Topic 8

Counting

Introduction

- Counting techniques are very useful in many applications.
  - Security - combinations of passwords or pins
  - Addresses counts
- Some counting techniques
  - Pigeonhole principle
  - Permutation
  - Combinations

Introduction

- In computer sciences many problems require the use of counting techniques:
  - how many bits are needed to store some data?
  - determine the computer memory space needed
  - counts the possible items to search for to find a solution

Counting the Elements of a List

- Given a continuous list numbers with the uniform incremental values…count the number in the list.

list:	5	6	7	8	9	10	11	12
	↕	↕	↕	↕	↕	↕	↕	↕
count:	1	2	3	4	5	6	7	8

Counting the Elements of a List

- If elements in a list are indexed with a continuous positive integer value, if *m* and *n* are integers $m \leq n$, how many integers are there from *m* through *n*?

list:	$m\ (= m + 0)$	$m + 1$	$m + 2$	$\ldots$	$n\ (= m + (n - m))$
count:	1	2	3	$\ldots$	$(n - m) + 1$

Theorem

- **The number of elements in a List.**

- If m and n are integers and $m \leq n$, then there are $n - m + 1$ integers from m to n inclusive.

Example

- How many three-digit integers (integers from 100 to 999 inclusive) are divisible by 5?

100	101	102	103	104	105	106	107	108	109	110	...	994	995	996	997	998	999
$\updownarrow$					$\updownarrow$					$\updownarrow$			$\updownarrow$				
$5 \cdot 20$					$5 \cdot 21$					$5 \cdot 22$			$5 \cdot 199$				

Solution:

There are 199-20+1 = 180 integers

Counting elements in an array

- Analysis of many computer algorithms requires skill at counting the elements of a one-dimensional array. Let $A[1], A[2],\ldots, A[n]$ be a one-dimensional array, where n is a positive integer.

Example:

a. Suppose the array is cut at a middle value $A[m]$ so that two subarrays are formed:

$A[1], A[2], \ldots , A[m]$ and (2) $A[m+1], A[m+2], \ldots, A[n]$. How many elements does each subarray have?

Solution:

Array (1) has the same number of elements as the list of integers from 1 through m. So by Theorem, it has m, or $m-1+1$, elements. Array (2) has the same number of elements as the list of integers from $m+1$ through n. So by Theorem, it has $n-m$, or $n-(m+1) - 1$, elements.

Multiplication Rule

- Consider the following example. Suppose a computer installation has four input/output units (*A*, *B*, *C*, and *D*) and three central processing units (*X*, *Y*, and *Z*). Any input/output unit can be paired with any central processing unit. How many ways are there to pair an input/output unit with a central processing unit?

To answer this question, imagine the pairing of the two types of units as a two-step

Operation

Step 1: Choose the input/output unit.

Step 2: Choose the central processing unit.

Thus the total number of ways to pair the two types of units is the same as the number of branches of the tree, which is 3+3+3+3=4 X 3

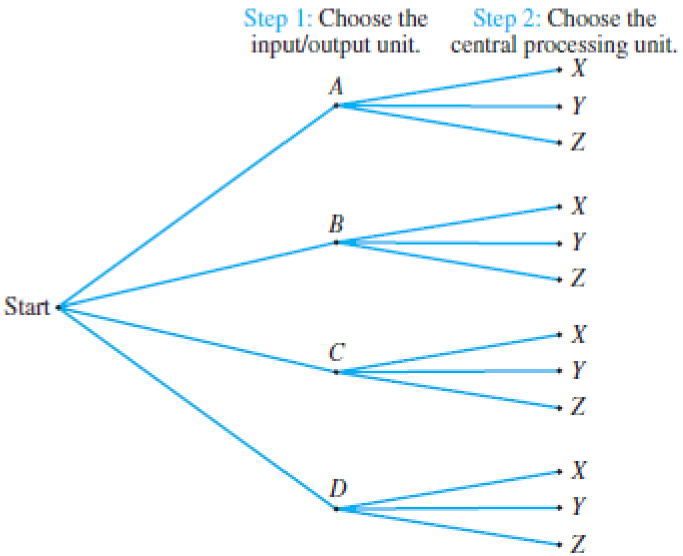


FIGURE 9.2.2 Pairing Objects Using a Possibility Tree

The Multiplication Rule

- If a task can be divided into m independent sub-tasks.

