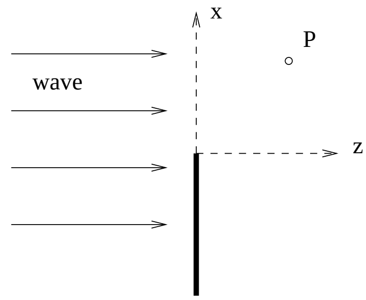
PH 4433/6433 HW 3, due Mon. Sept 14

1. Knife-edge Diffraction

In what is known as knife-edge diffraction, an incoming plane wave is partially blocked by a straight edge:



The intensity at the point P with coordinates (x,z) is given by

$$I = \frac{I_0}{8}\left([2C(u)+1]^2 + [2S(u)+1]^2\right)$$

where I_0 is the intensity of the incident wave and

$$u = x\sqrt{\frac{2}{\lambda z}} \qquad C(u) = \int_0^u \cos(\frac{1}{2}\pi t^2)dt \qquad S(u) = \int_0^u \sin(\frac{1}{2}\pi t^2)dt$$

1. Write a program to calculate I/I_0 using Gauss-Legendre quadrature to do the integrals. In your program, write a separate functions to calculate $C(u)$ and $S(u)$.
2. Assume $\lambda = 1$. Plot I/I_0 as a function of x for $-5 < x < 5$ and comparing (a) z=0.5, (b) z=1, (c) $z = 2$, and (d) $z = 3$. Discuss what value of N is necessary in the Gaussian quadrature to obtain accurate results.

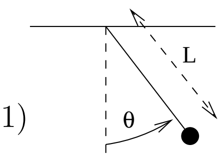
2. Differentiation

Use the 2–, 3–, and 5–point formulas to calculate the first derivatives of the functions $\sin(x)$ and e^x for x=0.1, 10, and 100.

1. Print the derivative and the relative error ϵ as a function of h. For what h is the relative error comparable to the machine precision for each x? (use double precision).
2. Plot $\log_{10}|\epsilon|$ as a function of $\log_{10} h$. Does the number of decimal places of accuracy agree with what we derived in class?
3. On your plots, identify regions where the approximation error dominates and where round-off error dominates. Use xmgrace to estimate the slopes of the plots.

3. Plane pendulum

The period of a plane pendulum of length L is given by



$$T = 2\sqrt{\frac{L}{g}} \int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}\sqrt{1-k^2x^2}} \tag{1}$$

where g is the acceleration of gravity. The constant k is determined by the highest angle the pendulum reaches, $\theta = \theta_0$:

$$k = \sin(\theta_0/2) \tag{2}$$

As is well known from introductory physics, for small oscillations (θ_0 small) the period $T_s = 2\pi\sqrt{\frac{L}{g}}$.

The form of integral in Eq. 1 is called an elliptic integral of the first kind. It is difficult to calculate using methods such as the trapezoid method because the integrand is singular at $x = \pm 1$. However, it is quite easy to do using Gaussian quadrature. Use Gauss-Chebyshev quadrature, appropriate for integrals of the form

$$S = \int_{-1}^{1} \frac{F(x)}{\sqrt{1-x^2}} dx \tag{3}$$

Gauss-Chebyshev quadrature is particularly easy because the weights and abscissas can be written *exactly*. For a N-point quadrature,

$$x_j = \cos\left[\frac{\pi(j-\frac{1}{2})}{N}\right] \qquad W_j = \frac{\pi}{N} \tag{4}$$

Then the approximate integral is simply

$$S \approx \sum_{j=1}^{N} W_j F(x_j) \tag{5}$$

1. Set $L = g = 1$. What is the function $F(x)$ for this problem?

2. Write a program to calculate T using Gauss-Chebyshev quadrature.

3. Check that you get the small-amplitude period when ϕ_0 is small. Modify the program so that N is chosen automatically to produce a given precision in T (say 10^{-6}). How does the required N depend on θ_0?

4. Make a plot of T as a function of ϕ_0 in the range $(0, \pi)$.

5. Make a plot of T/T_s showing how much the solution differs from the small oscillation result.