Lecture 2

07 Aug 2019

Chapter 1 Review:

Linear algebra is the study of vectors and linear functions.

Vectors are things you can add and scalar multiply.

Example: $\vec{x} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, $\vec{y} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $c = 2$, $d = 3$

$$c\vec{x} + d\vec{y} = 2\begin{bmatrix} 3 \\ 1 \end{bmatrix} + 3\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \end{bmatrix} + \begin{bmatrix} 3 \\ 6 \end{bmatrix} = \begin{bmatrix} 9 \\ 8 \end{bmatrix}$$

Example: $f(x) = e^x$, $g(x) = \sin x$, $c = 3$, $d = 5$

$$cf(x) + dg(x) = 3e^x + 5\sin x$$

A function is linear if it satisfies the following

a. (Additivity) $f(\vec{a}+\vec{b}) = f(\vec{a}) + f(\vec{b})$

b. (Homogeneity) $f(c\vec{a}) = cf(\vec{a})$

Nonexamples: $f(x) = \sin x$, $g(x) = \sqrt{x}$

$f(x+y) \neq \sin(x) + \sin(y)$, $g(x+y) = \sqrt{x+y} \neq \sqrt{x} + \sqrt{y}$

Chapter 2

Systems of Linear Equations

A collection of linear equations

Example:
$$x + y = 27$$
$$2x - y = 0$$

rewrite the linear system in matrix form

$$\begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 27 \\ 0 \end{bmatrix}$$

or as an augmented matrix

$$\left[\begin{array}{cc|c} 1 & 1 & 27 \\ 2 & -1 & 0 \end{array}\right]$$

or as a linear combination of the variables:

$$x\begin{bmatrix} 1 \\ 2 \end{bmatrix} + y\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 27 \\ 0 \end{bmatrix}$$

$x = 9$
$y = 18$

$$9\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 18\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 9 \\ 18 \end{bmatrix} + \begin{bmatrix} 18 \\ -18 \end{bmatrix} = \begin{bmatrix} 27 \\ 0 \end{bmatrix}$$

$$\begin{aligned} x+y &= 27 \\ 2x-y &= 0 \end{aligned} \qquad \left[\begin{array}{cc|c} 1 & 1 & 27 \\ 2 & -1 & 0 \end{array}\right]$$

$$\begin{aligned} -2x-2y &= -54 \\ 2x-y &= 0 \end{aligned} \qquad \left[\begin{array}{cc|c} -2 & -2 & -54 \\ 2 & -1 & 0 \end{array}\right]$$

$$\begin{aligned} -2x-2y &= -54 \\ -3y &= -54 \end{aligned} \qquad \left[\begin{array}{cc|c} -2 & -2 & -54 \\ 0 & -3 & -54 \end{array}\right]$$

$$\begin{aligned} -2x-2y &= -54 \\ y &= 18 \end{aligned} \qquad \left[\begin{array}{cc|c} -2 & -2 & -54 \\ 0 & 1 & 18 \end{array}\right]$$

$$\begin{aligned} -2x-2y &= -54 \\ 2y &= 36 \end{aligned} \qquad \left[\begin{array}{cc|c} -2 & -2 & -54 \\ 0 & 2 & 36 \end{array}\right]$$

$$\begin{aligned} -2x \quad &= -18 \\ 2y &= 36 \end{aligned} \qquad \left[\begin{array}{cc|c} -2 & 0 & -18 \\ 0 & 2 & 36 \end{array}\right]$$

$$\begin{aligned} x \quad &= 9 \\ y &= 18 \end{aligned} \qquad \left[\begin{array}{cc|c} 1 & 0 & 9 \\ 0 & 1 & 18 \end{array}\right]$$

Gaussian Elimination

$$\left[\begin{array}{cc|c} 1 & 1 & 27 \\ 2 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 9 \\ 0 & 1 & 18 \end{array}\right]$$

Elementary Row Operations (ERO):

- (Row Swap) Exchange any two rows
- (Scalar Multiplication) Multiply any row by a non-zero constant.
- (Row Addition) Add one row to another row.

ERO's preserve the solution.

Goal of Gaussian Elimination is to get augmented matrix into <u>Reduced Row Echelon Form</u> (RREF).

