$$g(x; \mu, \sigma) := \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x - \mu)^2}{2\sigma^2}\right)$$

- $\mu$  = expected value of  $x$ ,  $\sigma^2$  = variance,  $\sigma$  = standard deviation
- The quotient  $\frac{x - \mu}{\sigma}$  measures deviation of  $x$  from its expected value in units of  $\sigma$  (i.e.,  $\sigma$  defines the length scale).



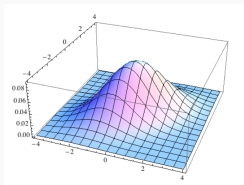
## Gaussian density in $d$ dimensions

The quadratic function

$$-\frac{(x - \mu)^2}{2\sigma^2} = -\frac{1}{2}(x - \mu)(\sigma^2)^{-1}(x - \mu)$$

is replaced by a quadratic form:

$$g(\mathbf{x}; \mu, \Sigma) := \frac{1}{\sqrt{2\pi|\Sigma|}} \exp\left(-\frac{1}{2}(\mathbf{x} - \mu)^T \Sigma^{-1}(\mathbf{x} - \mu)\right)$$



# Example: Gaussian Mean MLE

## Model: Multivariate Gaussians

The model  $\mathcal{P}$  is the set of all Gaussian densities on  $\mathbb{R}^d$  with *fixed* covariance matrix  $\Sigma$

$$\mathcal{P} = \{g(\cdot \mid \mu, \Sigma) \mid \mu \in \mathbb{R}^d\}$$

where  $g$  is the Gaussian density function. The parameter space is  $\Theta = \mathbb{R}^d$ .

## MLE equation

We have to solve the maximum likelihood equation

$$\sum_{i=1}^n \nabla_{\mu} \log g(x_i \mid \mu, \Sigma) = 0$$

for  $\mu$ .

## Example: Gaussian Mean MLE

$$\begin{aligned} 0 &= \sum_{i=1}^n \nabla_{\mu} \log \left[ \frac{1}{\sqrt{(2\pi)^d |\Sigma|}} \exp \left( -\frac{1}{2} (\mathbf{x} - \mu)^T \Sigma^{-1} (\mathbf{x} - \mu) \right) \right] \\ &= \sum_{i=1}^n \nabla_{\mu} \left[ \log \frac{1}{\sqrt{(2\pi)^d |\Sigma|}} \right] + \nabla_{\mu} \left[ \log \left( \exp \left( -\frac{1}{2} (\mathbf{x} - \mu)^T \Sigma^{-1} (\mathbf{x} - \mu) \right) \right) \right] \\ &= \sum_{i=1}^n \nabla_{\mu} \left( -\frac{1}{2} (\mathbf{x} - \mu)^T \Sigma^{-1} (\mathbf{x} - \mu) \right) = - \sum_{i=1}^n \Sigma^{-1} (\mathbf{x}_i - \mu) \end{aligned}$$

Multiplication by  $(-\Sigma)$  gives

$$0 = \sum_{i=1}^n (\mathbf{x}_i - \mu) \implies \mu = \frac{1}{n} \sum_{i=1}^n \mathbf{x}_i$$

## Conclusion

The maximum likelihood estimator of the Gaussian expectation parameter for fixed covariance is

$$\hat{\mu}_{\text{ML}} := \frac{1}{n} \sum_{i=1}^n \mathbf{x}_i$$

# Example: Gaussian with Unknown Mean, Covariance

## Model: Multivariate Gaussians

The model  $\mathcal{P}$  is now

$$\mathcal{P} = \{g(\cdot \mid \mu, \Sigma) \mid \mu \in \mathbb{R}^d, \Sigma \in \Delta_d\}$$

where  $\Delta_d$  is the set of positive definite  $d \times d$ -matrices. The parameter space is  $\Theta = \mathbb{R}^d \times \Delta_d$ .

## ML approach

Since we have just seen that the ML estimator of  $\mu$  does not depend on  $\Sigma$ , we can compute  $\hat{\mu}_{\text{ML}}$  first. We then estimate  $\Sigma$  using the criterion

$$\sum_{i=1}^n \nabla_{\Sigma} \log g(x_i \mid \hat{\mu}_{\text{ML}}, \Sigma) = 0$$

for  $\mu$ .

