PH4433/6433 HW 8, due Monday November 9

1. 10-dimensional integral

Write a program to do the following 10-dimensional integral using simple Monte Carlo integration, the "Stone throwing" method of Section 6.5.

$$I = \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 \cdots \int_{-\infty}^{\infty} dx_{10}\; e^{-\frac{1}{2}(x_1^2+x_2^2+\cdots+x_{10}^2)} \tag{1}$$

Since each integral evaluates to $\sqrt{2\pi}$ the exact answer is $(2\pi)^5$. With uniform random numbers it is not possible to extended the upper limit to infinity. However, since the integrand decreases rapidly for large x, using a finite upper limit does not cause a large error. Try lower/upper limits on all the integrals of ± 5.

(a) Make a plot of $|I_{\mathrm{MC}} - I_{\mathrm{exact}}|$ and the standard error versus N the number of Monte Carlo samples on a log-log plot. How do these quantities depend on N?

(b) Using uniform random numbers is inefficient because the integral is peaked about $\vec{x} = 0$. Instead, try taking x_i from an exponential distribution, $w(\vec{x}) \propto e^{-|x_1|}e^{-|x_2|} \ldots e^{-|x_{10}|}$ (this is a form of *importance sampling*). Here is an outline of the changes to make:

- If x_i is from a uniform density $[0,1]$, then $y_i = -\ln(1-x_i)$ is distributed according to the density $w(y_i) = e^{-y_i}$. To get $w(y_i) = e^{-|y_i|}$ you can simply multiply y_i by ± 1 with equal probabilities.
- In importance sampling, you will need to divide by the probability density you are sampling from: $f(\vec{x})/w(\vec{x})$.
- In the stone or dart throwing Monte Carlo, you multiply the average value of the function $f(\vec{x})$ by the volume of the sampled area. In the first part of this problem this volume was $(2 * x_{max})^{10}$, where x_{max} is cutoff on the lower/upper limit of the integral. With importance sampling, you need to change this factor to:

$$\int_{-\infty}^{\infty} dx_1 \cdots \int_{-\infty}^{\infty} dx_{10} e^{-|x_1|} \cdots e^{-|x_{10}|} \tag{2}$$

Plot the estimated and actual error versus N as in (a). The error for the same N should be much smaller.

2. Metropolis

Modify the Metropolis algorithm example from class to generate samples from the 10-dimensional probability distribution (with $x_i \in (-\infty, \infty)$):

$$P(\vec{x}) \propto e^{-\frac{1}{2}(x_1^2+x_2^2+\cdots+x_{10}^2)}$$

(a) Make a plot of the variable x_1 as generated by the algorithm. Do you see evidence for correlations?

(b) From the output of the algorithm, compute estimates for the following:

(a) $\langle x \rangle$

(b) $\langle x^2 \rangle$

(c) $\langle x^4 \rangle$

(the exact values should be 0, 1, and 3).