Homework 2 Key

Solutions by:
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1. $$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 1 & 2 & 1 & 3 \\ 1 & 1 & a^2-5 & a \end{array}\right] \underset{\sim}{\substack{R_2-R_1\to R_2 \\ R_3-R_1\to R_3}} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & a^2-4 & a-2 \end{array}\right]$$

(a) No solution

$$a^2-4=0 \quad \text{and} \quad a-2\neq 0$$
$$a^2=4 \qquad\qquad a\neq 2$$
$$a=\pm 2$$

Therefore, $\boxed{a=-2}$

(b) A unique solution

$$a^2-4\neq 0$$
$$a^2\neq 4$$
$$\boxed{a\neq\pm 2}$$

(c) Infinite solutions

$$a^2-4=0 \quad \text{and} \quad a-2=0$$
$$a^2=\pm 2 \qquad\qquad a=2$$
$$a=\pm 2$$

Therefore, $\boxed{a=2.}$

2. a. $$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 1 & 8 \\ 1 & 3 & 0 & 1 & 7 \\ 1 & 0 & 2 & 1 & 3 \end{array}\right] \underset{\sim}{\substack{R_2-R_1\to R_2 \\ R_3-R_1\to R_3}} \left[\begin{array}{cccc|c} 1 & 2 & 3 & 1 & 8 \\ 0 & 1 & -3 & 0 & -1 \\ 0 & -2 & -1 & 0 & -5 \end{array}\right] \underset{\sim}{\substack{R_1-2R_2\to R_1 \\ R_3+2R_2\to R_3}}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 9 & 1 & 10 \\ 0 & 1 & -3 & 0 & -1 \\ 0 & 0 & -7 & 0 & -7 \end{array}\right] \underset{-\frac{1}{7}R_3\to R_3}{\overset{\substack{R_1+\frac{9}{7}R_3\to R_1 \\ R_2-\frac{3}{7}R_3\to R_3}}{\sim}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 1 \end{array}\right]$$

Let $w=t$ $\quad x=1-t$

$y=2$

$z=1$

$$\boxed{\vec{x}=\begin{bmatrix} 1-t \\ 2 \\ 1 \\ t \end{bmatrix}=\begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}+t\begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix},\ t\in\mathbb{R}}$$

b. $\left[\begin{array}{ccc|c} 1 & -2 & 3 & 4 \\ 2 & -1 & -3 & 5 \\ 3 & 0 & 1 & 2 \\ 3 & -3 & 0 & 7 \end{array}\right] \begin{array}{c} R_2-2R_1\to R_2 \\ R_3-3R_3\to R_3 \\ \sim \\ R_4-3R_3\to R_4 \end{array} \left[\begin{array}{ccc|c} 1 & -2 & 3 & 4 \\ 0 & 3 & -9 & -3 \\ 0 & 6 & -8 & -10 \\ 0 & 3 & -9 & -5 \end{array}\right] \begin{array}{c} R_1+\frac{2}{3}R_2\to R_1 \\ \frac{1}{3}R_2\to R_2 \\ \sim \\ R_3-2R_2\to R_3 \\ R_4-R_2\to R_4 \end{array} \left[\begin{array}{ccc|c} 1 & 0 & -3 & 2 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 10 & -4 \\ 0 & 0 & 0 & -2 \end{array}\right]$

Last row has $0=2$, therefore $\boxed{\text{no solution.}}$

3\. a. $A\vec{u}=\vec{0}$, $A\vec{v}=\vec{0}$

$A\vec{u}+A\vec{v}=\vec{0}$

$A(\vec{u}+\vec{v})=\vec{0}$

Thus, $\vec{u}+\vec{v}$ is a solution.

b. $A\vec{u}=0$, $A\vec{v}=\vec{0}$

$A\vec{u}-A\vec{v}=\vec{0}$

$A(\vec{u}-\vec{v})=\vec{0}$

Thus, $\vec{u}-\vec{v}$ is a solution.

c. $A\vec{u}=\vec{0}$

$rA\vec{u}=\vec{0}$

$A(r\vec{u})=\vec{0}$

Thus $r\vec{u}$ is a solution.

d. From c., $r\vec{u}$ and $s\vec{v}$ are solutions.

