Lecture 1

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What is Linear Algebra?

Naively, the study of vectors and ~~matrices.~~ linear transformations.

Vector

something you can add

$\vec{x} = \begin{bmatrix} 3 \\ 5 \end{bmatrix} \qquad \vec{y} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad \vec{x} + \vec{y} = \begin{bmatrix} 3 \\ 5 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$= \begin{bmatrix} 3+1 \\ 5+2 \end{bmatrix} = \begin{bmatrix} 4 \\ 7 \end{bmatrix}$

$f(x,y) = 3x + 5y \qquad f(x,y) + g(x,y) = 3x + 5y + x + 2y$

$g(x,y) = x + 2y \qquad = 4x + 7y$

$f(x) = 1 + 2x - 2x^2 + 3x^3$

$g(x) = \quad x + 3x^2 - 3x^3 + x^4$

$f(x) + g(x) = 1 + 3x + x^2 + x^4$

$\vec{f} = \begin{bmatrix} 1 \\ 2 \\ -2 \\ 3 \\ 0 \end{bmatrix} \quad \vec{g} = \begin{bmatrix} 0 \\ 1 \\ 3 \\ -3 \\ 1 \end{bmatrix}$

Linear Transformations
Maps
Functions

input $\longrightarrow$ $\boxed{\text{Function}}$ $\longrightarrow$ output

$$\begin{bmatrix}2\\3\end{bmatrix} \longrightarrow \begin{bmatrix}3&0\\0&1\end{bmatrix} \rightarrow \begin{bmatrix}6\\3\end{bmatrix}$$

$x=2$	$f=3x+0y$	$f=6$
$y=3$	$g=0x+1y$	$g=3$

$$f(x)=3x^2 \longrightarrow \frac{d}{dx}\left[3x^2\right] \rightarrow 6x$$

$$\begin{bmatrix}0\\0\\3\end{bmatrix} \rightarrow \begin{bmatrix}0&1&0\\0&0&2\\0&0&0\end{bmatrix} \rightarrow \begin{bmatrix}0\\6\\0\end{bmatrix}$$

$$f(x,y) = 3x + 5y$$

$$f(x) = \sin(x) \qquad \sin(x) + \cos(x)$$

$$g(x) = \cos(x)$$

$$\frac{d}{dx}(\sin x) = \cos x$$

$$\frac{d}{dx}(\sin x + \cos x) = \frac{d}{dx}\sin x + \frac{d}{dx}\cos x$$

$$M : \overset{\text{input}}{\mathbb{R}^n} \to \overset{\text{output}}{\mathbb{R}^m}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \underbrace{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}_{\mathbb{R}^3} = \begin{bmatrix} 1\cdot 1 + 2\cdot 1 + 3\cdot 1 \\ 4\cdot 1 + 5\cdot 1 + 6\cdot 1 \\ 7\cdot 1 + 8\cdot 1 + 9\cdot 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 6 \\ 15 \\ 24 \end{bmatrix}}_{\mathbb{R}^3}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1\cdot 2 + 2\cdot 1 + 3\cdot 2 \\ 4\cdot 2 + 5\cdot 1 + 6\cdot 2 \\ 7\cdot 2 + 8\cdot 1 + 9\cdot 2 \end{bmatrix} = \begin{bmatrix} 10 \\ 25 \\ 40 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}} = \begin{bmatrix} 1\cdot 1 + 2\cdot 0 + 3\cdot 1 \\ 4\cdot 1 + 5\cdot 0 + 6\cdot 1 \\ 7\cdot 1 + 8\cdot 0 + 9\cdot 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 10 \\ 16 \end{bmatrix}$$

$$\begin{bmatrix} 4 \\ 10 \\ 16 \end{bmatrix} + \begin{bmatrix} 6 \\ 15 \\ 24 \end{bmatrix} = \begin{bmatrix} 10 \\ 25 \\ 40 \end{bmatrix}$$

$$M : \mathbb{R}^{3\times 3} \to \mathbb{R}^{3\times 3}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{I} = \begin{bmatrix} 1+0+0 & 0+2+0 & 0+0+3 \\ 4+0+0 & 0+5+0 & 0+0+6 \\ 7+0+0 & 0+8+0 & 0+0+9 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

I - identity matrix

For any square matrix M, $MI = M = IM$.