Ideally, we want an <u>identity matrix</u> on the left side.

$$I = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

<u>Example:</u>

$$x + y = 2$$
$$2x + 2y = 4$$

$$\left[\begin{array}{cc|c} 1 & 1 & 2 \\ 2 & 2 & 4 \end{array}\right] \sim \underbrace{\left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right]}_{\text{RREF}}$$

Example:

$$\begin{aligned} x+y&=2 \\ 2x+2y&=5 \end{aligned} \quad \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 2 & 2 & 5 \end{array}\right] \sim \underbrace{\left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 0 & 1 \end{array}\right]}_{\text{RREF}} \Rightarrow 0 \neq 1$$

$\Downarrow$

no solution

Algorithm for Obtaining RREF

- Make the leftmost entry in the toprow 1 by multiplication.*
- Then use that 1 as a <u>pivot</u> to eliminate everything below it.
- Then go to the next row and make the leftmost nonzero entry 1.
- Use that 1 as a <u>pivot</u> to eliminate everything below and above it!
- Go to the next row and make the leftmost nonzero entry 1, etc.

* If first element is zero, row swap to make not zero.

<u>Definition</u>: A <u>pivot</u> is the first nonzero element of a row.

RREF

1. In every row the leftmost non-zero entry (pivot) is 1.
2. The pivot of any given row is always to the right of the pivot of the row above it.
3. The pivot is the only non-zero entry in its column.

Example:

$$\left[\begin{array}{ccc|c} 1 & 0 & 7 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

O – pivot

Nonexample:

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

Solution Sets and RREF

The Standard Approach To Solution Sets

1. Write the augmented matrix.
2. Perform EROs to reach RREF
3. Express the pivot variables in terms of non pivot variables.

Example:

2 equations, 4 unknowns

$$\begin{bmatrix} 1 & 0 & 7 & 0 & | & 4 \\ 0 & 1 & 3 & 4 & | & 1 \\ 0 & 0 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & 0 & | & 0 \end{bmatrix} \Rightarrow \begin{aligned} x + 7z &= 4 \quad (1) \\ y + 3z + 4w &= 1 \quad (2) \end{aligned}$$

$$\underbrace{x \quad y}_{\text{pivot variables}} \quad \underbrace{z \quad w}_{\text{nonpivot variables}}$$

Solve for x, y

$$(1) \quad x = 4 - 7z$$
$$y = 1 - 3z - 4w$$

$$\begin{aligned} x &= 4 - 7t \\ y &= 1 - 3t - 4s \\ z &= t \\ w &= s \end{aligned} \qquad \begin{aligned} z &= t \\ w &= s \end{aligned}$$

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -7 \\ -3 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 \\ -4 \\ 0 \\ 1 \end{bmatrix}, \quad s, t \in \mathbb{R}$$

$$\vec{x} = \vec{x}^p + \vec{x}^h$$

Definition: A homogeneous solution to a linear equation $Lx = v$, with L and v known is a vector $\vec{x}^h$ such that $L\vec{x}^h = \vec{0}$ where $\vec{0}$ is the zero vector.

Let $t, s = 1$

$$\begin{bmatrix} 1 & 0 & 7 & 0 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -7 \\ -7 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1\cdot(-7) + 0 + 7\cdot 1 + 0 \\ 0 + (-7) + 3 + 4 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Elementary Row Operations

$$\left[\begin{array}{ccc|c} \multicolumn{3}{c|}{\overbrace{\hphantom{0 \quad 1 \quad 1}}^{M}} & \\ 0 & 1 & 1 & 7 \\ 2 & 0 & 0 & 4 \\ 0 & 0 & 1 & 4 \end{array}\right] \underset{\sim}{R_2 \leftrightarrow R_1} \left[\begin{array}{ccc|c} 2 & 0 & 0 & 4 \\ 0 & 1 & 1 & 7 \\ 0 & 0 & 1 & 4 \end{array}\right]$$

$$\begin{array}{c} \frac{1}{2}R_1 \to R_1 \\ \sim \\ R_2 - R_3 \to R_2 \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 4 \end{array}\right]$$

$R_2 \leftrightarrow R_1$

$$R_1' = 0R_1 + R_2 + 0R_3$$
$$R_2' = R_1 + 0R_2 + 0R_3$$
$$R_3' = 0R_1 + 0R_2 + R_3$$

$$E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{E_1} \underbrace{\begin{bmatrix} 0 & 1 & 1 & 7 \\ 2 & 0 & 0 & 4 \\ 0 & 0 & 1 & 4 \end{bmatrix}}_{M} = \left[\begin{array}{ccc|c} 2 & 0 & 0 & 4 \\ 0 & 1 & 1 & 7 \\ 0 & 0 & 1 & 4 \end{array}\right]$$