## Solution

The ML estimator of  $\Sigma$  is

$$\hat{\Sigma}_{\text{ML}} = \frac{1}{n} \sum_{i=1}^n (x_i - \hat{\mu}_{\text{ML}})(x_i - \hat{\mu}_{\text{ML}})^T$$



# Anomaly Detection

---

# Anomaly Detection with Multivariate Gaussians

Given a fitted Gaussian model, how can we assess the anomalousness of test data?

# Anomaly Detection with Multivariate Gaussians

Given a fitted Gaussian model, how can we assess the anomalousness of test data?

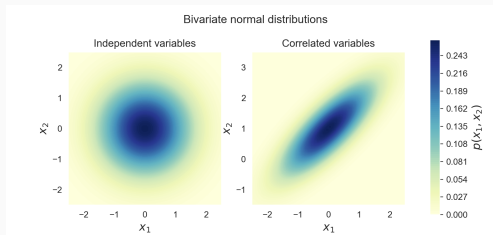


Image Credit: Peter Roelants

# Anomaly Detection with Multivariate Gaussians

Given a fitted Gaussian model, how can we assess the anomalousness of test data?

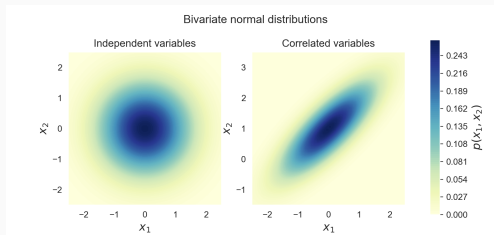


Image Credit: Peter Roelants

## Mahalanobis Distance

For a Gaussian random variable  $X \sim N(\mu, \Sigma)$ , the quadratic form (or *squared Mahalanobis distance*) has known distribution

$$\Delta^2 = (X - \mu)^T \Sigma^{-1} (X - \mu) \sim \chi^2(d)$$

This can be used to assess the anomalousness of test data.

# Optimization

---

# Optimization problem

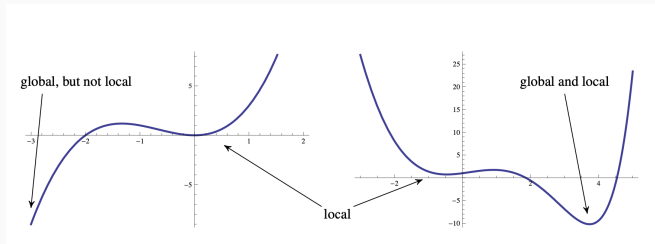
An *optimization problem* for a given function  $f : \mathbb{R}^d \rightarrow \mathbb{R}$  is a problem of the form

$$\min_{\mathbf{x}} f(\mathbf{x})$$

which we read as “find  $\mathbf{x}_0 = \arg \min_{\mathbf{x}} f(\mathbf{x})$ ”.

Note that finding the maximum likelihood requires minimizing the cost function that is the negative log likelihood.

# Local and global modes



## Local and global minima

A minimum of a function  $f$  at  $x$  is called

- **Global** if  $f$  assumes no smaller value on its domain.
- **Local** if there is some open neighborhood  $U$  of  $x$  such that  $f(x)$  is a global minimum of  $f$  restricted to  $U$ .

Image Credit: Peter Orbanz

## Analytic criteria for local minima

Recall that  $\mathbf{x}$  is a local minimum of  $f$  if

$$f'(\mathbf{x}) = 0 \quad \text{and} \quad f''(\mathbf{x}) > 0$$

In  $\mathbb{R}^d$ ,

$$\nabla f(\mathbf{x}) = 0 \text{ and } H_f(\mathbf{x}) = \left( \frac{\delta f}{\delta x_i \delta x_j}(\mathbf{x}) \right)_{i,j=1,\dots,n} \text{ positive definite}$$

The  $d \times d$ -matrix  $H_f(\mathbf{x})$  is called the **Hessian matrix** of  $f$  at  $\mathbf{x}$ .



# The MLE and Global Maximizers

You may have noticed that the maximum likelihood equation is only tracking a *local* maximality criterion. In fact, it also ignored the second-order condition. What gives?

- Many well-known distributions<sup>1</sup> have strictly concave likelihoods, in which case the MLE equation is sufficient to verify a global maximum.
- For many other distributions, it can be hard to find the global maximizer of the likelihood. Thus a local maximizer is often used and is called an MLE. The local optimizer is typically found by an optimization procedure, from which the second order condition generally follows.

