Lecture 3

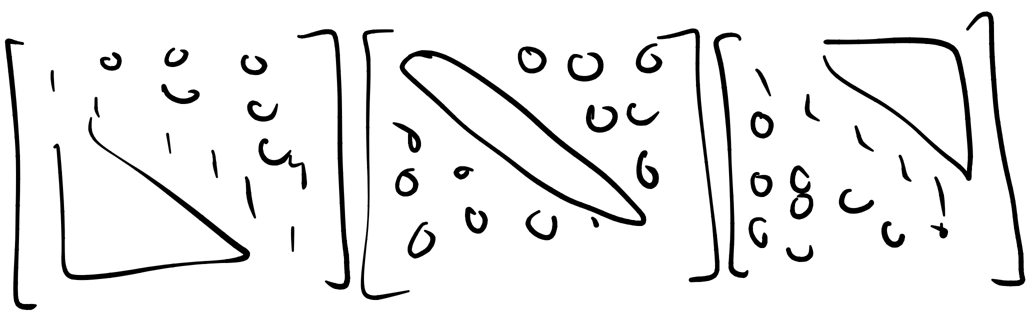
09 Aug 2019

Homework 1 due yesterday. If you had any problems submitting homework, see me after class.

Ungraded survey available in Quizzes section on Canvas for midterm logistics.

LU, LDU, LDPU

LU - lower and upper triangular matrix
By convention, L has 1 along diagonal, but not necessarily U.
If we want U to have ones along diagonal then store a D matrix.



L D U

$$\begin{bmatrix} 2 & 0 & -3 & 1 \\ 0 & 1 & 2 & 2 \\ -4 & 0 & 9 & 2 \\ 0 & -1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ 0 & -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & -3 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ 0 & -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} 1 & 0 & -\frac{3}{2} & \frac{1}{2} \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & \frac{4}{3} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

LDPU

$$\begin{bmatrix} 0 & 1 & 2 & 2 \\ 2 & 0 & -3 & 1 \\ -4 & 0 & 9 & 2 \\ 0 & -1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & -3 & 1 \\ 0 & 1 & 2 & 2 \\ -4 & 0 & 9 & 2 \\ 0 & -1 & 1 & -1 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} \end{bmatrix}}_{P} \begin{bmatrix} \end{bmatrix}$$

$A = LU$ $\qquad A\vec{x} = \vec{b}$

$$\underbrace{LU\vec{x}}_{\vec{y}} = b$$

Now we have

$U\vec{x} = \vec{y}$ to solve for $\vec{x}$ we need $\vec{y}$

Solve $L\vec{y} = \vec{b}$ to get $\vec{y}$

Solution Sets for Systems of Linear Equations

If A is a linear operator and $\vec{b}$ is known, then $A\vec{x}=\vec{b}$ has either:

1. One solution
2. No solutions
3. Infinitely many solutions.

Suppose $\vec{x}_1$ and $\vec{x}_2$ are solutions

$A\vec{x}_1=\vec{b}$, $A\vec{x}_2=\vec{b}$ Add equations

$$A\vec{x}_1+A\vec{x}_2=2\vec{b}$$
$$A(\vec{x}_1+\vec{x}_2)=2\vec{b}$$
$$\tfrac{1}{2}A(\vec{x}_1+\vec{x}_2)=\vec{b}$$

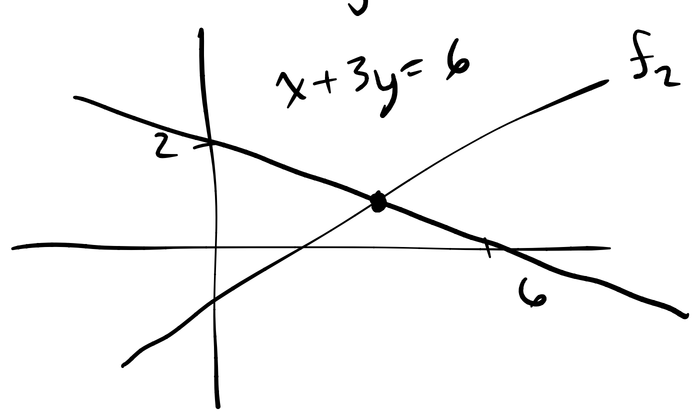
$A\left[\tfrac{1}{2}(\vec{x}_1+\vec{x}_2)\right]=\vec{b}$ so $\tfrac{1}{2}(\vec{x}_1+\vec{x}_2)$ is a new solution.

