Lecture 4

12 Aug 2019

Mistakes from Lecture 3

$$M = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ 0 & -1 & 1 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} 1 & 0 & -\frac{3}{2} & \frac{1}{2} \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & \frac{4}{3} \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{U}$$

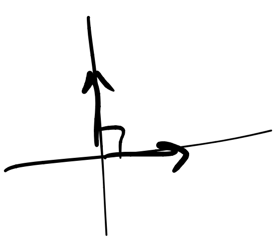
Book has mistakes. It is PLDU not LDPU.

Example 31 has the wrong order.

Chapter 4 (cont.)

<u>Definition</u>: Two vectors are <u>orthogonal</u> (or <u>perpendicular</u>) if their dot product is zero.

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 1 \cdot 0 + 0 \cdot 1 = 0$$



<u>Theorem 4.3.1.</u> Cauchy-Schwarz Inequality

$$\frac{|\langle u, v \rangle|}{\|u\| \|v\|} \leq 1 \qquad |\langle u, v \rangle| \leq \|u\| \|v\|$$

<u>Proof:</u> Let α be any real number.

$$0 \le \langle u+\alpha v, u+\alpha v\rangle = \langle u,u\rangle + 2\alpha\langle u,v\rangle + \alpha^2\langle v,v\rangle$$

$$\langle u, u+\alpha v\rangle + \langle \alpha v, u+\alpha v\rangle$$

$$\langle u,u\rangle + \langle u,\alpha v\rangle + \langle \alpha v, u\rangle + \langle \alpha v, \alpha v\rangle$$

$$\langle u,u\rangle + \alpha\langle u,v\rangle + \alpha\langle v,u\rangle + \alpha^2\langle v,v\rangle$$

$$\underbrace{\langle u,u\rangle}_{c} + \underbrace{2\langle u,v\rangle}_{b}\alpha + \underbrace{\langle v,v\rangle}_{a}\alpha^2 \qquad c + bx + ax^2$$

$$\frac{-b}{2a}$$

$$\alpha = \frac{-2\langle u,v\rangle}{2\langle v,v\rangle}$$

$$\langle u,u\rangle + 2\langle u,v\rangle\left(\frac{-2\langle u,v\rangle}{2\langle v,v\rangle}\right) + \langle v,v\rangle\left(\frac{-2\langle u,v\rangle}{2\langle v,v\rangle}\right)^2$$

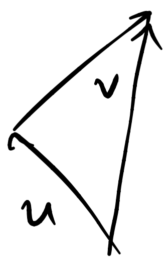
$$\langle u,u\rangle - \frac{2\langle u,v\rangle^2}{\langle v,v\rangle} + \frac{\cancel{\langle v,v\rangle}\langle u,v\rangle^2}{\langle v,v\rangle^{\cancel{2}}} \qquad \sqrt{\langle u,u\rangle} = \|u\|$$

$$0 \le \langle u,u\rangle - \frac{\langle u,v\rangle^2}{\langle v,v\rangle} \iff \langle u,v\rangle^2 \le \langle u,u\rangle\langle v,v\rangle$$

$$|\langle u,v\rangle| \le \|u\|\|v\|$$

Theorem 4.3.2. (Triangle Inequality) (in $\mathbb{R}^n$)

$$\|u+v\| \le \|u\| + \|v\|$$



$$\|u+v\|^2 = (u+v)\cdot(u+v)$$

$$= u\cdot u + 2u\cdot v + v\cdot v$$

$$= \|u\|^2 + \|v\|^2 + 2\|u\|\|v\|\cos\theta$$

$$= \|u\|^2 + 2\|u\|\|v\| + \|v\|^2 - 2\|u\|\|v\| + 2\|u\|\|v\|\cos\theta$$

$$= (\|u\| + \|v\|)^2 + 2\|u\|\|v\|\underbrace{(\cos\theta - 1)}_{\le 0}$$

$$\|u+v\|^2 \le (\|u\| + \|v\|)^2$$

$$\|u+v\| \le \|u\| + \|v\|$$

4.4. Vectors, Lists, Functions

$$f: D \to \mathbb{R} \Rightarrow \mathbb{R}^D$$

$$S = \{*, \star, \#\}$$

$$f: S \to \mathbb{R} \Leftrightarrow \mathbb{R}^S$$

$$* \to 1$$

$$\star \to 2$$

$$\# \to 3$$

$$S = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix} \right\}$$

$$f: S \to \mathbb{R}$$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \to 5$$

$$\begin{bmatrix} 3 \\ 4 \end{bmatrix} \to 6$$

Chapter 5 Vector Spaces

A vector space is a set that is closed under addition and scalar multiplication.