Counting task 1 — n_1 ways

Counting task 2 — n_2 ways

.

.

Counting task m — n_m ways

The number of ways to do task1, task2, …and taskm is = $n_1 \times n_2 \times \ldots \times n_m$.

Examples of independent events

- Number of ways to travel to Sibu town from Kuching. Choosing a hotel to stay there.

- Determine the number of ways to arrange books

The Multiplication Rule

- Example 1

How many ways are there to pick three students with different majors from among the 19 Math major, 17 Economic majors and 5 Law major.

Answer:

$19 \cdot 5 \cdot 17 = 1615.$

- Example 2

Given 50 different cards. How many ways are there to pick a sequence of 4 cards?

Answer: $50 \cdot 49 \cdot 48 \cdot 47$ ways.

Example

- A Personal Identification Number (PIN) is a sequence of any four symbols chosen from the 26 letters in the alphabet and the ten digits, with repetition is now allowed. How many different PINs are possible?

The Addition Rule

- If a first task can be done in n_1 ways and a second task in n_2 ways, and if these tasks cannot be done at the same time, then there are $n_1 + n_2$ ways to do one of these tasks.

Theorem 9.3.1 The Addition Rule

Suppose a finite set A equals the union of k distinct mutually disjoint subsets A_1, $A_2, \ldots, A_k$. Then

$$N(A) = N(A_1) + N(A_2) + \cdots + N(A_k).$$

The Addition Rule

- Example:

A student can choose a computer project from one of three lists. The three lists contain 23, 15, and 19 possible projects, respectively. How many possible projects are there to choose from?

- Solution:

The student can choose a project from the first list in 23 ways, from the second list in 15 ways, and from the third list in 19 ways. Hence, there are 23 + 15 + 15 = 57 projects to choose from.

The Addition Rule

- **Example:**

Suppose statement labels in a programming language must be a single letter or a single decimal digit. How many ways to choose a single letter label?

Since a label cannot be both at the same time,

The number of labels are

=the number of letters + the number of decimal digits

=26 + 10 = 36.

Principle of Inclusion-Exclusion

- In the addition rule, the different sets of solutions have to be independent. If two of more sets of the solutions overlap, that could overcount the possible solutions.

Theorem 9.3.3 The Inclusion/Exclusion Rule for Two or Three Sets

If A, B, and C are any finite sets, then

$$N(A \cup B) = N(A) + N(B) - N(A \cap B)$$

and

$$N(A \cup B \cup C) = N(A) + N(B) + N(C) - N(A \cap B) - N(A \cap C) - N(B \cap C) + N(A \cap B \cap C).$$

Principle of Inclusion-Exclusion

- Example:

  Count the number of bit strings of length 4 which begin with a 1 or end with a 00.

1	x	0	0

Solution:

The set can be expressed as the union of

a) the subset *S* of strings which begin with 1, and

b) the subset *B* that end in 00.

Principle of Inclusion-Exclusion

- For set S

1	x	x	x

- There are $2^3=8$ ways to construct bits string begin with 1 (using multiplication rule)
- For set B

x	x	0	0

- There are $2^2=4$ ways to construct bits string ended with 00 (using multiplication rule).

Principle of Inclusion-Exclusion

1	x	0	0

Overlap between set S and B. The number of intersection is $2^1=2$

So, the total number of bit strings which begin with 1 and end of 00 is 12-2=10.

Example:

- How many integers from 1 through 1,000 are multiples of 3 or multiples of 5?

Solution:

A=set of integers multiples of 3
B= set of integers multiples of 5

$A \cup B$ = the set of all integers from 1 through 1000 that are multiplies of 3 or multiples of 5.

$A \cap B$ = the set of all integers from 1 through 1000 that are multiples of both 3 and 5.
= the set of all integers from 1 through 1000 that are multiples of 15.

Continue

1	2	3	4	5	6	…	996	997	998	999	1000
		↑			↑		↑			↑	
		$3 \cdot 1$			$3 \cdot 2$		$3 \cdot 332$			$3 \cdot 333$	

From the list above, there are $333 - 1 + 1 = 333$ integers that are multiples of 3.

1	2	3	4	5	6	7	8	9	10	…	995	996	997	998	999	1000
				↑							↑					↑
				$5 \cdot$ 1							$5 \cdot$ 199					$5 \cdot$ 200

From the list above, there are $200-1+1=200$ integers that are multiples of 5.