Geometry of Solutions

$y=2-\frac{1}{3}x$

$f_1 := x+3y=6$

one solution
two lines
intersecting

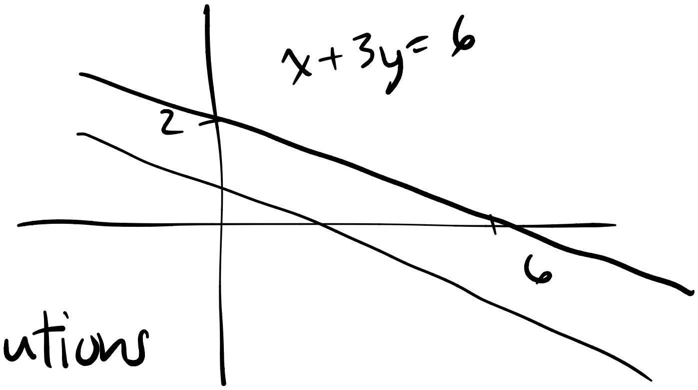


$f_2 := x+3y=4$

parallel

lines will never

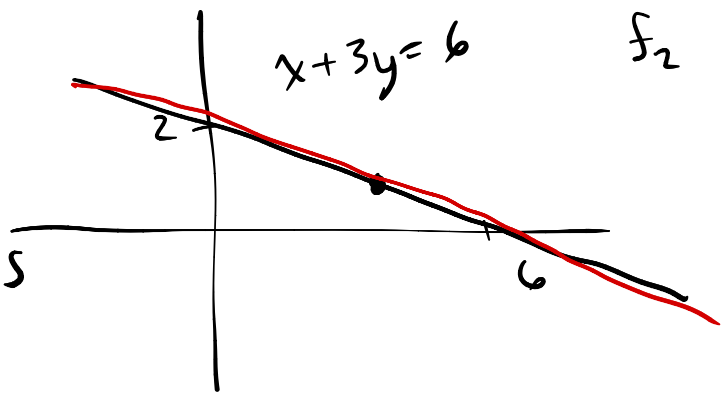
intersect ⇒ no solutions



$f_2 := 2x+6y=12$

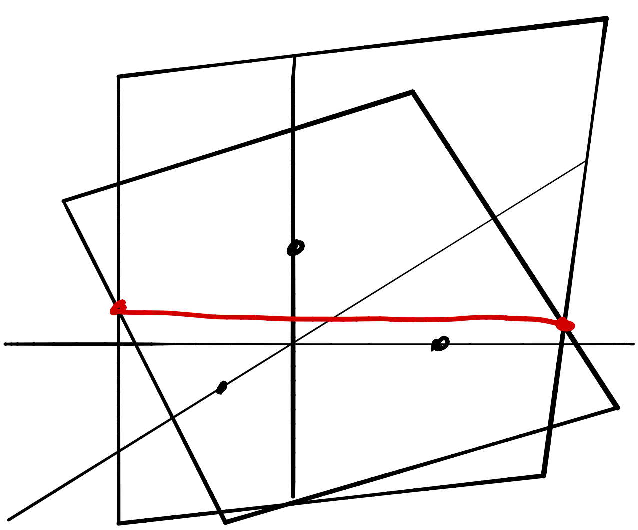
they overlap

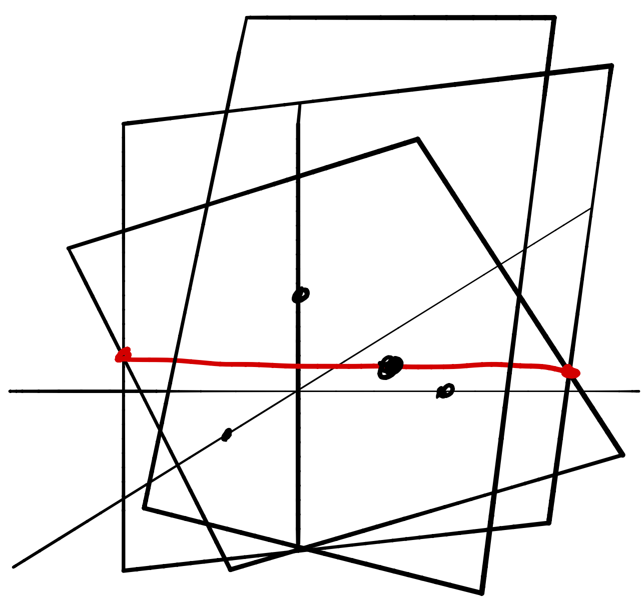
⇒ infinite solutions



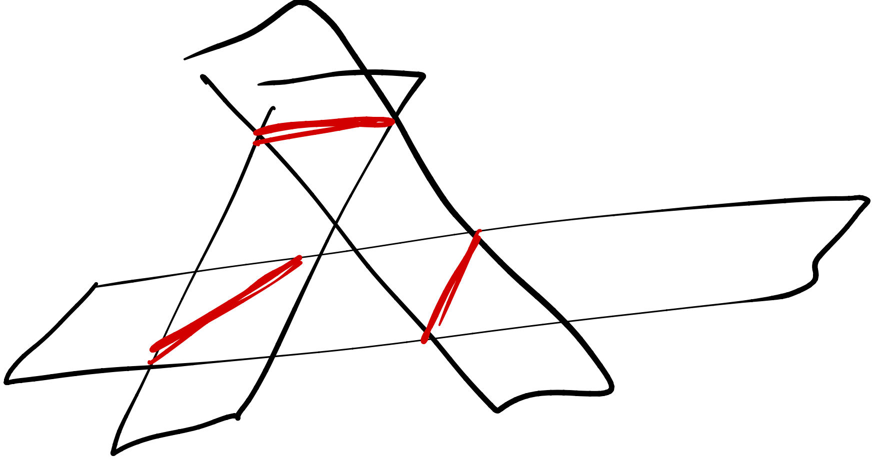
3D-dimension

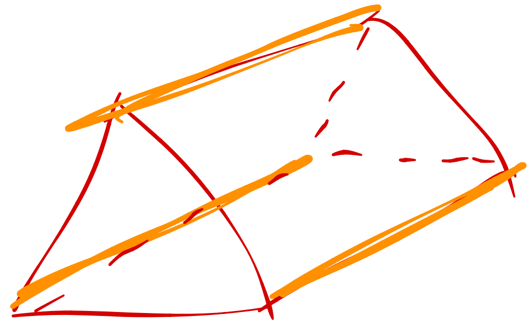
$x+y+z=1$





Two planes can intersect with a line.





parallel:

$$x + 3y + 2z = 4$$
$$2x + 6y + 4z = 9$$

not parallel:

$$x + 3y + 2z = 4$$
$$2x + 3y + z = 9$$

Hyperplanes are planes in higher than 3 dimensions.

3-hyperplane in 4D space.

$$\left.\begin{aligned} x + y + z + w &= 1 \\ y + z + w &= 3 \end{aligned}\right\}$$

Particular and Homogeneous Solutions

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & -1 & 1 \\ 0 & 1 & -1 & 1 & 1 \\ \underbrace{0 \quad 0}_{\text{2 pivot variables}} & & \underbrace{0 \quad 0}_{\text{2 free variables}} & & 0 \end{array}\right]$$

$$\begin{aligned} x_1 \qquad + x_3 - x_4 &= 1 \\ x_2 - x_3 + x_4 &= 1 \end{aligned}$$

$$y = mx + b$$

$\Downarrow$

2-D plane is the shape of our solution

$x_1 = 1 - x_3 + x_4 \qquad x_3 = t$

$x_2 = 1 + x_3 - x_4 \qquad x_4 = s$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1-t+s \\ 1+t-s \\ t \\ s \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{x^p} + \underbrace{t\begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}}_{x^h}, s,t \in \mathbb{R}$$

