MAT 022A Midterm 1

August 14, 2018

Name: Midterm 1 Key

SID:

Instructions: Do not turn the page until instructed to do so. No external help or calculators allowed. Please show your work and explain your answers if necessary. Box your answer if applicable.

1. Find the inverse of $A = \begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}$.

$$\left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 2 & 7 & 0 & 1 \end{array}\right] \underset{\sim}{R_2 - 2R_1 \to R_2} \left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{array}\right]$$

$$\underset{\sim}{R_1 - 3R_2 \to R_1} \left[\begin{array}{cc|cc} 1 & 0 & 7 & -3 \\ 0 & 1 & -2 & 1 \end{array}\right]$$

$$\boxed{A^{-1} = \begin{bmatrix} 7 & -3 \\ -2 & 1 \end{bmatrix}}$$

2. The steps of the LU-factorization of a matrix, A, is given below. Find E_1, E_2, E_3, and L. Then solve the system $A\vec{x} = \vec{b}$.

$$\vec{b} = \begin{bmatrix} 7 \\ -1 \\ 5 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & -2 \\ -1 & 4 & 1 \end{bmatrix}$$
$$= E_1^{-1} \begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & -6 \\ -1 & 4 & 1 \end{bmatrix}$$
$$= E_1^{-1} E_2^{-1} \begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & -6 \\ 0 & 3 & 3 \end{bmatrix}$$
$$= \underbrace{E_1^{-1} E_2^{-1} E_3^{-1}}_{L} \begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & -6 \\ 0 & 0 & 9 \end{bmatrix}$$

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix}$$

$y_1 = 7$

$2(7) + y_2 = -1$

$y_2 = -15$

$-(7) - 15 + y_3 = 5$

$y_3 = 27$

$$\vec{y} = \begin{bmatrix} 7 \\ -15 \\ 27 \end{bmatrix}$$

$9x_3 = 27$

$x_3 = 3$

$3x_2 - 6(3) = -15$

$3x_2 - 18 = -15$

$3x_2 = 3$

$x_2 = 1$

$x_1 - (1) + 2(3) = 7$

$x_1 + 5 = 7$

$x_1 = 2$

$$\boxed{\vec{x} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}}$$

3. Find all values of a and b such that the following system has (a) one solution, (b) infinitely many solutions, and (c) no solutions.

$$\begin{aligned} x + \ \ y + \qquad\quad z &= 2 \\ 2x + 3y + \qquad 2z &= b \\ 2x + 3y + (a-2)z &= a+1 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 2 & 3 & 2 & b \\ 2 & 3 & a-2 & a+1 \end{array}\right] \underset{R_3 - 2R_1 \to R_3}{\overset{R_2 - 2R_1 \to R_2}{\sim}} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 0 & b-4 \\ 0 & 1 & a-4 & a-3 \end{array}\right]$$

$$\overset{R_3 - R_2 \to R_2}{\sim} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 0 & b-4 \\ 0 & 0 & a-4 & a-b+1 \end{array}\right]$$

(a) $a - 4 \neq 0 \Rightarrow \boxed{a \neq 4}$

(b) $a - 4 = 0 \Rightarrow a = 4$

$4 - b + 1 = 0$

$\boxed{b = 5, a = 4}$

(c) $\boxed{a = 4, b \neq 5}$

4. Let $R_\varphi = \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}$ and $\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$. Show that the angle between $R_\varphi\vec{x}$ and $\vec{x}$ is φ.

$$R_\varphi\vec{x} = \begin{bmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{bmatrix}$$

$$\|\vec{x}\| = \sqrt{x^2+y^2}$$

$$\|R_\varphi\vec{x}\| = \sqrt{(x\cos\varphi - y\sin\varphi)^2 + (x\sin\varphi + y\cos\varphi)^2}$$

$$= \sqrt{x^2\cos^2\varphi \cancel{- 2xy\cos\varphi\sin\varphi} + y^2\sin^2\varphi + x^2\sin^2\varphi + \cancel{2xy\sin\varphi\cos\varphi} + y^2\cos^2\varphi}$$

$$= \sqrt{x^2(\cos^2\varphi + \sin^2\varphi) + y^2(\cos^2\varphi + \sin^2\varphi)}$$

$$= \sqrt{x^2+y^2}$$

$$R_\varphi\vec{x}\cdot\vec{x} = x(x\cos\varphi - y\sin\varphi) + y(x\sin\varphi + y\cos\varphi)$$

$$= x^2\cos\varphi - \cancel{xy\sin\varphi} + \cancel{xy\sin\varphi} + y^2\cos\varphi$$

$$= (x^2+y^2)\cos\varphi$$

$$\|R_\varphi\vec{x}\|\,\|\vec{x}\|\cos\theta = R_\varphi\vec{x}\cdot\vec{x}$$

$$\cancel{(\sqrt{x^2+y^2})(\sqrt{x^2+y^2})}\cos\theta = \cancel{(x^2+y^2)}\cos\varphi$$

$$\cos\theta = \cos\varphi$$

$$\boxed{\theta = \varphi}$$

5. Suppose that $\vec{u}$ is orthogonal to both $\vec{v}$ and $\vec{w}$. Show that $\vec{u}$ is orthogonal to $r\vec{v} + s\vec{w}$ for any scalar $r, s \in \mathbb{R}$.

$$\begin{aligned}
\vec{u} \cdot (r\vec{v} + s\vec{w}) &= \vec{u} \cdot r\vec{v} + \vec{u} \cdot s\vec{w} \\
&= r(\vec{u} \cdot \vec{v}) + s(\vec{u} \cdot \vec{w}) \\
&= 0 + 0 \\
&= 0
\end{aligned}$$

6. Explain why each of the following are not vector spaces.

a. The set of polynomials with coefficients greater than or equal to zero.

$f(x) = x$ is in the set but not its additive inverse $g(x) = -x$.

b. The set of diverging sequences: $\left\{ f : \mathbb{N} \to \mathbb{R} \mid \lim_{n \to \infty} f(n) \text{ does not exist} \right\}$.

The zero sequence $(0, 0, 0, \ldots)$ is not in the set since it converges.

c. The set of all 2-vectors with norm less than 1: $\{\vec{x} \in \mathbb{R}^2 \mid \|\vec{x}\| < 1\}$.

Take $\vec{u} = \begin{bmatrix} 1/2 \\ 0 \end{bmatrix}$. Then $2\vec{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ which is not in the set.

7. Let $L : \mathbb{R}^2 \to \mathbb{R}^{\mathbb{R}}$ be a linear transformation such that

$$L\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right)(x) = \sin 5x \quad \text{and} \quad L\left(\begin{bmatrix} -3 \\ 4 \end{bmatrix}\right)(x) = 2e^{3x}.$$

Find $L\left(\begin{bmatrix} 7 \\ 4 \end{bmatrix}\right)(x)$.

$x - 3y = 7 \Rightarrow x = 7 + 3y$

$2x + 4y = 4$

$x = 7 + 3(-1)$

$x = 4$

$2(7 + 3y) + 4y = 4$

$14 + 6y + 4y = 4$

$10y = -10$

$y = -1$

$$L\left(\begin{bmatrix} 7 \\ 4 \end{bmatrix}\right)(x) = L\left(4\begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} -3 \\ 4 \end{bmatrix}\right)(x)$$

$$= 4L\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right)(x) - L\left(\begin{bmatrix} -3 \\ 4 \end{bmatrix}\right)(x)$$

$$\boxed{L\left(\begin{bmatrix} 7 \\ 4 \end{bmatrix}\right)(x) = 4\sin(5x) - 2e^{3x}}$$