Problem Set 3

AE 598: Formal Methods in Aerospace Robotics

Due on April 23

- You are welcome to collaborate, but you must write up your own solutions.
- You can use any programming language, but you must add comments to each section of the codes to clearly explain your intent.
- If you use any resources other than your brain and lecture notes to solve the problems, you must cite them.
- Submit your write-ups and codes on Canvas as a **single PDF** (unless specifically instructed otherwise) clearly indicating your final solutions.
- In the following, $\|\cdot\|$ denotes the Euclidean norm for vectors and the induced 2-norm for matrices, i.e., $\|\cdot\| = \|\cdot\|_2$, unless otherwise noted.

1 Introduction to Risk-Aware Motion Planning

In this problem, we consider a motion planning problem with the following discretized double-integrator dynamics:

$$x_{k+1} = \mathbb{I}_{4\times 4} x_k + \Delta t \begin{bmatrix} \mathbb{O}_{2\times 2} & \mathbb{I}_{2\times 2} \\ \mathbb{O}_{2\times 2} & \mathbb{O}_{2\times 2} \end{bmatrix} x_k + \Delta t \begin{bmatrix} \mathbb{O}_{2\times 2} \\ \mathbb{I}_{2\times 2} \end{bmatrix} u_k$$

where $\mathbb{O}$ is the zero matrix, $\mathbb{I}$ is the identity matrix, k is the time step index, Δt is the discretization interval, $x_k = [p_k^\top, \dot{p}_k^\top]^\top \in \mathbb{R}^4$ is the state, $p_k \in \mathbb{R}^2$ is the 2D position, and $u_k \in \mathbb{R}^2$ is the control input. We have three circular obstacles in our environment located at

$$p_{\text{obs},1} = \begin{bmatrix} 5.0 \\ 0.0 \end{bmatrix} \text{ with } r_{\text{obs},1} = 1.0,\ p_{\text{obs},2} = \begin{bmatrix} 5.0 \\ 2.6 \end{bmatrix} \text{ with } r_{\text{obs},2} = 1.3,\ p_{\text{obs},3} = \begin{bmatrix} 5.0 \\ -3.0 \end{bmatrix} \text{ with } r_{\text{obs},3} = 1.5$$

where $p_{\text{obs},i}$ and $r_{\text{obs},i}$, $i = 1, \cdots, N_{\text{obs}}$, $N_{\text{obs}} = 3$, denote the positions and radii of the obstacles, respectively. The directory `data_HW3` contains

- `all_trajectories.npy`: N sampled trajectories with SCP with $dt = 0.1$, i.e.,

  `all_trajectories` = `(N, K+1, 4) array`

  `all_trajectories[n]` = `(K+1, 4) array` for the n-th state trajectory $x_k^{(n)} \in \mathbb{R}^4 = \begin{bmatrix} x_0^{(n)} & \cdots & x_K^{(n)} \end{bmatrix}^\top$

  where $N = 27$ is the number of the sampled trajectories, $K = 100$ is the number of the time steps, and $n = 0, \cdots, N-1$ is the index for the samples.

- `all_costs.npy`: Cumulative control efforts of N sampled trajectories, i.e.,

  `all_costs` = `(N,) array`

  `all_costs[n]` = `() array` for the control effort of the n-th state trajectory $= \sum_{k=0}^{K} \|u_k^{(n)}\|^2 \Delta t$.

  where $N = 27$ is again the number of the sampled trajectories and $n = 0, \cdots, N-1$ is the index.

Note that the sampled trajectories can be visualized by `HW3_risk_metrics.py` as in Figure 1. **The objective of this problem is to demonstrate that different risk metrics lead to different optimal trajectories.**

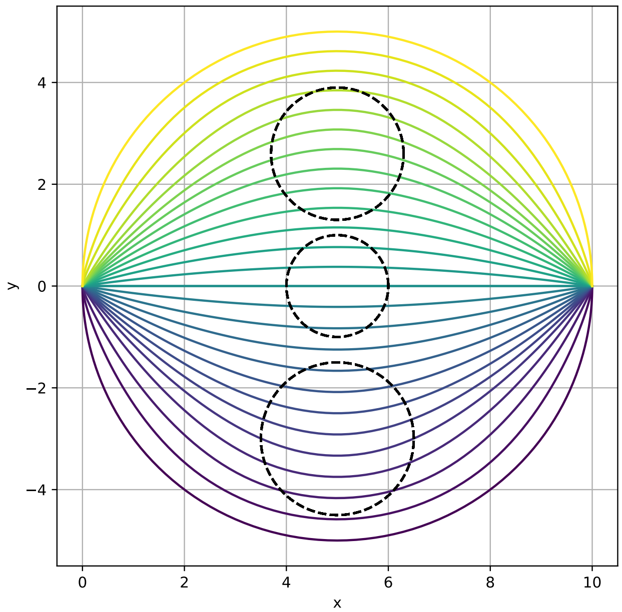

Figure 1: Sampled trajectories in `all_trajectories.npy`.