↑ same number of rows and columns

$$3 \times 2 \cdot 2 \times 1 \qquad\qquad 3 \times 1$$

$$\begin{array}{c c} \text{3 row} & \left\{ \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \right. \\ & \text{2 columns} \end{array} \begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 1\cdot 3 + 5 \cdot 2 \\ 3 \cdot 3 + 4 \cdot 5 \\ 5 \cdot 3 + 6 \cdot 5 \end{bmatrix} = \begin{bmatrix} 13 \\ 29 \\ 45 \end{bmatrix}$$

$$M : \mathbb{R}^2 \to \mathbb{R}^3 \qquad f(g(x))$$

$$\underset{A}{\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}} \underset{B}{\begin{bmatrix} 2 & 6 \\ 4 & 8 \end{bmatrix}} \underset{\vec{x}}{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}$$

$$\underset{B}{\begin{bmatrix} 2 & 6 \\ 4 & 8 \end{bmatrix}} \underset{\vec{x}}{\begin{bmatrix} 1 \\ 2 \end{bmatrix}} = \begin{bmatrix} 2 \cdot 1 + 6 \cdot 2 \\ 4 \cdot 1 + 8 \cdot 2 \end{bmatrix} = \begin{bmatrix} 14 \\ 20 \end{bmatrix}$$

$$\underset{A}{\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}} \underset{\vec{y}}{\begin{bmatrix} 14 \\ 20 \end{bmatrix}} = \begin{bmatrix} 1 \cdot 14 + 2 \cdot 20 \\ 0 \cdot 14 + 1 \cdot 20 \end{bmatrix} = \underset{\vec{z}}{\begin{bmatrix} 54 \\ 20 \end{bmatrix}}$$

$$B(x,y) = \begin{cases} 2x + 6y \\ 4x + 8y \end{cases}$$

$$A(x,y) = \begin{cases} x + 2y \\ y \end{cases}$$

$$A(\underline{2x+6y}, 4x+8y) = \begin{cases} (2x+6y) + 2(4x+8y) \\ 4x+8y \end{cases}$$

$$= \begin{cases} 10x + 22y \\ 4x + 8y \end{cases}$$

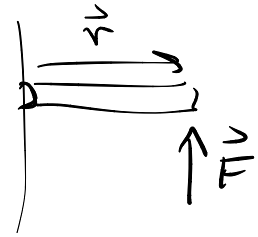
$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 6 \\ 4 & 8 \end{bmatrix} = \begin{bmatrix} 1\cdot 2 + 2\cdot 4 & 1\cdot 6 + 2\cdot 8 \\ 0\cdot 2 + 1\cdot 4 & 0\cdot 6 + 1\cdot 8 \end{bmatrix}$$

$$= \begin{bmatrix} 10 & 22 \\ 4 & 8 \end{bmatrix}$$

Chapter 1.

2. cross product

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \times \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} yz' - zy' \\ zx' - xz' \\ xy' - yx' \end{bmatrix}$$



torque, $\tau = \vec{r} \times \vec{F}$

$$\vec{r} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad \tau = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \vec{F} = ? = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \times \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} c - 0 \\ 0 - c \\ b - a \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\left.\begin{aligned} c &= 0 \\ -c &= 0 \\ b - a &= 1 \end{aligned}\right\} \Rightarrow \boxed{\begin{aligned} c &= 0 \\ b &= 1 + a \end{aligned}}$$

Let $a = t$

$$\begin{bmatrix} t \\ 1+t \\ 0 \end{bmatrix}$$

for any t

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 \cdot a + 0 \cdot b + c \\ 0 \cdot a + 0 \cdot b - c \\ -a + b + 0 \cdot c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Chapter 2

Systems of Linear Equations

$x + y = 27$

$2x - y = 0 \Rightarrow y = 2x$

$x + 2x = 27 \Rightarrow 3x = 27$

$x = 9$

$y = 18$

$$\begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 27 \\ 0 \end{bmatrix}$$

solution: $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 9 \\ 18 \end{bmatrix}$

Augmented Matrix

$$\left[\begin{array}{cc|c} 1 & 1 & 27 \\ 2 & -1 & 0 \end{array}\right]$$

$-2(x + y = 27)$

$2x - y = 0$

$-\cancel{2x} - 2y = -54$

$\cancel{2x} - y = 0$

$0 - 3y = -54$

$y = 18$

$x = 9$

$$\left[\begin{array}{cc|c} 1 & 1 & 27 \\ 2 & -1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 0 & ? \\ 0 & 1 & ? \end{array}\right]$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ? \\ ? \end{bmatrix}$$

$-2R_1 + R_2 \rightarrow R_2$

$$\sim \left[\begin{array}{cc|c} 1 & 1 & 27 \\ 0 & -3 & -54 \end{array}\right]$$

$-\frac{1}{3}R_2 \rightarrow R_2$

$$\sim \left[\begin{array}{cc|c} 1 & 1 & 27 \\ 0 & 1 & 18 \end{array}\right]$$

$-R_2 + R_1 \rightarrow R_1$

$$\sim \left[\begin{array}{cc|c} 1 & 0 & 9 \\ 0 & 1 & 18 \end{array}\right] \Rightarrow \begin{array}{l} x = 9 \\ y = 18 \end{array}$$