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21 April 2017
Assignment 4 STA 243 Lee

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$\int_a^b f(x)h(x)dx$ can be approximated by

$$\begin{aligned}\hat{\mu}_{MC} &= \frac{1}{n}\sum_{i=1}^{n} h(X_i) \\ \operatorname{Var}(\hat{\mu}_n) &= \frac{1}{n}\sum_{i=1}^{n} \left(h(X_i) - \hat{\mu}_{MC}\right)^2\end{aligned}$$

where X_i is sampled from $f(x)$.

1.a $\int_0^1 x^2 dx$

$$\begin{aligned}f(x) &= \mathbb{1}_{[0,1]}(x) \\ h(x) &= x^2 \\ n &= 1000\end{aligned}$$

$$\begin{aligned}\hat{\mu}_n &= 0.33910520 \\ \operatorname{Var}(\hat{\mu}_n) &= 0.08896713\end{aligned}$$

1.b $\int_0^1 \int_{-2}^2 x^2 \cos(xy)\, dxdy$

$$\begin{aligned}f(x,y) &= \frac{1}{4}\mathbb{1}_{[-2,2]}(x)\mathbb{1}_{[0,1]}(y) \\ h(x,y) &= 4x^2\cos(xy) \\ n &= 1000\end{aligned}$$

$$\begin{aligned}\hat{\mu}_n &= 3.501649 \\ \operatorname{Var}(\hat{\mu}_n) &= 13.390987\end{aligned}$$

1.c $\int_0^\infty \frac{3}{4}x^4 e^{-x^3/4} dx$

$$\begin{aligned}f(x) &= \frac{1}{4}\mathbb{1}_{[0,\infty)}(x)e^{-x/4} \\ h(x) &= 3x^4 e^{-x^3/4+x/4} \\ n &= 1000\end{aligned}$$

$$\begin{aligned}\hat{\mu}_n &= 2.188157 \\ \operatorname{Var}(\hat{\mu}_n) &= 12.954084\end{aligned}$$

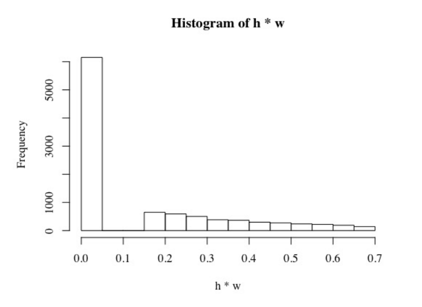
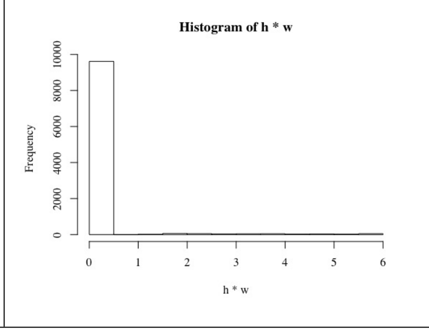
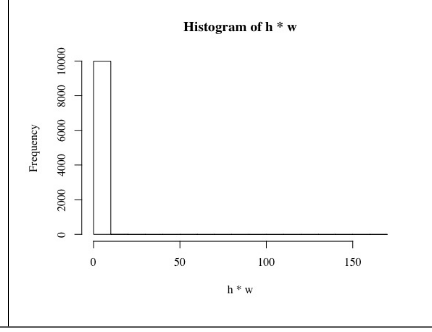
2 $\quad I = \frac{1}{\sqrt{2\pi}} \int_1^2 e^{-x^2/2} dx$

$$\begin{aligned} f(x) &= \mathbb{1}_{[1,2]} \\ h(x) &= \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \\ g(x) &= \frac{1}{\sqrt{2\pi\nu^2}} e^{-(x-1.5)^2/2\nu^2} \\ n &= 10000 \end{aligned}$$

$\nu = 0.1, \quad \hat{I}_n = 0.1169773, \quad \operatorname{Var}\left(\hat{I}_n\right) = 3.1721584$

