The Power Method

Any matrix $M \in \mathbb{R}^{n\times n}$ has n eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, which we assume to be numbered with decreasing absolute value:

$$|\lambda_1| \geq |\lambda_2| \geq |\lambda_3| \geq \cdots \geq |\lambda_n|\,. \tag{1}$$

If in addition $|\lambda_1| > |\lambda_2|$, then λ_1 is said to be the *dominant* eigenvalue of M. Note that the dominant eigenvalue of a real matrix M is guaranteed to be a real number.

The dominant eigenvalue λ and a corresponding eigenvector x, i.e., a vector $x \in \mathbb{R}^n$, $x \neq 0$, such that $Mx = \lambda x$, can be computed with a simple iterative method called the *power method*. One important feature of this method is that the matrix M is only used to compute matrix-vector products. This makes it easy to exploit sparsity or other structures of the matrix M.

In the following, we denote by

$$\|x\|_\infty := \max_{i=1,2,\ldots,n} |x_i|$$

the *infinity norm* of vectors

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n.$$

This definition implies that at least one of the entries of x has absolute value $\|x\|_\infty$, i.e.,

$$|x_j| = \|x\|_\infty \quad \text{for some} \quad j \in \{\, 1, 2, \ldots, n \,\}. \tag{2}$$

Note that in general, the x_j in (2) can be negative. In particular, simply setting $x_j = \|x\|_\infty$ in the following statement of the power method will lead to incorrect results in general.

Algorithm (Power Method):

Input: A function for computing matrix-vector products Mx with M;
An initial vector $x \in \mathbb{R}^n$, $x \neq 0$;
A parameter $\varepsilon > 0$ for the convergence check;
An iteration limit $k_{\max}$ to safeguard against nonconvergence

1) Determine an entry x_j of x such that $|x_j| = \|x\|_\infty$, and set $\lambda^{(0)} = x_j$.

2) For $k = 1, 2, \ldots, k_{\max}$ do the following:

   a) Set $x = Mx$.
   b) Determine an entry x_j of x such that $|x_j| = \|x\|_\infty$, and set $\lambda^{(k)} = x_j$.
   c) Set $x = \dfrac{1}{\lambda^{(k)}}\, x$
   d) If $\left|\lambda^{(k)} - \lambda^{(k-1)}\right| \leq \varepsilon \left|\lambda^{(k-1)}\right|$, stop.

Output: The approximation $\lambda^{(k)}$ of the dominant eigenvalue λ of M and a corresponding approximate eigenvector x

Notes:

1) Practical choices for the initial vector x of the power method are

$$x = e := \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \in \mathbb{R}^n \qquad \text{and} \qquad \text{random vectors } x \in \mathbb{R}^n. \tag{3}$$

   Typical values for the parameter ε, which is used in the convergence check, range from $\varepsilon = 10^{-15}$ to $\varepsilon = 10^{-8}$.

2) Provided that M has indeed a dominant eigenvalue, i.e., $|\lambda_1| > |\lambda_2|$ in (1), the power method is guaranteed to convergence in exact arithmetic. Furthermore, the speed of convergence depends on the ratio

$$0 \leq \left| \frac{\lambda_2}{\lambda_1} \right| < 1. \tag{4}$$

   More precisely, the smaller the ratio (4), the faster the convergence of the power method. In finite-precision arithmetic, however, the power method may not convergence if the ratio (4) is very close to 1.

3) The power method is the main tool for computing the ranking of websites discussed in class. In this case, the algorithm is applied to $M = A^T$, where $A \in \mathbb{R}^{n \times n}$ is a row-stochastic matrix of the form

$$A = Q + \frac{1}{n} v\, e^T$$

   and e is the vector defined in (3). However, in finite-precision arithmetic, convergence can be exceedingly slow or the algorithm may not converge at all. A standard approach to speed up convergence is to apply the algorithm to $M = A_\alpha^T$, where

$$A_\alpha = \alpha\, A + (1 - \alpha)\, \frac{1}{n}\, e\, e^T.$$

   and $0 \leq \alpha \leq 1$ is a parameter. A typical choice is $\alpha = 0.85$.