$\mathbb{R}^2 = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mid x_1, x_2 \in \mathbb{R} \right\}$

Definition: A vector space $(V, +, \cdot, \mathbb{R})$ with $\forall u, v \in V$, $c, d \in \mathbb{R}$, we have:

$(+i)$ (Additive closure) $u+v \in V$

$(+ii)$ (Additive Commutativity) $u+v = v+u$

$(+iii)$ (Additive Associativity) $(u+v)+w = u+(v+w)$

$(+iv)$ (Zero) $0_V \in V$ such that $u+0_V = u$

$(+v)$ (Additive Inverse) For every $u \in V$, $\exists\, w \in V$ s.t. $u+w = 0_V$.

$(\cdot i)$ (Multiplicative Closure) $c \cdot v \in V$

$(\cdot ii)$ (Distributivity) $(c+d)\cdot v = c\cdot v + d\cdot v$

$(\cdot iii)$ (Distributivity) $c\cdot(u+v) = c\cdot u + c\cdot v$

$(\cdot iv)$ (Associativity) $(cd)\cdot v = c\cdot(d\cdot v)$

$(\cdot v)$ (Unity) $1\cdot v = v$, $\forall v \in V$

Example: $\mathbb{R}^{\mathbb{N}} = \{ f \mid f : \mathbb{N} \to \mathbb{R} \}$

$(0, 1, 2, 3, 4, \ldots)$ $\quad (3, \pi, 6, 1, 5, \ldots)$

$(1, 4, 1, 5, 9, 2, 6, 5, \ldots)$

$1 \mapsto 0$

$2 \mapsto 1$

$3 \mapsto 2$

$\vdots$

$(0,1,2,3,4,\ldots) + (3,\pi,6,1,5,\ldots)$

$(0+3, 1+\pi, 2+6, 3+1, 4+5, \ldots)$

$(f_1+f_2)(n) = f_1(n) + f_2(n)$

$3(0,1,2,3,4,\ldots)$

$(0\cdot 3, 1\cdot 3, 2\cdot 3, 3\cdot 3, 4\cdot 3, \ldots)$

$f(x) = \sin x$ $\quad (f+g)(x) =$
$g(x) = e^x$ $\quad \sin x + e^x$

Example: The space of all differentiable functions.

$C = \{ f : \mathbb{R} \to \mathbb{R} \mid \frac{d}{dx} f \text{ exists} \}$

$(+ii), (+iii), (\cdot ii), (\cdot iii), (\cdot iv)$ follow from usual addition and scalar multiplication.

$(+i)$ $f, g \in C$ $\quad f+g \in C$? yes because the sum of two differentiable functions is differentiable.

$(+iv)$ $f(x) = 0$

$(+v)$ $f(x), \; -f(x) \in C$

$(\cdot i)$ $cf(x) \in C$? $\quad \frac{d}{dx} cf(x) = c \frac{d}{dx} f(x)$

$(\cdot v)$ $1 \cdot f(x) = f(x)$

Example: $M = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix}$

$$\left\{ c_1 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \;\middle|\; c_1, c_2 \in \mathbb{R} \right\}$$

$$a_1 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + a_2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + b_1 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + b_2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} =$$

$$(a_1+b_1)\begin{bmatrix}-1\\1\\0\end{bmatrix}+(a_2+b_2)\begin{bmatrix}-1\\0\\1\end{bmatrix}$$

Nonexample:

$$\begin{bmatrix}0\\0\end{bmatrix}\notin\left\{\begin{bmatrix}1\\0\end{bmatrix}+c\begin{bmatrix}-1\\1\end{bmatrix}\;\middle|\;c\in\mathbb{R}\right\}$$

$\mathbb{C}$

Chapter 6 Linear Transformations

Definition: A function $L: V\to W$ is linear if V and W are vector spaces and

$$L(ru+sv)=rL(u)+sL(v)$$

for all $u,v\in V$ and $r,s\in\mathbb{R}$.