$\frac{1}{2}R_1 \to R_1$

E_2

$R_1' = \frac{1}{2}R_1 + 0 + 0$

$R_2' = 0 \quad R_2 \quad 0$

$R_3' = 0 \quad 0 \quad R_3$

$$\begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\overset{E_2}{\begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \overset{E_1 M}{\begin{bmatrix} 2 & 0 & 0 & 4 \\ 0 & 1 & 1 & 7 \\ 0 & 0 & 1 & 4 \end{bmatrix}}$$

$R_2 - R_3 \to R_2$

E_3

$R_1' = R_1 + 0 + 0$

$R_2' = 0 + R_2 - R_3$

$R_3' = 0 \quad 0 \quad 1$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$E_3 E_2 E_1 M$

Matrix Inverses

$$M^{-1} \qquad [M \mid I] \sim [I \mid M^{-1}]$$

$$\begin{bmatrix} 0 & 1 & 1 \\ 2 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \underbrace{\begin{bmatrix} 7 \\ 4 \\ 4 \end{bmatrix}}_{\vec{b}}$$

$$E_1 M \vec{x} = E_1 \vec{b}$$

$$E_3 E_2 E_1 M \vec{x} = E_3 E_2 E_1 \vec{b}$$

$$\vec{x} = \underbrace{E_3 E_2 E_1}_{M^{-1}} \vec{b}$$

$$\left[\begin{array}{ccc|ccc} 0 & 1 & 1 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \qquad \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \qquad = \begin{bmatrix} 0 & \frac{1}{2} & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & \frac{1}{2} & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \qquad = \begin{bmatrix} 0 & \frac{1}{2} & 0 \\ 1 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

Inverse Matrix $MM^{-1} = M^{-1}M = I$

$$\begin{bmatrix} 0 & 1 & 1 \\ 2 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{2} & 0 \\ 1 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & \frac{1}{2} & 0 \\ 1 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 2 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Definition: A matrix is <u>invertible</u> if its RREF is an identity matrix.

LU, LDU, PLDU Factorization

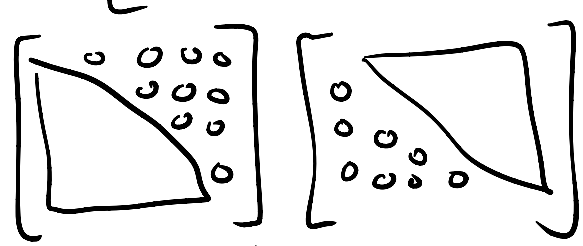
L - lower triangular

U - upper triangular

D - diagonal

P - permutation

L U



above diagonal is 0 below diagonal is 0

$M = LU$

Example:

$$\overset{E_1}{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}} \begin{bmatrix} 2 & 0 & -3 & 1 \\ 0 & 1 & 2 & 2 \\ -4 & 0 & 9 & 2 \\ 0 & -1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & -3 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 3 & 4 \\ 0 & -1 & 1 & -1 \end{bmatrix}$$

$$\overset{E_2}{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}} \begin{bmatrix} \quad & \quad & \quad & \quad \\ & & & \\ & & & \\ & & & \end{bmatrix} = \begin{bmatrix} 2 & 0 & -3 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 3 & 1 \end{bmatrix}$$

$$\overset{E_3}{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}} \begin{bmatrix} \quad & \quad & \quad & \quad \\ & & & \\ & & & \\ & & & \end{bmatrix} = \begin{bmatrix} 2 & 0 & -3 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & -3 \end{bmatrix}$$

$$E_3 E_2 E_1 M = \qquad U$$

$$M = E_1^{-1} E_2^{-1} E_3^{-1} U$$

$$E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

$$E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$E_1^{-1} E_2^{-1} E_3^{-1}$$

$$\underset{L}{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ 0 & -1 & 1 & 1 \end{bmatrix}} \underset{U}{\begin{bmatrix} 2 & 0 & -3 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & -3 \end{bmatrix}}$$

$$E_4 = \begin{bmatrix} 2 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

$$E_4^{-1} \begin{bmatrix} \frac{1}{2} & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

To solve $A\vec{x} = \vec{b}$ with LU factorization.

Given $A = LU$, then $LU\vec{x} = \vec{b}$.

If we set $U\vec{x} = \vec{y}$, then $L\vec{y} = \vec{b}$.

First solve $L\vec{y} = \vec{b}$ for $\vec{y}$ using forward substitution.

Then solve $U\vec{x} = \vec{y}$ for $\vec{x}$ using back substitution.