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Assignment 3 STA 243 Lee

1

1.a

$$
\begin{aligned}
\ell(\theta) &= x_1 \log(2+\theta) + (x_2+x_3)\log(1-\theta) + x_4 \log\theta + c \\
&= 125\log(2+\theta) + (18+19)\log(1-\theta) + 35\log\theta + c \\
\ell'(\theta) &= \frac{125}{2+\theta} + \frac{37}{1-\theta} + \frac{35}{\theta} \\
&= \frac{-197\theta^2 + 16\theta + 70}{(\theta+2)(1-\theta)\theta}
\end{aligned}
$$

Thus the MLE is found by setting the numerator of the derivative of the log-likelihood to zero.

$$
\begin{aligned}
0 &= -197\theta^2 + 16\theta + 70 \\
\theta &= \frac{1}{197}\left(8 \pm \sqrt{13854}\right) \\
&\approx -0.556868, 0.638086
\end{aligned}
$$

Since $0 \le \theta \le 1$, then the MLE is $\theta = \frac{1}{197}\left(8+\sqrt{13854}\right) \approx 0.638086$.

1.b

E-Step: Calculate

$$
\begin{aligned}
Q(\theta \,|\, \theta^{(k)}) &= \mathbb{E}_{\theta^{(k)}}\left\{\ell_c(\theta) \,|\, y\right\} \\
&= \mathbb{E}_{\theta^{(k)}}\left\{(x_{12}+x_4)\log\theta + (x_2+x_3)\log(1-\theta) \,|\, y\right\} \\
&= \mathbb{E}_{\theta^{(k)}}\left\{x_{12} \,|\, y\right\}\log\theta + x_4\log\theta + (x_2+x_3)\log(1-\theta) \\
&= \frac{\theta^{(k)}\,(1+x_2+x_3+x_4)!}{\left(1-\theta^{(k)}\right)^2 (x_2+x_3+x_4)!}\log\theta + x_4\log\theta + (x_2+x_3)\log(1-\theta)
\end{aligned}
$$

M-Step: Maximize $Q(\theta \,|\, \theta^{(k)})$ with respect to θ.

$$
\begin{aligned}
\frac{dQ(\theta^{(k+1)}, \,|\, \theta^{(k)})}{d\theta} &= 0 \\
&= \frac{1}{\theta^{(k+1)}}\frac{\theta^{(k)}}{\left(1-\theta^{(k)}\right)^2}\frac{(1+x_2+x_3+x_4)!}{(x_2+x_3+x_4)!} + \frac{x_4}{\theta^{(k+1)}} - \frac{x_2+x_3}{1-\theta^{(k+1)}} \\
\theta^{(k+1)} &= \frac{\theta^{(k)}\,(1+x_2+x_3+x_4)! + x_4\left(1-\theta^{(k)}\right)^2 (x_2+x_3+x_4)!}{\theta^{(k)}\,(1+x_2+x_3+x_4)! + (x_2+x_3+x_4)\left(1-\theta^{(k)}\right)^2 (x_2+x_3+x_4)!}
\end{aligned}
$$

1.c

I don't know actually how to calculate $Q\dots$ so $\theta^{(k)} \to 1$ as $k \to \infty$ which is not right...

2

2.a

First, find the CDF of the Weibull distribution,

$$\begin{aligned}
w\left(x \,|\, \alpha\right) &= \int_0^x W\left(t \,|\, \alpha\right) dt \\
&= \int_0^x \frac{2t}{\alpha^2} e^{-\frac{t^2}{\alpha^2}} dt \\
&= -e^{-\frac{t^2}{\alpha^2}} \Big|_0^x \\
&= 1 - e^{-\frac{x^2}{\alpha^2}}
\end{aligned}$$

Then, find the complete log-likelihood function

$$\begin{aligned}
\ell_c\left(\alpha\right) &= \sum_u \log\left(W\left(x_i \,|\, \alpha\right)\right) + \sum_c \log\left(1 - w\left(x_i \,|\, \alpha\right)\right) \\
&= \sum_u \log\left(\frac{2x_i}{\alpha^2} e^{-\frac{x_i}{\alpha^2}}\right) + \sum_c \log\left(e^{-\frac{x_i^2}{\alpha^2}}\right) \\
&= \sum_u \log\left(\frac{2x_i}{\alpha^2}\right) - \sum_{i=1}^n \frac{x_i^2}{\alpha^2}
\end{aligned}$$

Then, the E-step is

$$Q(\alpha \,|\, \alpha^{(k)}) = \mathbb{E}_{\alpha^{(k)}} \left\{\ell_c\left(\theta\right) | \, y\right\}$$

and the M-step is

$$\frac{dQ(\alpha^{(k+1)}, | \, \alpha^{(k)})}{d\alpha} = 0$$

2.b

3

$$f(x) = c\, e^{-x}, \qquad 0 < x < 2$$

$$\begin{aligned}
c^{-1} &= \int_0^2 e^{-x} dx \\
&= -e^{-x}\Big|_0^2 \\
&= 1 - e^{-2}
\end{aligned}$$

$$\begin{aligned}
F(x) &= \frac{\int_0^x e^{-t} dt}{1 - e^{-2}} \\
&= \frac{1 - e^{-x}}{1 - e^{-2}}
\end{aligned}$$