6.1. Consequences of Linearity

Example 69: $L\left(\begin{bmatrix}1\\0\end{bmatrix}\right)=\begin{bmatrix}5\\3\end{bmatrix}$

$$L\left(\begin{bmatrix}5\\0\end{bmatrix}\right)=\begin{bmatrix}25\\15\end{bmatrix}\qquad L\left(5\begin{bmatrix}1\\0\end{bmatrix}\right)=5L\left(\begin{bmatrix}1\\0\end{bmatrix}\right)=5\begin{bmatrix}5\\3\end{bmatrix}$$

$$=\begin{bmatrix}25\\15\end{bmatrix}$$

$$L\left(\begin{bmatrix}\pi\\0\end{bmatrix}\right)$$

Example: $L\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}5\\3\end{bmatrix}$ and $L\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}2\\2\end{bmatrix}$

$$L\left(\begin{bmatrix}1\\1\end{bmatrix}\right) = \begin{bmatrix}7\\5\end{bmatrix} = L\left(\begin{bmatrix}1\\0\end{bmatrix} + \begin{bmatrix}0\\1\end{bmatrix}\right)$$

$$= L\left(\begin{bmatrix}1\\0\end{bmatrix}\right) + L\left(\begin{bmatrix}0\\1\end{bmatrix}\right)$$

$$= \begin{bmatrix}5\\3\end{bmatrix} + \begin{bmatrix}2\\2\end{bmatrix}$$

$$\begin{bmatrix}x\\y\end{bmatrix} = x\begin{bmatrix}1\\0\end{bmatrix} + y\begin{bmatrix}0\\1\end{bmatrix}$$

$$L\left(\begin{bmatrix}x\\y\end{bmatrix}\right) = L\left(x\begin{bmatrix}1\\0\end{bmatrix} + y\begin{bmatrix}0\\1\end{bmatrix}\right)$$

$$= xL\left(\begin{bmatrix}1\\0\end{bmatrix}\right) + yL\left(\begin{bmatrix}0\\1\end{bmatrix}\right)$$

$$= x\begin{bmatrix}5\\3\end{bmatrix} + y\begin{bmatrix}2\\2\end{bmatrix}$$

$$= \begin{bmatrix}5x+2y\\3x+2y\end{bmatrix}$$

6.3. Linear Differential Operators

$$\frac{d}{dx} : C \to \mathbb{R}^{\mathbb{R}}$$

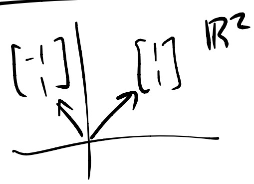
$$V = \{a_0 + a_1x + a_2x^2 \mid a_0, a_1, a_2 \in \mathbb{R}\}$$

$$\frac{d}{dx} : V \to V$$

$$\frac{d}{dx}\left(a_0 + a_1 x + a_2 x^2\right) = a_1 + 2a_2 x$$

$$\underbrace{\left(\frac{d}{dx} + 2\right)} f = f' + 2f$$

6.4 Bases



$$\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$$

We can write any vector in $\mathbb{R}^2$ as a linear combination of elements in the basis.

$$x\begin{bmatrix} 0 \\ 1 \end{bmatrix} + y\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Example:

$$L\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 4 \end{bmatrix} \qquad L\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} 6 \\ 8 \end{bmatrix}$$

$L\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$ Find a, b in terms of x, y

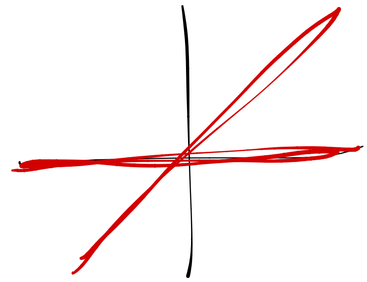
$$\begin{bmatrix} x \\ y \end{bmatrix} = a\begin{bmatrix} 1 \\ 1 \end{bmatrix} + b\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 1 & 1 & x \\ 1 & -1 & y \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & \frac{x+y}{2} \\ 0 & 1 & \frac{x-y}{2} \end{array}\right]$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{x+y}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{x-y}{2}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$L\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \frac{x+y}{2}\begin{bmatrix} 2 \\ 4 \end{bmatrix} + \frac{x-y}{2}\begin{bmatrix} 6 \\ 8 \end{bmatrix}$$

$\left\{ c \begin{bmatrix} 1 \\ 0 \end{bmatrix} \;\middle|\; c \in \mathbb{R} \right\}$



$\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$ is a basis for $\left\{ c \begin{bmatrix} 1 \\ 0 \end{bmatrix} \;\middle|\; c \in \mathbb{R} \right\}$