1. For each sampled trajectory with $x_k^{(n)} = \left[\left(p_k^{(n)}\right)^\top, \left(\dot{p}_k^{(n)}\right)^\top\right]$ and the obstacles with the positions $p_{\text{obs},i}$ and radii $r_{\text{obs},i}$, let us define $\bar{d}_{k,i}^{(n)}$ as

$$\bar{d}_{k,i}^{(n)} = \|p_k^{(n)} - p_{\text{obs},i}\|. \tag{1}$$

Consider the case where the nominal trajectory is perturbed and the true distance $d_{\text{mean},k,i}^{(n)}$ is given by

$$d_{\text{mean},k,i}^{(n)} = \bar{d}_{k,i}^{(n)} + \omega_k^{(n)} \text{ with } \omega_k^{(n)} \sim \mathcal{N}(\mu, \sigma^2) \text{ for } n = 0, \cdots, N-1 \text{ and } i = 1, \cdots, N_{\text{obs}} \tag{2}$$

where $\mathcal{N}(\mu, \sigma^2)$ is the Gaussian distribution with the mean $\mu = 0$ and standard deviation $\sigma = 0.2$. Define the expected collision risk $r_{\text{mean}}^{(n)}$ as

$$r_{\text{mean}}^{(n)} = \sum_{k=0}^{K} \sum_{i=1}^{N_{\text{obs}}} \mathbb{1}\left[\mathbb{E}\left[d_{\text{mean},k,i}^{(n)}\right] < r_{\text{obs},i}\right] + \sum_{k=0}^{K} \|u_k^{(n)}\|^2 \Delta t \text{ for } n = 0, \cdots, N-1$$

where $\mathbb{1}[\cdot]$ is the indicator function, and find the optimal index

$$n_{\text{mean}}^* = \arg\min_{n=0,\cdots,N-1} r_{\text{mean}}^{(n)}. \tag{3}$$

2. Consider the case where the nominal trajectory is perturbed and the true distance $d_{\text{wrst},k,i}^{(n)}$ is given by

$$d_{\text{wrst},k,i}^{(n)} = \bar{d}_{k,i}^{(n)} + \omega_k^{(n)} \text{ with } |\omega_k^{(n)}| \leq \bar{\omega} \text{ for } n = 0, \cdots, N-1 \text{ and } i = 1, \cdots, N_{\text{obs}}$$

where $\bar{\omega} = 0.6$ and $\bar{d}_{k,i}^{(n)}$ is given by (1). Define the worst-case collision risk $r_{\text{wrst}}^{(n)}$ as

$$r_{\text{wrst}}^{(n)} = \sum_{k=0}^{K} \sum_{i=1}^{N_{\text{obs}}} \mathbb{1}\left[\inf_{|\omega_k^{(n)}| \leq \bar{\omega}} \left(d_{\text{wrst},k,i}^{(n)}\right) < r_{\text{obs},i}\right] \text{ for } n = 0, \cdots, N-1$$

where $\mathbb{1}[\cdot]$ is the indicator function, and find the optimal index

$$n_{\text{wrst}}^* = \arg\min_{n=0,\cdots,N-1} r_{\text{wrst}}^{(n)} + \sum_{k=0}^{K} \|u_k^{(n)}\|^2 \Delta t. \tag{4}$$

3. Consider again the case where the perturbed distance is given by (2). Define the expected shortfall risk $r_{\text{cvar}}^{(n)}$ as

$$r_{\text{cvar}}^{(n)} = \sum_{k=0}^{K} \sum_{i=1}^{N_{\text{obs}}} \mathbb{1}\left[\text{CVaR}_{\alpha}^{\text{lower}}\left[d_{\text{mean},k,i}^{(n)}\right] < r_{\text{obs},i}\right] \text{ for } n = 0, \cdots, N-1$$

where $\mathbb{1}[\cdot]$ is the indicator function, $\alpha = 0.3$, and

$$\text{CVaR}_{\alpha}^{\text{lower}}[d] = \frac{1}{\alpha}\int_0^{\alpha} \text{VaR}_{1-\tau}[d]\, \mathrm{d}\tau = \frac{1}{\alpha}\int_0^{\alpha} \inf\left\{\mathsf{d} \in \mathbb{R} \mid \mathbb{P}[d > \mathsf{d}] \leq 1-\tau\right\} \mathrm{d}\tau. \tag{5}$$

Find the optimal index

$$n_{\text{cvar}}^{*} = \arg\min_{n=0,\cdots,N-1} r_{\text{cvar}}^{(n)} + \sum_{k=0}^{K} \|u_k^{(n)}\|^2 \Delta t. \tag{6}$$

Hint: For the Gaussian random variable (2), we can analytically compute (5) using the PDF and CDF of the standard Gaussian random variable $Z \sim \mathcal{N}(0,1)$.