$M\vec{x} = \vec{b}$

$\vec{x} = \vec{x}^p + \vec{x}^h \qquad M\vec{x}^h = \vec{0}$

$M(\vec{x}^p + \vec{x}^h) = \vec{b}$

$M\vec{x}^p + \underbrace{M\vec{x}^h}_{\vec{0}} = \vec{b}$

$M\vec{x}^p + \vec{0} = \vec{b}$

$M\vec{x}^p = \vec{b}$

Chapter 4

Vectors in Space, n-Vectors

The set of all n-vectors is denoted $\mathbb{R}^n$.

$$\mathbb{R}^n := \left\{ \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \;\middle|\; a_1, \dots, a_n \in \mathbb{R} \right\}$$

$\mathbb{R}$ - line

$\mathbb{R}^2$ - xy-plane

$\mathbb{R}^3$ - real space

<u>Definition</u> Given two n-vectors $\vec{a}$ and $\vec{b}$ whose components are given by

$$\vec{a} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \qquad \vec{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

their <u>sum</u> is

$$\vec{a} + \vec{b} := \begin{bmatrix} a_1 + b_1 \\ \vdots \\ a_n + b_n \end{bmatrix}$$

Given a scalar, c, the <u>scalar multiple</u>

$$c\vec{a} = \begin{bmatrix} ca_1 \\ \vdots \\ ca_n \end{bmatrix}$$

Remember that n-vectors are <u>ordered lists</u>.

$$\begin{bmatrix} 7 \\ 4 \\ 2 \\ 5 \end{bmatrix} \neq \begin{bmatrix} 7 \\ 2 \\ 4 \\ 5 \end{bmatrix}$$

The <u>zero vector</u> is

$$\vec{0} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

may be written as just 0 or $\mathbf{0}$.

$$A\vec{x} = 0$$

understood from context without formatting

<u>Definition:</u> A set of $k+1$ vectors $\vec{p}, \vec{v}_1, \ldots, \vec{v}_k$, in $\mathbb{R}^n$ with $k \leq n$ determines a k-dimensional <u>hyperplane</u>

$$\left\{ \vec{p} + \sum_{i=1}^{k} \lambda_i \vec{v}_i \;\middle|\; \lambda_i \in \mathbb{R} \right\}$$

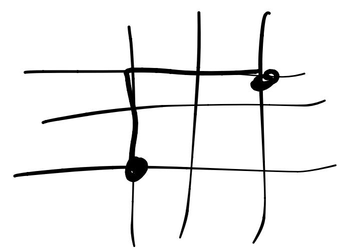
$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \lambda_1 \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + \lambda_2 \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix} \;\middle|\; \lambda_1, \lambda_2 \in \mathbb{R} \right\}$$

$$\vec{p} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad \vec{v}_1 = \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \quad \vec{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

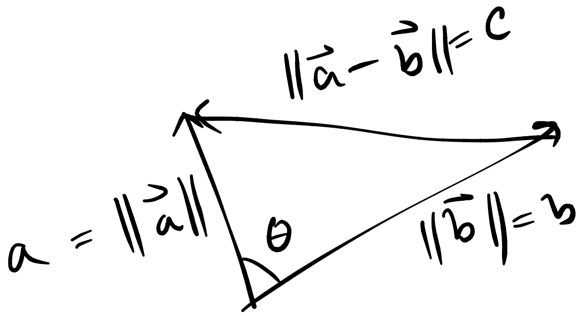
$2+1=3$ vectors which describe 2-dimensional hyperplane $\quad k=2$