$\nu = 1, \quad \hat{I}_n = 0.13391863, \quad \operatorname{Var}\left(\hat{I}_n\right) = 0.03766775$

$\nu = 10, \quad \hat{I}_n = 0.1344222, \quad \operatorname{Var}\left(\hat{I}_n\right) = 0.5202285$

$\nu = 0.1$	$\nu = 1$	$\nu = 10$
		

3 $\quad I = \int_0^1 \frac{1}{1+x} dx$

3.a

$$\begin{aligned} f(x) &= \mathbb{1}_{[0,1]} \\ h(x) &= \frac{1}{1+x} \\ n &= 1500 \end{aligned}$$

$$\hat{I}_{MC} = 0.69635548$$
$$\operatorname{Var}\left(\hat{I}_{MC}\right) = 0.02066273$$

3.b

$$\begin{aligned} b &= \frac{\mathrm{Cov}\left(\hat{I}_{MC}, \hat{\theta}_{MC}\right)}{\mathrm{Var}\left(\hat{\theta}_{MC}\right)}, \qquad \hat{\theta}_{MC} = \frac{1}{n}\sum_{i=1}^{n} c\left(U_i\right) \\ \mathrm{Cov}\left(\hat{I}_{MC}, \hat{\theta}_{MC}\right) &= \frac{1}{n-1}\sum_{i=1}^{n}\left[h\left(U_i\right) - \hat{I}_{MC}\right]\left[c\left(U_i\right) - \hat{\theta}_{MC}\right] \\ \mathrm{Var}\left(\hat{\theta}_{MC}\right) &= \frac{1}{n-1}\sum_{i=1}^{n}\left(c\left(U_i\right) - \hat{\theta}_{MC}\right)^2 \end{aligned}$$

$$\begin{aligned} E\left\{c\left(U\right)\right\} &= \int_0^1 (1+x)\,dx \\ &= x + \frac{1}{2}x^2\Big|_0^1 = \frac{3}{2} \end{aligned}$$

$$\begin{aligned} b &= -0.4793439 \\ \hat{I}_{CV} &= \quad 0.6944337896 \\ \mathrm{Var}\left(\hat{I}_{CV}\right) &= \quad 0.0006408379 \end{aligned}$$

3.c

The variance for $\hat{I}_{CV}$ is 2 orders of magnitude smaller than for $\hat{I}_{MC}$.

$$\begin{aligned} \mathrm{Var}\left(\hat{I}_{MC}\right) &= 0.02066273 \\ \mathrm{Var}\left(\hat{I}_{CV}\right) &= 0.0006408379 \end{aligned}$$

3.d

Define two control variates $c_1(x), c_2(x)$ so

$$\begin{aligned} \hat{I}_{CV} &= \frac{1}{n}\sum_{i=1}^{n} h\left(U_i\right) - b_1\left[\frac{1}{n}\sum_{i=1}^{n} c_1\left(U_i\right) - E\left\{c_1\left(U\right)\right\}\right] - b_2\left[\frac{1}{n}\sum_{i=1}^{n} c_2\left(U_i\right) - E\left\{c_2\left(U\right)\right\}\right] \\ \mathrm{Var}\left(\hat{I}_{CV}\right) &= \mathrm{Var}\left(\hat{I}_{MC}\right) + b_1^2\mathrm{Var}\left(\hat{\theta}_{1MC}\right) + b_2^2\mathrm{Var}\left(\hat{\theta}_{2MC}\right) - 2b_1\mathrm{Cov}\left(\hat{I}_{MC}, \hat{\theta}_{1MC}\right) - \\ &\qquad 2b_2\mathrm{Cov}\left(\hat{I}_{MC}, \hat{\theta}_{2MC}\right) - 2b_1b_2\mathrm{Cov}\left(\hat{\theta}_{1MC}, \hat{\theta}_{2MC}\right) \end{aligned}$$