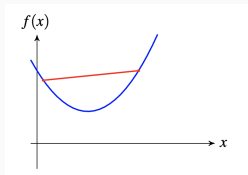
---

<sup>1</sup>In particular, those in the exponential family. We will cover this in the next slide deck.

# Convex Functions

## Definition

A function  $f$  is **convex** if every line segment between function values lies above the graph of  $f$



## Analytic criterion

A twice differentiable function is convex if  $f''(\mathbf{x}) \geq 0$  (or  $H_f(\mathbf{x})$  positive semidefinite) for all  $\mathbf{x}$ .

## Implications for optimization

If  $f$  is convex, then:

- $f'(\mathbf{x}) = 0$  is a sufficient criterion for a minimum.
- Local minima are global.
- If  $f$  is strictly convex ( $f'' > 0$  or  $H_f$  positive definite), there is only one minimum (which is both global and local).

# Exponential Family

---

# Exponential Family Models

## Definition

We consider a model  $\mathcal{P}$  for data in a sample space  $\mathcal{X}$  with parameter space  $\Theta \subset \mathbb{R}^m$ . Each distribution in  $\mathcal{P}$  has density  $p(x \mid \theta)$  for some  $\theta \in \Theta$ .

The model is called an **exponential family model** (EFM) if  $p$  can be written as

$$p(x \mid \theta) = h(x) \exp\{\eta(\theta)^T s(x) - a(\theta)\}$$

where we refer to

- $h$  as the base measure
- $\eta$  as the natural parameter
- $s$  as the sufficient statistics
- $a$  as the log normalizer.

## Exponential families are important because

- The special form of  $p$  gives them many nice properties.<sup>2</sup>
- Many important parametric models (Gaussian, Poisson, beta, gamma, etc.) are EFM's.
- Many algorithms and methods can be formulated generically for all EFM's.

## An observation

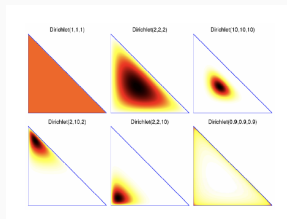
The data and the parameter interact only through the linear term  $\eta(\theta)^T s(x)$  in the exponent.

---

<sup>2</sup>Indeed, we have already seen one. More will come up in this workshop.

# Example: The Dirichlet Distribution

We can write the density of the Dirichlet distribution in exponential form:



$$\begin{aligned} p(\pi \mid \alpha) &= \frac{\Gamma(\sum_k \alpha_k)}{\prod_k \Gamma(\alpha_k)} \pi_1^{\alpha_1-1} \dots \pi_K^{\alpha_K-1} \\ &= \exp \left\{ \sum_{k=1}^K (\alpha_k - 1) \log \pi_k - \left[ \sum_k \log \Gamma(\alpha_k) - \log \Gamma(\sum_k \alpha_k) \right] \right\} \end{aligned}$$

with natural parameter  $\eta(\alpha) = [\alpha_1 - 1, \dots, \alpha_K - 1]^T$ , sufficient statistics  $s(\pi) = \log \pi = [\log \pi_1, \dots, \log \pi_K]^T$ , base measure  $h(\pi) = 1$ , and log normalizer  $a(\alpha) = \sum_k \log \Gamma(\alpha_k) - \log \Gamma(\sum_k \alpha_k)$ .  $\square$

## Exercise: The Bernoulli distribution

As an example, let's put the Bernoulli (in its usual form) into exponential family form.  
The Bernoulli you are used to seeing is:

$$p(x \mid \pi) = \pi^x (1 - \pi)^{1-x} \quad x \in \{0, 1\}$$

## Exercise: The Bernoulli distribution

As an example, let's put the Bernoulli (in its usual form) into exponential family form. The Bernoulli you are used to seeing is:

$$p(x | \pi) = \pi^x (1 - \pi)^{1-x} \quad x \in \{0, 1\}$$

In exponential family form:

$$\begin{aligned} p(x | \pi) &= \exp \left( \log \left[ \pi^x (1 - \pi)^{1-x} \right] \right) \\ &= \exp \left( x \log \pi + (1 - x) \log(1 - \pi) \right) \\ &= \exp \left( x \log \pi - x \log(1 - \pi) + \log(1 - \pi) \right) \\ &= \exp \left( x \log(\pi / (1 - \pi)) + \log(1 - \pi) \right) \end{aligned}$$

which reveals the exponential family where

$$\begin{aligned} \eta &= \log(\pi / (1 - \pi)) \\ s(x) &= x \\ a(\eta) &= -\log(1 - \pi) = \log(1 + e^\eta) \\ h(x) &= 1 \end{aligned}$$

---

Note that the relationship between  $\pi$  and  $\eta$  is invertible

$$\pi = 1 / (1 + e^{-\eta})$$

This is the *logistic function*.