First sample u from unif$(0,1)$, then find $F^{-1}(u)$. Equivalently solve $F(x) - u = 0$.

$$\begin{aligned}
0 &= F(x) - u \\
0 &= \frac{1-e^{-x}}{1-e^{-2}} - u \\
0 &= 1 - e^{-x} - \left(1 - e^{-2}\right) u \\
x &= -\log\left(1 - \left(1 - e^{-2}\right) u\right)
\end{aligned}$$

The results are displayed in Figure 1. The gray bars represent frequency. The white bars represent the cumulative frequency. The lines represent the estimated densities.

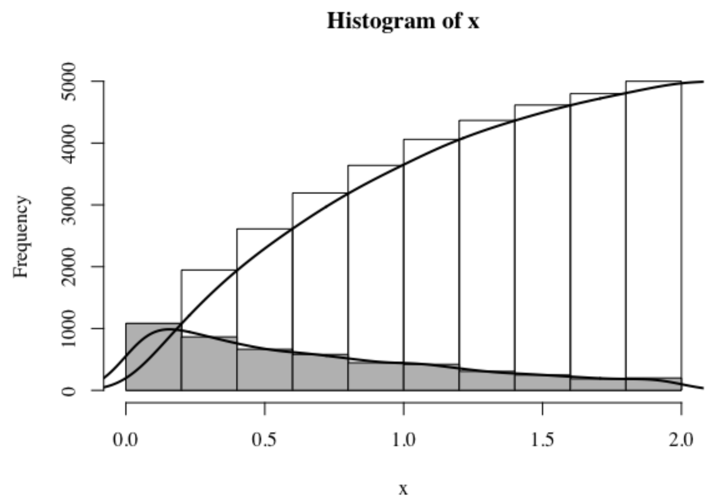


Figure 1: **Problem 3.**

4

4.a

$$\begin{aligned}
\alpha_1 &= \sup_{0<x} \frac{q(x)}{g_1(x)} = \sup_{0<x} \frac{\frac{e^{-x}}{1+x^2}}{e^{-x}} = \sup_{x>0} \frac{1}{1+x^2} = 1 \\
\alpha_2 &= \sup_{0<x} \frac{q(x)}{g_2(x)} = \sup_{0<x} \frac{\frac{e^{-x}}{1+x^2}}{\frac{2}{\pi\left(1+x^2\right)}} = \sup_{x>0} \frac{\pi}{2} e^{-x} = \frac{\pi}{2}
\end{aligned}$$

Use the inverse transform to sample x from $g_1(x)$ and $g_2(x)$ using u from unif$(0,1)$.

$$
\begin{aligned}
G_1(x) &= \int_0^x e^{-t} dt & \qquad G_2(x) &= \frac{2}{\pi} \int_0^x \frac{1}{1+t^2} dt \\
&= 1 - e^{-x} & &= \frac{2}{\pi} \tan^{-1} x \\
x &= -\log(1-u) & x &= \tan\left(\frac{\pi}{2} u\right)
\end{aligned}
$$

The results are shown in Table 1.

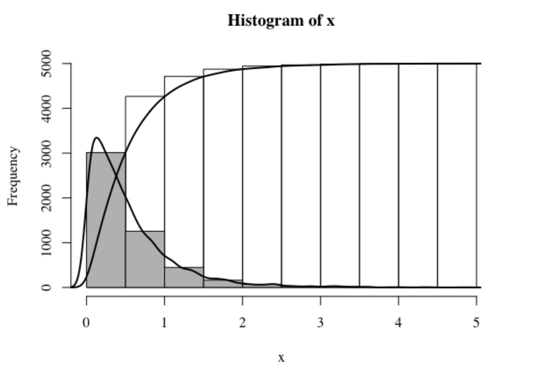
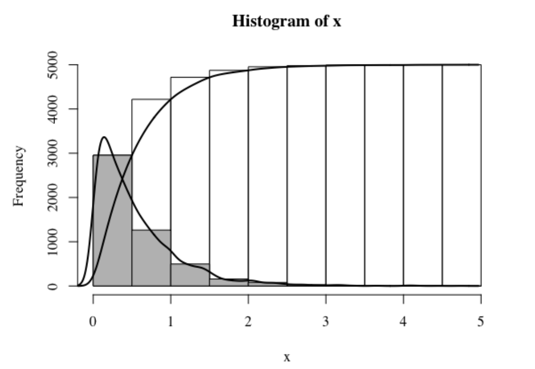
<table>
<tr><th>Sampling $f(x)$ using $g_1(x)$</th><th>Sampling $f(x)$ from $g_2(x)$</th></tr>
<tr><td>
</td><td>
</td></tr>
</table>

Table 1: **Problem 4.**

4.b

The number of iterations needed to produce 5,000 data points required approximately 8,000 iterations using $g_1(x)$ and 11,000 iterations using $g_2(x)$. This makes sense since the $\alpha_1 < \alpha_2$ and the acceptance probability, which is inversely proportional to the α value, is higher for $g_1(x)$.

