Midterm 1 Review

A **vector space** $(V, +, \cdot, \mathbb{R})$ is a set V with two operations $+$ and $\cdot$ satisfying the following properties for all $u,v,w \in V$ and $c,d \in \mathbb{R}$.

(+i) (Additive Closure) $u+v \in V$

(+ii) (Additive Commutativity) $u+v = v+u$

(+iii) (Additive Associativity) $(u+v)+w = u+(v+w)$

(+iv) (Zero) $\exists\, 0_V \in V$ such that $u + 0_V = u$

(+v) (Additive Inverse) $\forall v \in V$, $\exists w \in V$ such that $v+w = 0_V$

(·i) (Multiplicative Closure) $c \cdot v \in V$

(·ii) (Distributivity) $(c+d)\cdot v = c\cdot v + d \cdot v$

(·iii) (Distributivity) $c\cdot(u+v) = c\cdot u + c\cdot v$

(·iv) (Associativity) $(cd)\cdot v = c\cdot(d\cdot v)$

(·v) (Unity) $1 \cdot v = v$

A function $L: V \to W$ is **linear** if V and W are vector spaces and

$$L(ru+sv) = rL(u) + sL(v)$$

For all $u,v \in V$ and $r,s \in \mathbb{R}$.

A **pivot** is the first nonzero element of a row.

A matrix is in **Row Reduced Echelon Form (RREF)**:

1. In every row the pivot is one.
2. The pivot of any row is always to the right of the pivot of the row above it.
3. The pivot is the only nonzero entry in its column.

Elementary Row Operations (ERO) preserve the solution and are

1. (Row swap) Exchange any two rows.
2. (Scalar Multiplication) Multiply any row by a nonzero constant.
3. (Row Addition) Add one row to another.

Elementary Row Matrices are matrices that perform EROs when multiplying.

A **homogeneous solution** to a linear equation $Lx = v$ with L and v known is a vector x^H such that $Lx^H = 0$ where 0 is the zero vector.

A matrix is **invertible** if its RREF is the identity matrix.

LU-Factorization factors a matrix, A, into a lower triangular matrix, L, and upper triangular matrix, U, such that $A = LU$.

A set of $k+1$ vectors $P, v_1, \dots, v_k$ in $\mathbb{R}^n$ with $k \leq n$ determines a k-dimensional **hyperplane** $\{P + \sum_{i=1}^{k} \lambda_i v_i \mid \lambda_i \in \mathbb{R}\}$.

The **Euclidean length** of an n-vector is $\|v\| := \sqrt{\sum_{i=1}^{n} v_i^2}$.

The **dot** (or **inner**) **product** of $u, v \in \mathbb{R}^n$ is $u \cdot v = \langle u, v \rangle = \sum_{i=1}^{n} u_i v_i$ and satisfies the following properties:

1. (Symmetric) $\langle u, v \rangle = \langle v, u \rangle$
2. (Distributive) $\langle u, v + w \rangle = \langle u, v \rangle + \langle u, w \rangle$
3. (Bilinear) $\langle u, cv + dw \rangle = c\langle u, v \rangle + d\langle u, w \rangle$
4. (Positive Definite) $\langle u, u \rangle \geq 0$ and $\langle u, u \rangle = 0$ if and only if $u = 0$.

The **length** (or **norm** or **magnitude**) of a vector v is $\|v\| := \sqrt{\langle v, v \rangle}$.

The **angle** θ between two vectors is given by $\langle u, v \rangle = \|u\| \|v\| \cos \theta$.

Two vectors, u and v, are **orthogonal** (or **perpendicular**) if $\langle u, v \rangle = 0$.

Cauchy-Schwarz Inequality For any nonzero vectors u, v with inner product $\langle \cdot, \cdot \rangle$, then $|\langle u, v \rangle| \leq \|u\| \|v\|$

Triangle Inequality For any $u, v \in \mathbb{R}^n$, $\|u + v\| \leq \|u\| + \|v\|$

The set of all functions from a set S to $\mathbb{R}$ is denoted by $\mathbb{R}^S = \{ f : S \to \mathbb{R} \}$.