{ Surreal Numbers }

1 What's that?

The field of surreal numbers, usually denoted as **No**, was introduced in 1969 by John H. Conway. It is an extention of the field of Real numbers **R**.

The field of Surreals houses the Reals, the Hyperreals, all of Cantor's Ordinal numbers and their reciprocals. In a sense, it is the largest possible extention of the Real numbers.

2 { L | R }

Every surreal number consists of a left set and a right set (**L** and **R**), which contain other surreal numbers. Each half can be empty, finite or infinite, but must always satisfy the following condition: ***every number in L must be less than every number in R***.

$$\forall x \in L,\ \forall y \in R : x < y$$

Such a form **{ L | R }** represents the simplest surreal number that lies between the sets **L** and **R**.

It is possible for the same number to have different, but equivalent representations.

3 Constructing the numbers

$$\{\ |\ \} = 0$$

To get **0**, keep both sides empty.

$$\{ n \mid \ \} = n + 1 \qquad \{\ \mid n \} = n - 1$$

To get **n + 1**, put **n** in the left set and keep the right set empty. To get **n - 1**, put **n** in the right set instead. This way, we can obtain every integer.

$$\{ a \mid b \} = \frac{a+b}{2}$$

A number exactly between **a** and **b** is expressed as **{ a | b }**, for example:

$$\{ 0 \mid 1 \} = \frac{1}{2} \qquad \{ 0 \mid \tfrac{1}{2} \} = \frac{1}{4}$$

This gives us all the dyadic rational numbers, which have the form $\frac{a}{2^b}$.

What about $\frac{1}{3}$ or $\sqrt{2}$? We can get any real number **x** by using an increasing and a decreasing sequence of binary fractions, both of which converge to **x**.

$$\sqrt{2} = \{ 1, \tfrac{5}{4}, \tfrac{11}{8}, \tfrac{45}{32} \dots \mid \dots \tfrac{91}{64}, \tfrac{23}{16}, \tfrac{3}{2} \}$$

increasing ⟶ ⟵ *decreasing*

We can obtain a number greater than any integer by placing all of them in **L**:

$$\omega = \{ 1, 2, 3, 4, 5 \dots \mid \ \}$$

This is Cantor's first ordinal - **ω**. We can also get the first infinitesimal **ε** :

$$\varepsilon = \{ 0 \mid \dots \tfrac{1}{32}, \tfrac{1}{16}, \tfrac{1}{8}, \tfrac{1}{4}, \tfrac{1}{2} \}$$

4 Comparison and order

Suppose we have two surreal numbers, **X = { Lx | Rx }** and **Y = { Ly | Ry }**.

We say that **X ≤ Y** *if and only if everything in* **Lx** *is less than* **Y**, *and everything in* **Ry** *is greater than* **X**.

If **X ≤ Y**, but **Y ≰ X**, then we say that **X < Y**. For example, consider **2 = { 1 | }** and **ω = { 1, 2, 3, 4, ... | }**. **1** is less than **ω**, so everything in the left set of **2** is less than **ω**. The right side of **ω** is empty, therefore both conditions are met and **2 ≤ ω**. However, the reverse statement **ω ≤ 2** is false, since the left set of **ω** contains **3**, which is greater than **2**.

If **X ≤ Y** and **Y ≤ X**, then we say that **X = Y**. For example, **{ | }** and **{ -1 | 1 }** are equivalent ways to write **0**, since **{ | } ≤ { -1 | 1 }** and **{ -1 | 1 } ≤ { | }**.

5 Addition and subtraction

Let's add two surreal numbers together: **X = { Lx | Rx }** and **Y = { Ly | Ry }**.

$$X + Y = \{ L_x + Y, X + L_y \mid R_x + Y, X + R_y \}$$

For each element in **Lx**, add with **Y**, and put the result in the ***left*** set of the sum. Also take every element in **Ly**, add it with **X**, and put the result in the ***left*** set of the sum. For the ***right*** set, we take every element in **Rx** plus **Y**, and every element in **Ry** plus **X**.

For example, let **X = (ω + 1)** and **Y = 1**. We should expect to get **(ω + 2)**:

$$\begin{aligned} &(\omega + 1) + 1 = \\ &= \{ \omega \mid \ \} + \{ 0 \mid \ \} \\ &= \{ \omega + 1,\ \omega + 1 \mid \ \} \\ &= \{ \omega + 1 \mid \ \} = \omega + 2 \end{aligned}$$

Write **X** as **{ ω | }**, and **Y** as **{ 0 | }**.

For the left part, we take **X** and add it to every element in **Ly**. Since **Ly** only contains **0**, this gets us **X + 0**, which is **(ω + 1)**. We also add **Y** to the elements of **Lx** (which only contains **ω**), and get **(ω + 1)** again.

Since **Rx** and **Ry** are empty, there is nothing to add for the right part of the sum - it's empty.

What about subtraction? Let's see how to negate a number **X = { Lx | Rx }**:

$$-X = \{ -R_x \mid -L_x \}$$

To get **-X**, swap the right and left sets, then negate every element in both sets. For example, **-2 = - { 1 | } = { | -1 }**. To subtract **Y** from **X**, just take the negative of **Y** and add it to **X**:

$$X - Y = X + (-Y)$$

6 Multiplication

$$\begin{aligned} X \cdot Y &= \{ L_x \mid R_x \} \cdot \{ L_y \mid R_y \} = \\ &= \{ L_x \cdot Y + X \cdot L_y - L_x \cdot L_y,\ \ R_x \cdot Y + X \cdot R_y - R_x \cdot R_y \mid \\ &\quad\ L_x \cdot Y + X \cdot R_y - L_x \cdot R_y,\ \ X \cdot L_y + R_x \cdot Y - R_x \cdot L_y \} \end{aligned}$$

Multiplication, just like addition, involves multiplying pairs of surreal numbers drawn from both sides of **X** and **Y**.

For example, to compute the term **$(L_x \cdot Y + X \cdot L_y - L_x \cdot L_y)$**, pick an element **$l_x \in L_x$**, an element **$l_y \in L_y$**, and compute **$(l_x Y + X \cdot l_y - l_x \cdot l_y)$**. Then repeat for all such pairs of elements, and aggregate the results.

This multiplication formula is consistent with the real numbers:

$$\frac{1}{2}\ 4 = 2 \qquad \{ 0 \mid 1 \} \{ 3 \mid \ \} = \{ 1 \mid \ \}$$

References: Conway, John H. [1976] On Numbers and Games; Knuth, Donald E. [1974] Surreal Numbers