From a., $r\vec{u}+s\vec{v}$ is a solution.

4\. $A\vec{u}=b$, $A\vec{v}=\vec{b}$

$A\vec{u}-A\vec{v}=\vec{b}-\vec{b}$

$A(\vec{u}-\vec{v})=\vec{0}$

Thus, $\vec{u}-\vec{v}$ is a solution to $A\vec{x}=\vec{b}$.

5\. a. $\|\vec{v}\|=\sqrt{(1)^2+(\sqrt{3})^2}=\sqrt{1+3}=\sqrt{4}=2$ $\qquad \theta=\cos^{-1}\left(\frac{1}{2\cdot 1}\right)$

$\|\vec{w}\|=\sqrt{(1)^2+(0)^2}=\sqrt{1+0}=\sqrt{1}=1$ $\qquad \theta=\cos^{-1}\left(\frac{1}{2}\right)$

$\vec{v}\cdot\vec{w}=(1)(1)+(\sqrt{3})(0)=1+0=1$ $\qquad \theta=\boxed{\frac{\pi}{3}}$

b. $\|\vec{v}\|=\sqrt{(2)^2+(2)^2+(-1)^2}=\sqrt{4+4+1}=\sqrt{9}=3$ $\qquad \theta=\cos^{-1}\left(\frac{0}{3\cdot 3}\right)$

$\|\vec{w}\|=\sqrt{(2)^2+(-1)^2+(2)^2}=\sqrt{4+1+4}=\sqrt{9}=3$ $\qquad \theta=\cos^{-1}(0)$

$\vec{v}\cdot\vec{w}=(2)(2)+(2)(-1)+(-1)(2)=0$ $\qquad \theta=\boxed{\frac{\pi}{2}}$

c. $\|\vec{v}\|=\sqrt{(1)^2+(\sqrt{3})^2}=\sqrt{1+3}=\sqrt{4}=2$ $\qquad \theta=\cos^{-1}\left(\frac{2}{2\cdot 2}\right)$

$\|\vec{w}\| = \sqrt{(-1)^2 + (\sqrt{3})^2} = \sqrt{1+3} = \sqrt{4} = 2$ $\qquad \theta = \cos^{-1}\left(\frac{1}{2}\right)$

$\vec{v} \cdot \vec{w} = (1)(-1) + (\sqrt{3})(\sqrt{3}) = -1 + 3 = 2$ $\qquad \theta = \boxed{\frac{\pi}{3}}$

d. $\|\vec{v}\| = \sqrt{(3)^2 + (1)^2} = \sqrt{9+1} = \sqrt{10}$ $\qquad \theta = \cos^{-1}\left(\frac{-5}{\sqrt{5}\cdot\sqrt{10}}\right)$

$\|\vec{w}\| = \sqrt{(-1)^2 + (-2)^2} = \sqrt{1+4} = \sqrt{5}$ $\qquad \theta = \cos^{-1}\left(\frac{-5}{\sqrt{50}}\right)$

$\vec{v} \cdot \vec{w} = (3)(-1) + (1)(-2) = -3 - 2 = -5$ $\qquad \theta = \cos^{-1}\left(\frac{-1}{\sqrt{2}}\right) = \boxed{\frac{3\pi}{4}}$

6. (+i) The sum of two polynomials is a polynomial.

(+ii), (+iii) follow from addition of functions.

(+iv) $h(x) = 0$ is a polynomial and $f + h = f$.

(+v) $-f$ is clearly a polynomial and $f + (-f) = 0$.