1	2	…	15	…	30	…	990	…	1000
			↑		↑		↑		
			$15 \cdot 1$		$15 \cdot 2$		$15 \cdot 66$		

From the list above, there are 66-1+1 = 66 integers that multiples of 15.
Therefore, the total number of integers multiplies of 3 or 5 is:
$N(A) + N(B) - N(A \cup B) = 333 + 200 - 66 = 467$ integers

Permutations and Combinations

Permutations

- A permutation of a set of distinct objects is an ordered arrangement of these objects.
- An ordered arrangement of *r* elements of a set is called an *r*-permutation.

- For example, the set of elements a, b, and c has six permutations.

abc, acb, cba, bac, bca, cab.

Permutations

- Example

Let S = {1, 2, 3}. The arrangement 3, 1, 2 is a 3-permutation of S. The arrangement 3, 2 is a 2-permutation of S.

Computing no of permutations

- In general, given a set of n objects, how many permutations does the set have?

  Step 1: Choose an element to write first (n ways);

  Step 2: Choose an element to write second (n-1 ways);

  …

  Step n: Choose an element to write nth (1 way).

  - Hence, by the multiplication rule, there are
  - $n \times (n-1) \times (n-2) \times \ldots 2 \times 1 = n!$ ways to perform the entire operation.

Counting r-permutations

- The number of *r*-permutations of a set with *n* distinct elements is

$$P(n, r) = n(n-1)(n-2)\ldots(n-r+1)$$

or

$$P(n, r) = \frac{n!}{(n-r)!}$$

Permutations

- Example:

How many ways are there to select a first prize winner, a second-prize winner, and a third-prize winner from 100 different people who have entered the contest?

Solution:

P(100, 3) = 100 * 99 * 98 = 970,200.

Exercise

- How many permutations of the letters ABCDEFGH contain the string ED?

- Solution:
- If “ED” has to be appeared “ED”, then the total number of letters left is 7 inclusive of “ED”
- Number of permutations contain ED is
- 7 x 6 x 5 x 4 x 3 x 2 x 1=5,040

Combinations

- An *r*-combination of elements of a set is an unordered selection of r elements from the set.

- The number of *r*-combinations of a set with *n* distinct elements is denoted by C(*n, r*). Where

$$C(n,r) = \frac{n!}{r!(n-r)!}$$

$0 \le r \le n$

Combinations

- How many ways are there to select five players from a 10-member tennis team to make a trip for a match at another school?

Solution:

This is a 5-combinations from a set with then elements.

$$C(10,5) = \frac{10!}{5!5!} = 252$$

Combinations

- Example

How many bit strings of length 10 contain exactly four 1s?

Solution:

The positions of four 1s in a bit string of length 10 is a 4-combination of the set {1, 2, 3, …, 10}. Hence, there are C(10, 4) bit strings of length 10 that contain exactly 4 1s.

C(10, 4)=210.