## Exercise: The Gaussian distribution

The familiar form of the univariate Gaussian is

$$p(x \mid \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2} \frac{(x - \mu)^2}{\sigma^2}\right)$$

## Exercise: The Gaussian distribution

The familiar form of the univariate Gaussian is

$$p(x \mid \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2} \frac{(x - \mu)^2}{\sigma^2}\right)$$

We put it in exponential family form by expanding the square

$$p(x \mid \mu, \sigma^2) = \frac{1}{\sqrt{2\pi}} \exp\left(\frac{\mu}{\sigma^2}x - \frac{1}{2\sigma^2}x^2 - \frac{1}{2\sigma^2}\mu^2 - \log \sigma\right)$$

which reveals the exponential family where

$$\eta = [\mu/\sigma^2, -1/2\sigma^2]$$

$$s(x) = [x, x^2]$$

$$a(\eta) = \mu^2/2\sigma^2 + \log \sigma$$

$$h(x) = 1/\sqrt{2\pi}$$

# Examples of Exponential Families

Model	Sample space	Sufficient statistic
Gaussian	$\mathbb{R}^d$	$S(\mathbf{x}) = (\mathbf{x}\mathbf{x}^t, \mathbf{x})$
Gamma	$\mathbb{R}_+$	$S(x) = (\ln(x), x)$
Poisson	$\mathbb{N}_0$	$S(x) = x$
Multinomial	$\{1, \dots, K\}$	$S(x) = x$
Wishart	Positive definite matrices	(requires more details)
Mallows	Rankings (permutations)	(requires more details)
Beta	$[0, 1]$	$S(x) = (\ln(x), \ln(1 - x))$
Dirichlet	Probability distributions on $d$ events	$S(\mathbf{x}) = (\ln x_1, \dots, \ln x_d)$
Bernoulli	$\{0, 1\}$	$S(x) = x$
...	...	...

## Roughly speaking

On every sample space, there is a “natural” statistic of interest. On a space with Euclidean distance, for example, it is natural to measure both location and correlation; on categories (which have no “distance” from each other), it is more natural to measure only expected numbers of counts.

# The Exponential Family and Maximum Likelihood

## i.i.d samples from an exponential family distribution

If  $\mathbf{x} = (x_1, \dots, x_n)$  are  $n$  independent samples from the same exponential family distribution, then

$$p(\mathbf{x} \mid \theta) = \prod_{i=1}^n h(x_i) \exp \left\{ \eta(\theta)^T \sum_{i=1}^n s(x_i) - n a(\eta(\theta)) \right\}$$

## Maximum likelihood with exponential families

The goal for maximum likelihood is to determine parameter

$$\theta_{ML} = \operatorname{argmax}_{\theta} \log p(\mathbf{x} \mid \theta)$$

Let us assume that  $\mathbf{x} = (x_1, \dots, x_n)$  are i.i.d observations from a fixed exponential family, so that the likelihood has form above.

# The Exponential Family and Maximum Likelihood

Let us compute the gradient with respect to the natural parameter  $\eta$  of  $\ell(\eta) := \log p(\mathbf{x} \mid \eta)$

$$\nabla_{\eta} \ell(\eta) = \sum_{i=1}^n s(x_i) - n \nabla_{\eta} a(\eta)$$

Setting the gradient to zero, we obtain

$$\nabla_{\eta} a(\eta) = \frac{1}{n} \sum_{i=1}^n s(x_i)$$

But<sup>3</sup>  $\nabla_{\eta} a(\eta) = \mathbb{E}[s(X)]$ . Thus, we should set  $\theta_{ML}$  such that

$$\mu(\theta_{ML}) = \frac{1}{n} \sum_{i=1}^n s(x_i)$$

where  $\mu := \mathbb{E}[s(x)]$  refers to the mean parametrization of the likelihood.

---

<sup>3</sup>A useful fact about exponential families. The proof is straightforward.