(·i) $cf(x)$ is clearly a polynomial

(·ii), (·iii), (·iv), (·v) follow from scalar multiplication of functions

7. (+i) $(x_1, x_2) + (y_1, y_2) = (x_1 + y_2, x_2 + y_1) \in \mathbb{R}^2$ ✓

$\boxed{(+ii)}$ $(x_1, x_2) + (y_1, y_2) = (x_1 + y_2, x_2 + y_1) \neq (y_1 + x_2, y_2 + x_1) = (y_1, y_2) + (x_1, x_2)$ X

$\boxed{(+iii)}$ $[(x_1, x_2) + (y_1, y_2)] + (z_1, z_2) = (x_1 + y_2, x_2 + y_1) + (z_1, z_2) = (x_1 + y_2 + z_2, x_2 + y_1 + z_1)$ X

$(x_1, x_2) + [(y_1, y_2) + (z_1, z_2)] = (x_1, x_2) + (y_1 + z_2, y_2 + z_1) = (x_1 + y_2 + z_1, x_2 + y_1 + z_2)$

(+iv) $(0,0) \in \mathbb{R}^2$, $(x_1, x_2) + (0,0) = (x_1, x_2)$ ✓

(+v) $-(x_1, x_2) = (-x_2, -x_1)$, $(x_1, x_2) + (-x_2, -x_1) = (0,0)$ ✓

(·i) $c(x_1, x_2) = (cx_1, cx_2) \in \mathbb{R}^2$ ✓

$\boxed{(\cdot ii)}$ $(c+d)(x_1, x_2) = ((c+d)x_1, (c+d)x_2) = (cx_1 + dx_1, cx_2 + dx_2)$ X

$c(x_1, x_2) + d(x_1, x_2) = (cx_1, cx_2) + (dx_1, dx_2) = (cx_1 + dx_2, cx_2 + dx_1)$

(·iii) $c[(x_1, x_2) + (y_1, y_2)] = c(x_1 + y_2, x_2 + y_1) = (cx_1 + cy_2, cx_2 + cy_1)$ ✓

$c(x_1, x_2) + c(y_1, y_2) = (cx_1, cx_2) + (cy_1, cy_2) = (cx_1 + cy_2, cx_2 + cy_1)$

(·iv) $(cd)(x_1, x_2) = (cdx_1, cdx_2) = (cdx_1, cdx_2) = c[(dx_1, dx_2)] = c[d(x_1, x_2)]$ ✓

(·v) $1(x_1, x_2) = (x_1, x_2)$ ✓

8. a. $L(cx_1+dx_2,\ cy_1+dy_2,\ cz_1+dz_2) = (cx_1+dx_2+cy_1+dy_2,\ 0,\ 2(cx_1+dx_2)+cz_1+dz_2)$

$$= (cx_1+cy_1+dx_2+dy_2,\ 0,\ 2cx_1+2dx_2+cz_1+dz_2)$$

$$= (cx_1+cy_1+dx_2+dy_2,\ 0,\ 2cx_1+cz_1+2dx_2+dz_2)$$

$$= (cx_1+cy_1,\ 0,\ 2cx_1+cz_1) + (dx_2+dy_2,\ 2dx_2+dz_2)$$

$$= c(x_1+y_1,\ 0,\ 2x_1+z_1) + d(x_2+y_2,\ 2x_2+z_2)$$

$$= cL(x_1,y_1,z_1) + dL(x_2,y_2,z_2) \checkmark$$

Thus, L is $\boxed{\text{linear.}}$

b. $L\left(c\begin{bmatrix}x_1\\y_1\end{bmatrix}+d\begin{bmatrix}x_2\\y_2\end{bmatrix}\right) = L\left(\begin{bmatrix}cx_1+dx_2\\cy_1+dy_2\end{bmatrix}\right)$

$$= \begin{bmatrix}(cx_1+dx_2)^2-(cy_1+dy_2)^2\\(cx_1+dx_2)^2+(cy_1+dy_2)^2\end{bmatrix}$$

$$= \begin{bmatrix}c^2x_1^2+2cdx_1x_2+d^2x_2^2-c^2y_1^2-2cdy_1y_2-d^2y_2^2\\c^2x_1^2+2cdx_1x_2+d^2x_2^2+c^2y_1^2+2cdy_1y_2+d^2y_2^2\end{bmatrix} \times$$

$$cL\left(\begin{bmatrix}x_1\\y_1\end{bmatrix}\right)+dL\left(\begin{bmatrix}x_2\\y_2\end{bmatrix}\right) = c\begin{bmatrix}x_1^2-y_1^2\\x_1^2+y_1^2\end{bmatrix}+d\begin{bmatrix}x_2^2-y_2^2\\x_2^2+y_2^2\end{bmatrix}$$

$$= \begin{bmatrix}cx_1^2-cy_1^2+dx_2^2-dy_2^2\\cx_1^2+cy_1^2+dx_2^2+dy_2^2\end{bmatrix} \times$$