where the optimal b_1, b_2 are

$$\begin{aligned} b_1 &= \frac{\mathrm{Cov}\left(\hat{\theta}_{1MC}, \hat{\theta}_{2MC}\right)\mathrm{Cov}\left(\hat{I}_{MC}, \hat{\theta}_{2MC}\right) + \mathrm{Cov}\left(\hat{I}_{MC}, \hat{\theta}_{1MC}\right)\mathrm{Var}\left(\hat{\theta}_{2MC}\right)}{\mathrm{Var}\left(\hat{\theta}_{1MC}\right)\mathrm{Var}\left(\hat{\theta}_{2MC}\right) - \left(\mathrm{Cov}\left(\hat{\theta}_{1MC}, \hat{\theta}_{2MC}\right)\right)^2} \\ b_2 &= \frac{\mathrm{Cov}\left(\hat{\theta}_{1MC}, \hat{\theta}_{2MC}\right)\mathrm{Cov}\left(\hat{I}_{MC}, \hat{\theta}_{1MC}\right) + \mathrm{Cov}\left(\hat{I}_{MC}, \hat{\theta}_{2MC}\right)\mathrm{Var}\left(\hat{\theta}_{1MC}\right)}{\mathrm{Var}\left(\hat{\theta}_{1MC}\right)\mathrm{Var}\left(\hat{\theta}_{2MC}\right) - \left(\mathrm{Cov}\left(\hat{\theta}_{1MC}, \hat{\theta}_{2MC}\right)\right)^2} \end{aligned}$$

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4.a

H_0: If $\mu_j = \sum_{i=1}^{n} y_{ij}$, then all the μ_j's are the same.
H_a: $\exists\, i, j$ such that $\mu_i \neq \mu_j$

Simulate the experiment from by generating e_{ij} and thus a set of data y_{ij}. Then compute the average of all the μ_j's. Repeat 998 times. Generate 1 more average of the μ_j's. If the last μ_j is among the smallest 2.5% or largest 2.5%, then reject the null hypothesis.

4.b

Similar to Problem 4.a except repeat the entire process for various distributions for e_{ij}.

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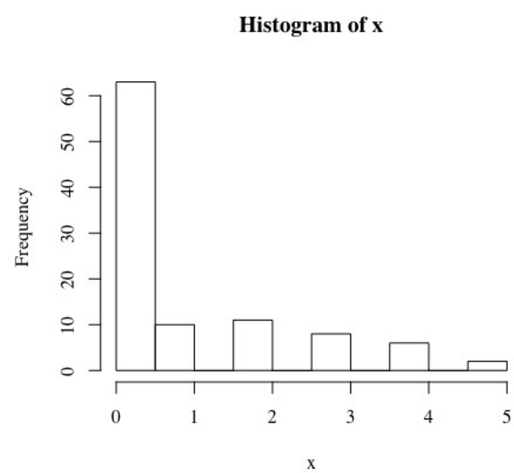
5.a



5.b

5.b.1

$$\begin{aligned}
f(\lambda \mid p, \mathbf{r}, \mathbf{x}) &= \frac{b^a \lambda^{a-1} e^{-b\lambda}}{\Gamma(a)} \prod_{i=1}^{n} \frac{e^{-\lambda r_i} (\lambda r_i)^{x_i}}{x_i!} p^{r_i} (1-p)^{1-r_i} \\
&= \frac{b^a \lambda^{a-1} e^{-b\lambda}}{\Gamma(a)} e^{\lambda \Sigma_i r_i} \lambda^{\Sigma_i x_i} \prod_{i=1}^{n} \frac{r_i^{x_i}}{x_i!} p^{r_i} (1-p)^{1-r_i} \\
&= \frac{b^a \lambda^{(a+\Sigma_i x_i)-1} e^{-(b+\Sigma_i r_i)\lambda}}{\Gamma(a)} \sum_{i=1}^{n} \frac{r_i^{x_i}}{\Gamma(x_i)} p^{r_i} (1-p)^{1-r_i} \\
&\vdots \\
&\stackrel{?}{=} \frac{(b+\Sigma_i r_i)^{(a+\Sigma_i x_i)} \lambda^{(a+\Sigma_i x_i)-1} e^{-(b+\Sigma_i r_i)\lambda}}{\Gamma(a+\Sigma_i x_i)} \\
(\lambda \mid p, \mathbf{r}, \mathbf{x}) &\sim \text{Gamma}(a+\Sigma_i x_i, b+\Sigma_i r_i)
\end{aligned}$$