The pigeonhole principle

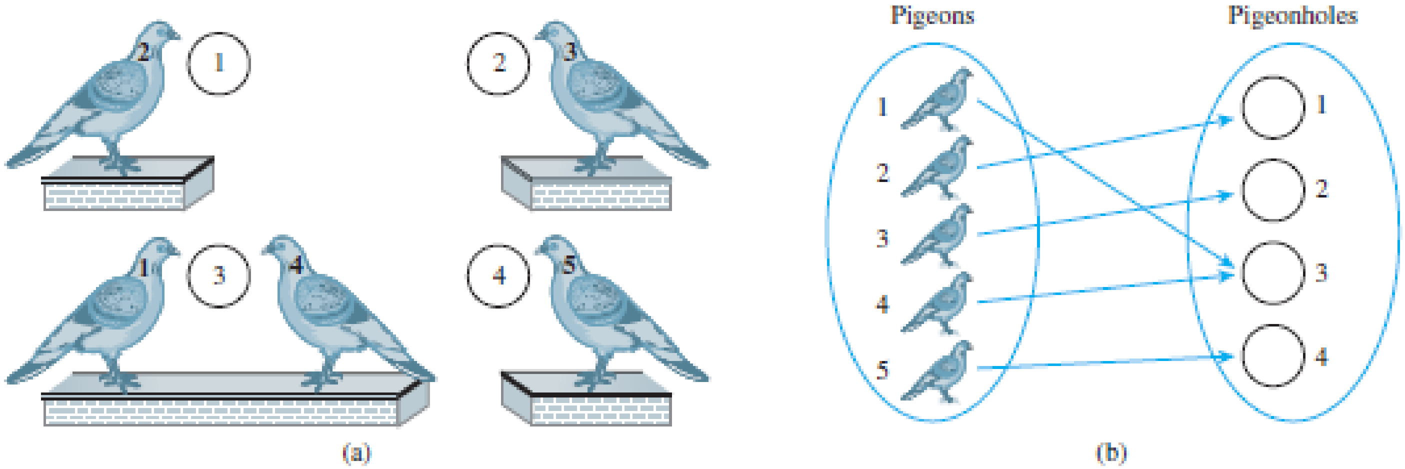


FIGURE 9.4.1

The Pigeonhole Principle

- If k is a positive integer and $k + 1$ or more objects are placed into k boxes, then there is at least one box containing two or more of the objects.

Pigeonhole Principle

A function from one finite set to a smaller finite set cannot be one-to-one: There must be at least two elements in the domain that have the same image in the co-domain.

The Pigeonhole Principle

- Let the number of holes is 16 and the number of pigeons is 17.

Θ	Θ	Θ	Θ
Θ	Θ	Θ	$\Theta\Theta$
Θ	Θ	Θ	Θ
Θ	Θ	Θ	Θ

Possible placement 1

Θ	Θ	Θ	
Θ	Θ	$\Theta\Theta$ Θ	
Θ	Θ	Θ	Θ
Θ	Θ	Θ	Θ

Possible placement 2

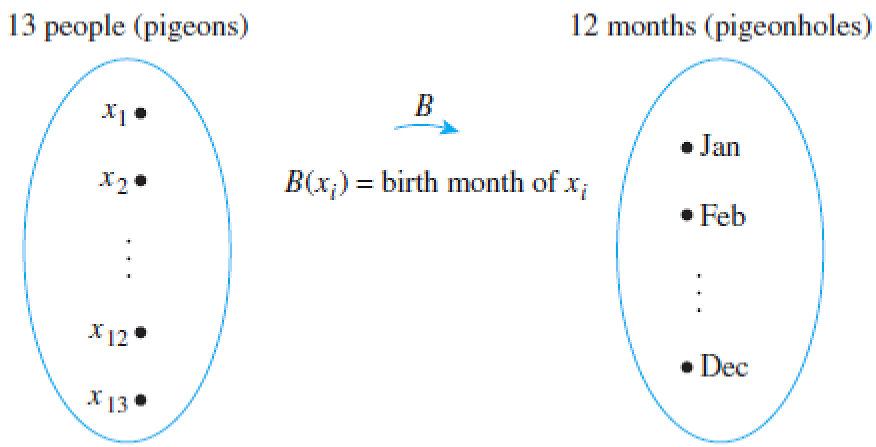
The Pigeonhole Principle

- Example:
- In any group of 27 English words, there must be at least two that begin with the same letter, since there are 26 letters in English alphabet.

The Pigeonhole Principle

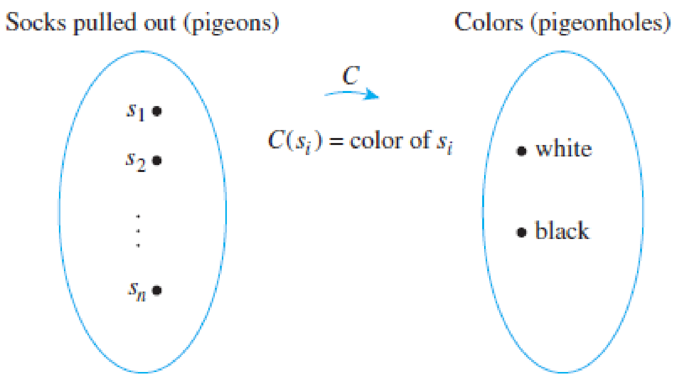
- Example:
- In a group of six people, must there be at least two who were born in the same month? In a group of thirteen people, must there be at least two who were born in the same month? Why?

Solution:



The Pigeonhole Principle

- Example:
- A drawer contains ten black and ten white socks. You reach in and pull some out without looking at them. What is the *least* number of socks you must pull out to be sure to get a matched pair? Explain how the answer follows from the pigeonhole principle.
- Solution:
- Let the socks pulled out be denoted $s1, s2, s3, \ldots, sn$ and consider the function C that sends each sock to its color.