4.3 Directions and Magnitudes

Euclidean length



$$\|\vec{v}\| = \sqrt{v_1^2 + \cdots + v_n^2} = \sqrt{\sum_{i=1}^{n} v_i^2}$$



$$(a_1 - b_1)^2 = a_1^2 - 2a_1b_1 + b_1^2$$

Law of Cosines

$$c^2 = a^2 + b^2 - 2ab\cos\theta$$

$$\|\vec{a}-\vec{b}\|^2 = \|\vec{a}\|^2 + \|\vec{b}\|^2 - 2\|\vec{a}\|\|\vec{b}\|\cos\theta$$

$$\begin{aligned}\|\vec{a}-\vec{b}\|^2 - \|\vec{a}\|^2 - \|\vec{b}\|^2 &= (a_1-b_1)^2 + \cdots (a_n-b_n)^2 - \\ &\quad -\left[a_1^2 + \cdots + a_n^2\right] - \\ &\quad -\left[b_1^2 + \cdots + b_n^2\right] \\ &= -2a_1b_1 - 2a_2b_2 - \cdots - 2a_nb_n\end{aligned}$$

$$-2\|\vec{a}\|\|\vec{b}\|\cos\theta = -2a_1b_1 - 2a_2b_2 - \cdots - 2a_nb_n$$

$$\|\vec{a}\|\|\vec{b}\|\cos\theta = a_1b_1 + \cdots + a_nb_n$$

Definition The dot (inner) product of $\vec{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$ is

$$\left.\begin{matrix} \vec{u}\cdot\vec{v} \\ \langle \vec{u}, \vec{v} \rangle \end{matrix}\right\} = u_1v_1 + u_2v_2 + \cdots + u_nv_n$$

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 4 \end{bmatrix} = 1\cdot 3 + 2\cdot 4 = 3 + 8 = 11$$

Remember dot product is a scalar.

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 4 \end{bmatrix} \neq \begin{bmatrix} 1\cdot 3 \\ 2\cdot 4 \end{bmatrix} = \begin{bmatrix} 3 \\ 8 \end{bmatrix}$$

Definition: The length (or norm or magnitude) of an n-vector is

$$\|\vec{v}\| = \sqrt{\vec{v}\cdot\vec{v}}$$

Definition: The angle θ between two vectors is

$$\vec{u}\cdot\vec{v} = \|\vec{u}\|\|\vec{v}\|\cos\theta$$

$$\underbrace{\begin{bmatrix}1\\2\\3\\\vdots\\100\end{bmatrix}}_{\vec{u}} \cdot \underbrace{\begin{bmatrix}1\\1\\1\\\vdots\\1\end{bmatrix}}_{\vec{v}} = 1+2+3+\cdots+100 = \frac{1}{2}(100)(100+1) = 5050$$

$$\|\vec{u}\| = \sqrt{\sum_{i=1}^{100} i^2} = \sqrt{338350}$$

$$\|\vec{v}\| = \sqrt{\sum_{i=1}^{100} 1^2} = \sqrt{100} = 10$$

$$\frac{5050}{10\cdot\sqrt{338350}} = \cos\theta$$

$$\theta = \cos^{-1}\left(\frac{5050}{10\cdot\sqrt{338350}}\right)$$

The dot product has some important properties:

1. symmetric

$$\vec{u}\cdot\vec{v} = \vec{v}\cdot\vec{u}$$

2. distributive

$$\vec{u}\cdot(\vec{v}+\vec{w}) = \vec{u}\cdot\vec{v} + \vec{u}\cdot\vec{w}$$

3. bilinear

$$\vec{u}\cdot(c\vec{v}+d\vec{w}) = c\vec{u}\cdot\vec{v} + d\vec{u}\cdot\vec{w}$$

4. positive definite

$$\vec{u}\cdot\vec{u} \geq 0$$

and $\vec{u}\cdot\vec{u} = 0$ only if $\vec{u} = \vec{0}$.

Theorem 4.3.1 (Cauchy-Schwarz Inequality)

$$\frac{|\langle \vec{u}, \vec{v} \rangle|}{\|\vec{u}\| \, \|\vec{v}\|} \leq 1 \iff |\langle u, v \rangle| \leq \|\vec{u}\| \|\vec{v}\|$$

Theorem 4.3.2 (Triangle Inequality)

$$\|\vec{u} + \vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|$$