5.b.2

$$\begin{aligned}
f(p \mid \lambda, \mathbf{r}, \mathbf{x}) &= \frac{b^a \lambda^{a-1} e^{-b\lambda}}{\Gamma(a)} \prod_{i=1}^{n} \frac{e^{-\lambda r_i} (\lambda r_i)^{x_i}}{x_i!} p^{r_i} (1-p)^{1-r_i} \\
&\vdots \\
&\stackrel{?}{=} \frac{\Gamma((1+\Sigma_i r_i)+(n+1-\Sigma_i r_i))}{\Gamma(1+\Sigma_i r_i)\Gamma(n+1-\Sigma_i r_i)} x^{(1+\Sigma_i r_i)-1} (1-x)^{(n+1-\Sigma_i r_i)-1} \\
(p \mid \lambda, \mathbf{r}, \mathbf{x}) &\sim \text{Beta}(1+\Sigma_i r_i, n+1-\Sigma_i r_i)
\end{aligned}$$

5.b.3

$$\begin{aligned}
f(\mathbf{r} \mid \lambda, p, \mathbf{x}) &= \frac{b^a \lambda^{a-1} e^{-b\lambda}}{\Gamma(a)} \prod_{i=1}^{n} \frac{e^{-\lambda r_i} (\lambda r_i)^{x_i}}{x_i!} p^{r_i} (1-p)^{1-r_i} \\
&\vdots \\
f(r_i \mid \lambda, p, \mathbf{x}) &\stackrel{?}{=} \left(\frac{pe^{-\lambda}}{pe^{-\lambda}+(1-p) I_{x_i=0}}\right)^{r_i} \left(1-\left(\frac{pe^{-\lambda}}{pe^{-\lambda}+(1-p) I_{x_i=0}}\right)\right)^{1-r_i} \\
(r_i \mid \lambda, p, \mathbf{x}) &\sim \text{Bernoulli}\left(\frac{pe^{-\lambda}}{pe^{-\lambda}+(1-p) I_{x_i=0}}\right)
\end{aligned}$$

5.c

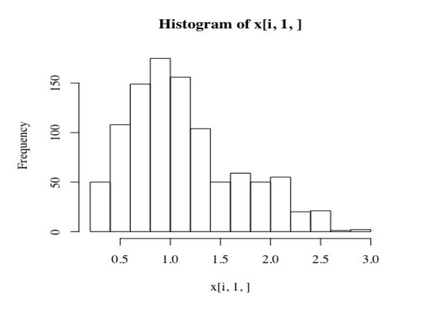
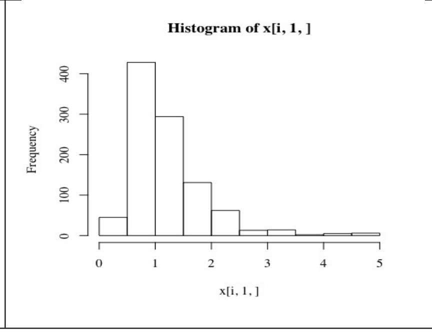
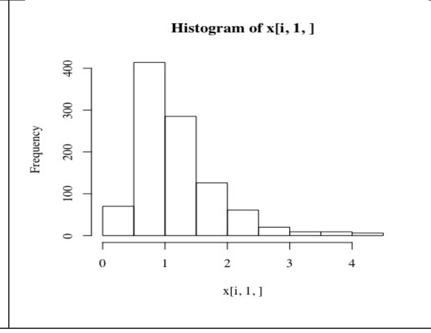
6