$L\left(c\begin{bmatrix}x_1\\y_1\end{bmatrix}+d\begin{bmatrix}x_2\\y_2\end{bmatrix}\right) \neq cL\left(\begin{bmatrix}x_1\\y_1\end{bmatrix}\right)+dL\left(\begin{bmatrix}x_2\\y_2\end{bmatrix}\right)$ Thus, L is $\boxed{\text{not linear.}}$

c. $L(cx_1+dx_2,\ cy_1+dy_2) = (cx_1+dx_2-cy_1-dy_2,\ 0,\ 2(cx_1+dx_2)+3)$

$$= (cx_1-cy_1+dx_2-dy_2,\ 0,\ 2cx_1+2dx_2+3) \times$$

$cL(x_1,y_1)+dL(x_2,y_2) = c(x_1-y_1,\ 0,\ 2x_1+3)+d(x_2-y_2,\ 0,\ 2x_2+3)$

$$= (cx_1-cy_1\ 0,\ 2cx_1+3c)+(dx_2-dy_2,\ 0,\ 2dx_2+3d)$$

$$= (cx_1-cy_1+dx_2-dy_2,\ 0,\ 2cx_1+2dx_2+3c+3d) \times$$

$L(cx_1+dx_2,\ cy_1+dy_2) \neq L(cx_1+dx_2,\ cy_1+dy_2)$ Thus, L is $\boxed{\text{not linear.}}$

9. $\vec{v} = x(1,1) + y(2,0) = (x+2y, x)$

$$\begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \vec{v} \qquad \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}^{-1} \vec{v}$$

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{array}\right] \underset{\sim}{R_2 - R_1 \to R_2} \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -2 & -1 & 1 \end{array}\right] \underset{-\frac{1}{2}R_2 \to R_2}{\overset{R_1 + R_2 \to R_1}{\sim}} \left[\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 0 & 1 & \frac{1}{2} & -\frac{1}{2} \end{array}\right]$$

a. $$\begin{bmatrix} 0 & 1 \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}\begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$$(2,2) = 2(1,1) + 0(2,0)$$
$$T(2,2) = 2T(1,1) + 0T(2,0)$$
$$= 2(2,2) + 0(0,0)$$
$$= \boxed{(4,4)}$$

b. $$\begin{bmatrix} 0 & 1 \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$(3,1) = 1(1,1) + 1(2,0)$$
$$T(3,1) = T(1,1) + T(2,0)$$
$$= (2,2) + (0,0)$$
$$= \boxed{(2,2)}$$

c. $$\begin{bmatrix} 0 & 1 \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$(-1,1) = (1,1) - (2,0)$$
$$T(-1,1) = T(1,1) - T(2,0)$$
$$= (2,2) - (0,0)$$
$$= \boxed{(2,2)}$$

d. $$\begin{bmatrix} 0 & 1 \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} b \\ \frac{1}{2}a - \frac{1}{2}b \end{bmatrix}$$

$$(a,b) = b(1,1) + \left(\tfrac{1}{2}a - \tfrac{1}{2}b\right)(2,0)$$
$$T(a,b) = bT(1,1) + \left(\tfrac{1}{2}a - \tfrac{1}{2}b\right)T(2,0)$$
$$= b(2,2) + \left(\tfrac{1}{2}a - \tfrac{1}{2}b\right)(0,0)$$
$$= \boxed{(2b,2b)}$$

10. $I(r\vec{u} + s\vec{v}) = r\vec{u} + s\vec{v} = r\,I(\vec{u}) + s\,I(\vec{v})$