If $n = 2$, C could be a one-to-one correspondence (if the two socks pulled out were of different colors). But if $n > 2$, then the number of elements in the domain of C is larger than the number of elements in the co-domain of C. Thus by the pigeonhole principle, C is not one-to-one: $C(s_i) = C(s_j)$ for some $s_i \neq s_j$. This means that if at least three socks are pulled out, then at least two of them have the same color. ■

The Generalized Pigeonhole Principle

A generalization of the pigeonhole principle states that if n pigeons fly into m pigeonholes and, for some positive integer k, $km < n$, then at least one pigeonhole contains k+1 or more pigeons.

Example: for m = 4, n =9, and k = 2. Since 2x4 < 9, at least one pigeonhole contains three (2+1) or more pigeons. (In this example, pigeonhole 3 contains three pigeons.)

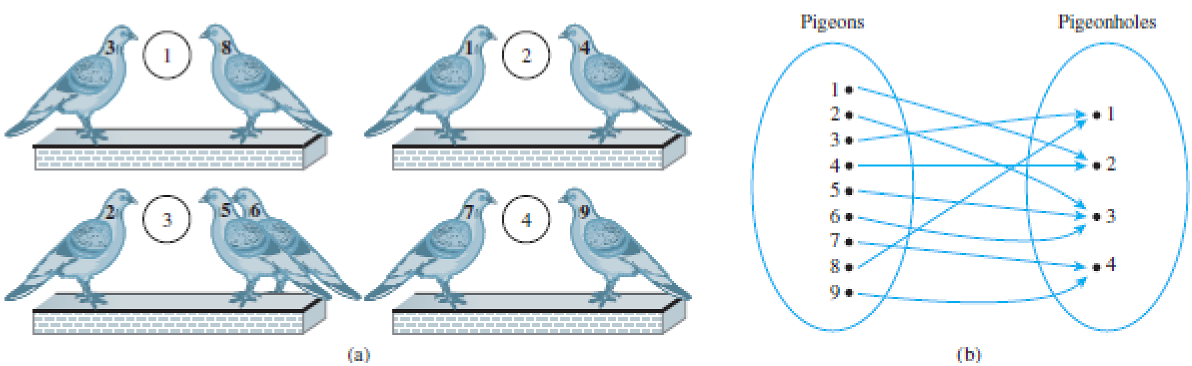
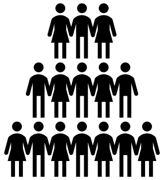


FIGURE 9.4.2

The Generalized Pigeonhole Principle

- Example:

  Among 100 people there are at least 9 who were born in the same month.



100 people
(pigeons)

1	2	3	12	(pigeonholes)

$n = 12$
$m = 100$
$k = 8$

$8 \times 12 = 96 < 100$

At least one of the pigeonhole has $8 + 1 = 9$ people

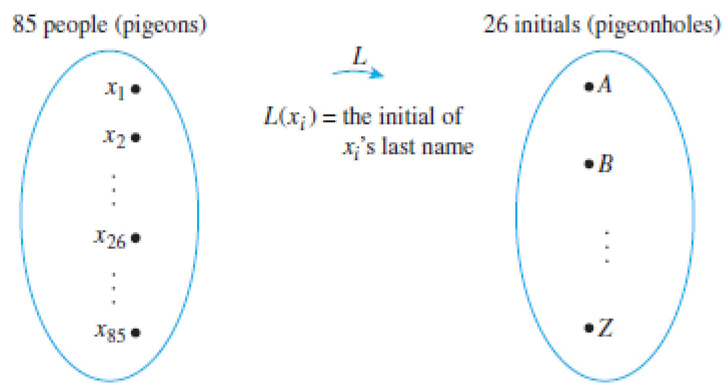
Example

- Show how the generalized pigeonhole principle implies that in a group of 85 people, at least 4 must have the same last initial.

- Solution:

In this example the pigeons are the 85 people and the pigeonholes are the 26 possible last initials of their names.

Consider the function L from people to initials defined by the following arrow diagram.



Since $3 \cdot 26 = 78 < 85$, the generalized pigeonhole principle states that some initial must be the image of at least four $(3 + 1)$ people. Thus at least four people have the same last initial.