$X_0 = 1$. Sample $Y_i \sim \text{Gamma}(X_{i-1}, \beta)$ and $U_i \sim \text{Unif}(0,1)$. To assess the accuracy, 100 sets of samples for each beta were found. The means $\mu_{\beta,j}(X), \mu_{\beta,j}(1/X), 1 \leq j \leq 100$ of each set was determined. Then $\bar{\mu}_\beta(X)$, $\bar{\mu}_\beta(1/X)$ was found to be the means of $\{\mu_{\beta,1}(X_i), \ldots, \mu_{\beta,100}(X)\}$,

$\{\mu_{\beta,1}(1/X), \ldots, \mu_{\beta,100}(1/X)\}$ and $\sigma^2_\beta(X), \sigma^2_\beta(1/X)$ was determined to be the mean and variance.

$$E(Z) = \sqrt{\frac{\theta_2}{\theta_1}} \approx 1.15470054 \quad \text{and} \quad E\left(\frac{1}{Z}\right) = \sqrt{\frac{\theta_1}{\theta_2} + \frac{1}{2\theta_2}} \approx 1.11602540$$

$$\begin{array}{rrr}
\beta = 0.5 & \beta = 1.0 & \beta = 2.0 \\
\bar{\mu}_\beta(X) = 1.14311467 & \bar{\mu}_\beta(X) = 1.15240804 & \bar{\mu}_\beta(X) = 1.14157197 \\
\sigma^2_\beta(X) = 0.00401173 & \sigma^2_\beta(X) = 0.00225613 & \sigma^2_\beta(X) = 0.00463643 \\
\bar{\mu}_\beta(1/X) = 1.11463973 & \bar{\mu}_\beta(1/X) = 1.11431215 & \bar{\mu}_\beta(1/X) = 1.112364733 \\
\sigma^2_\beta(1/X) = 0.00220111 & \sigma^2_\beta(1/X) = 0.00254056 & \sigma^2_\beta(1/X) = 0.00564459
\end{array}$$

$\beta = 0.5$	$\beta = 1$	$\beta = 2$
		

```
par(family = 'serif')
setwd("/Users/mikhailgaerlan/Box Sync/Education/UC Davis/2016-2017
Spring/STA 243 Computational Statistics/Assignments/Assignment 4")
rm(list=ls())

n = 1000
x = runif(n)
h = x^2
mu = mean(h)
var = sum((h-mu)^2)/(n-1)
print(c(mu,var))

n = 1000
x = cbind(runif(n,-2,2),runif(n,0,1))
h = 4*x[,1]^2*cos(x[,1]*x[,2])
mu = mean(h)
var = sum((h-mu)^2)/(n-1)
print(c(mu,var))

n = 1000
x = rexp(n,1/4)
h = 3*x^4*exp(-x^3/4+x/4)
mu = mean(h)
var = sum((h-mu)^2)/(n-1)
print(c(mu,var))
```

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par(family = 'serif')
setwd("/Users/mikhailgaerlan/Box Sync/Education/UC Davis/2016-2017
Spring/STA 243 Computational Statistics/Assignments/Assignment 4")
rm(list=ls())

nu = c(0.1,1,10)

for(i in nu){
  m = 10000
  x = rnorm(m,1.5,i)
  f = 0*x
  for(j in 1:m){
    if(x[j]>=1&&x[j]<=2){
      f[j] = 1
    }
  }
  g = dnorm(x,1.5,i)
  h = dnorm(x,0,1)
  w = f/g
  mu = mean(h*w)
  var = sum((h*w-mu)^2)/(m-1)
  hist(h*w)
  print(c(mu,var))
}
```

```
par(family = 'serif')
setwd("/Users/mikhailgaerlan/Box Sync/Education/UC Davis/2016-2017
Spring/STA 243 Computational Statistics/Assignments/Assignment 4")
rm(list=ls())