5

First, the problem can be converted into polar coordinates with the form

$$f(r,\theta) \propto r^\alpha \cos^\alpha \theta \, r \sin\theta = r^{\alpha+1} \cos^\alpha \theta \sin\theta = q(r,\theta), \quad 0 < r \leq 1, \quad 0 < \theta < \frac{\pi}{2}$$

where $\alpha > -1$ since otherwise $f(r,\theta)$ would not be a probability distribution since the integral would not converge over the domain for $\alpha \leq -1$. $f(r,\theta)$ is now easily integrable where $f(r,\theta) = c\, q(r,\theta)$ and

$$
\begin{aligned}
c^{-1} &= \int_0^{\frac{\pi}{2}} \int_0^1 r^{\alpha+1} \cos^\alpha \theta \sin\theta \, r \, dr \, d\theta \\
&= \frac{1}{\alpha+3} \, r^{\alpha+3}\Big|_0^1 \frac{1}{\alpha+1} \cos^{\alpha+1} \theta\Big|_0^{\frac{\pi}{2}} \\
&= \frac{1}{\alpha+3} \frac{1}{\alpha+1} \\
c &= (\alpha+3)(\alpha+1) > 1, \quad \alpha > -1
\end{aligned}
$$

Then sample (x, y) from $[\,0,1\,] \times [\,0,1\,]$. If $x^2 + y^2 \leq 1$, then accept (x, y). Now sample u uniformly from $[\,0,1\,]$ and accept (x, y) if $u \leq f(x,y)/\,(\beta\, g\,(x,y))$ where $\beta = \sup\limits_{(x,y)\in\mathbb{R}^2} f(x,y)/g(x,y)$ and $f(x,y) \leq c\, g(x,y)$ for all (x,y) in the upper right quarter of the unit disc and $c > 0$

```
par(family = 'serif')
setwd("/Users/mikhailgaerlan/Box Sync/Education/UC Davis/2016-2017
Spring/STA 243 Computational Statistics/Assignments/Assignment 2")
rm(list=ls())

n = 5000
u = runif(n)
x = -log(1-(1-exp(-2))*u)

h = hist(x,plot=F)
hcum = h
hcum$counts=cumsum(hcum$counts)
plot(hcum)
plot(h,add=T,col='grey')

d = density(x)
lines(d$x,d$y*length(x)*diff(h$breaks)[1],lwd=2)
lines(d$x,cumsum(d$y)/max(cumsum(d$y))*length(x),lwd=2)
```

```
par(family = 'serif')
setwd("/Users/mikhailgaerlan/Box Sync/Education/UC Davis/2016-2017
Spring/STA 243 Computational Statistics/Assignments/Assignment 2")
rm(list=ls())

alpha1 = 1
alpha2 = pi/2

q = function(x){
  return(exp(-x)/(1+x^2))
}
g1 = function(x){
  return(exp(-x))
}
g2 = function(x){
  return(2/(pi*(1+x^2)))
}
sampleg1 = function(n){
  x = 0*(1:n)
  for(i in 1:n){
    u = runif(1)
    x[i] = -log(1-u)
  }
  return(x)
}
sampleg2 = function(n){
  x = 0*(1:n)
  for(i in 1:n){
    u = runif(1)
    x[i] = tan(pi*u/2)
  }
  return(x)
}
samplef = function(n,o){
  x2 = 0*(1:n)
  j = 0
  for (i in 1:n){
    test = T
    while (test){
      if (o == 1){
        test2 = T
        while (test2){
          x = sampleg1(1)
          if (x < 5){
            test2 = F
          }
        }
        j = j + 1
        if (runif(1) < q(x)/(alpha1*g1(x))){
          x2[i] = x
```

```
          test = F
        }
      } else {
        test2 = T
        while (test2){
          x = sampleg2(1)
          if (x < 5){
            test2 = F
          }
        }
        j = j + 1
        if (runif(1) < q(x)/(alpha2*g2(x))){
          x2[i] = x
          test = F
        }
      }
    }
  }
  print(j)
  return(x2)
}

x = samplef(5000,1)
h = hist(x,plot=F)
hcum = h
hcum$counts=cumsum(hcum$counts)
plot(hcum)
plot(h,add=T,col='grey')
d = density(x)
lines(d$x,d$y*length(x)*diff(h$breaks)[1],lwd=2)
lines(d$x,cumsum(d$y)/max(cumsum(d$y))*length(x),lwd=2)

x = samplef(5000,2)
h = hist(x,plot=F)
hcum = h
hcum$counts=cumsum(hcum$counts)
plot(hcum)
plot(h,add=T,col='grey')
d = density(x)
lines(d$x,d$y*length(x)*diff(h$breaks)[1],lwd=2)
lines(d$x,cumsum(d$y)/max(cumsum(d$y))*length(x),lwd=2)
```