#=======================
#       3 (a)
#=======================
n = 1500
u = runif(n)

h = 1/(1+u)
Imc = mean(h)
Imcvar = sum((h-Imc)^2)/n
print(c(Imc,Imcvar,log(2)))

c = 1+u
thetamc = mean(c)
thetavar = sum((c-thetamc)^2)/(n-1)

covar = sum((h-Imc)*(c-thetamc))/(n-1)
b = covar/thetavar
print(b)
Icv = mean(h)-b*(mean(c)-3/2)
Icvvar = Imcvar+b^2*thetavar-2*b*covar
print(c(Icv,Icvvar,log(2)))

par(family = 'serif')
setwd("/Users/mikhailgaerlan/Box Sync/Education/UC Davis/2016-2017
Spring/STA 243 Computational Statistics/Assignments/Assignment 4")
rm(list=ls())

#=======================
#       3 (d)
#=======================
n = 1500

u = runif(n)
h = 1/(1+u)
Imc = mean(h)
var = sum((h-Imc)^2)/n

c1 = 1+u
theta1 = mean(c1)
var1 = sum((c1-theta1)^2)/(n-1)

c2 = 1-u/2
theta2 = mean(c2)
var2 = sum((c2-theta2)^2)/(n-1)
```

```
covar1 = sum((h-Imc)*(c1-theta1))/(n-1)
covar2 = sum((h-Imc)*(c2-theta2))/(n-1)
covar12 = sum((c1-theta1)*(c2-theta2))/(n-1)
b1 = (covar12*covar2+covar1*var2)/(var1*var2-covar12^2)
b2 = (covar12*covar1+covar2*var1)/(var1*var2-covar12^2)
Icv = mean(h)-b1*(mean(c1)-(3/2))-b2*(mean(c2)-(3/4))
Icvvar = var+b1^2*var1+b2^2*var2-2*b1*covar1-2*b2*covar2-2*b1*b2*covar12
print(c(b1,b2))
print(c(Icv,Icvvar,log(2)))
```

```
par(family = 'serif')
setwd("/Users/mikhailgaerlan/Box Sync/Education/UC Davis/2016-2017
Spring/STA 243 Computational Statistics/Assignments/Assignment 4")
rm(list=ls())

#========================
#       4 (a)
#========================
require("Rlab")

n = 100
lambda = 2
p = 0.3
y = rpois(n,lambda)
r = rbern(n,p)
x = y*r
hist(x)
```

```r
par(family = 'serif')
setwd("/Users/mikhailgaerlan/Box Sync/Education/UC Davis/2016-2017 Spring/STA 243 Computational
Statistics/Assignments/Assignment 4")
rm(list=ls())

options(digits=6)

betas = c(0.5,1,2)
m = 100  #Number of experiments
n = 1000 #Number of samples
x = array(0,c(length(betas),m,n))
theta1 = 1.5
theta2 = 2
for(i in 1:length(betas)){
  means = 1:m
  means1 = 1:m
  for (j in 1:m){
    f = function(z){
      return((z^(-3/2))*exp(-theta1*z-theta2/z+2*sqrt(theta1*theta2)+log(sqrt(2*theta2))))
    }

    g = function(x,y){
      alpha = y
      return(dgamma(x,alpha,betas[i]))
    }

    r = function(x,y){
      return(min(c(f(y)*g(x,y)/(f(x)*g(y,x)),1)))
    }

    x[i,j,1] = 1
    for(k in 2:n){
      y = rgamma(1,x[i,j,k-1],betas[i])
      if(runif(1) <= r(x[i,j,k-1],y)){
        x[i,j,k] = y
      }else{
        x[i,j,k]=x[i,j,k-1]
      }
    }
    means[j] = mean(x[i,j,])
    means1[j] = mean(1/x[i,j,])
    #print(c(betas[i],mu,vr,sqrt(2/1.5),mu1,vr1,sqrt(1.5/2)+1/(2*2)))
  }
  hist(x[i,1,])

print(c(betas[i],mean(means),var(means),sqrt(theta2/theta1),mean(means1),var(means1),sqrt(theta1/theta2)+1/(2*theta2)))

}
```