4. Plot the trajectories of the optimal indices n_{mean}^{*}, n_{wrst}^{*}, and n_{cvar}^{*} in (3), (4), and (6), respectively, along with the circular obstacles, and explain the reasons for the observed differences.

2 Brief Review of Adaptive Robot Control

Let us consider the following robot dynamics:

$$\mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{C}(\mathbf{q},\dot{\mathbf{q}})\dot{\mathbf{q}} + \mathbf{D}(\mathbf{q},\dot{\mathbf{q}}) + \mathbf{G}(\mathbf{q}) = \mathbf{B}(\mathbf{q},\dot{\mathbf{q}})\mathbf{u} + \mathbf{J}^{(c)\top}\mathbf{F}^{(c)}$$

where $\mathbf{q} \in \mathbb{R}^n$ is the generalized coordinate, $\dot{\mathbf{q}} \in \mathbb{R}^n$ is the generalized velocity, and $\ddot{\mathbf{q}} \in \mathbb{R}^n$ is the generalized acceleration, $\mathbf{M}(\mathbf{q}) \in \mathbb{R}^{n\times n}$ is the mass/inertia matrix, $\mathbf{C}(\mathbf{q},\dot{\mathbf{q}}) \in \mathbb{R}^{n\times n}$ is the Coriolis and centripetal force matrix, $\mathbf{D}(\mathbf{q},\dot{\mathbf{q}}) \in \mathbb{R}^n$ is the dissipative/frictional force, $\mathbf{G}(\mathbf{q}) \in \mathbb{R}^n$ is the potential force, $\mathbf{u} \in \mathbb{R}^m$ is the control input, $\mathbf{F}^{(c)} \in \mathbb{R}^k$ is the contact force, $\mathbf{B}(\mathbf{q},\dot{\mathbf{q}}) \in \mathbb{R}^{n\times m}$ is the actuation matrix, and $\mathbf{J}^{(c)} \in \mathbb{R}^{k\times n}$ is the Jacobian matrix for the contact force.

We suppose that $\mathbf{M} \succ 0$, $\dot{\mathbf{M}} - 2\mathbf{C}$ is skew-symmetric, and the target trajectory $(\mathbf{q}_d, \dot{\mathbf{q}}_d, \ddot{\mathbf{q}}_d)$ is given. We design a controller as

$$\begin{aligned}
\dot{\mathbf{q}}_r &= \dot{\mathbf{q}}_d - \mathbf{\Lambda}_r(\mathbf{q} - \mathbf{q}_d) \\
\mathbf{u} &= \mathbf{B}^{\dagger}\left(\mathbf{M}\ddot{\mathbf{q}}_r + \mathbf{C}\dot{\mathbf{q}}_r + \mathbf{D} + \mathbf{G} - \mathbf{J}^{(c)\top}\hat{\mathbf{F}}^{(c)} - \mathbf{K}_r\mathbf{s}_r\right) \\
\dot{\hat{\mathbf{F}}}^{(c)} &= \mathbf{\Gamma}_a^{-1}\mathbf{J}^{(c)}\mathbf{s}_r
\end{aligned}$$

where $\mathbf{\Lambda}_r \succ 0$, $\mathbf{K}_r \succ 0$, $\mathbf{\Gamma}_a \succ 0$, $\mathbf{s}_r = \dot{\mathbf{q}} - \dot{\mathbf{q}}_r$ and the arguments are omitted for notational simplicity.

1. Suppose that $\mathbf{B}\mathbf{B}^{\dagger} = \mathbb{I}$ and $\mathbf{F}^{(c)}$ is constant. Show that $\lim_{t\to\infty}\|\mathbf{s}_r(t)\| = 0$ asymptotically. Hint: Use Barbalat's lemma.

3 Final Project Report

This problem is intended to help track the progress of your project and will not be graded.

1. The final project report, **which is due on May 14**, should include the following four sections.

   (a) *Abstract*: A clear and concise verbal description of your problem formulation, including key technical challenges (and potential novelties, if any) in 150-250 words.

   (b) *Problem Formulation*: A mathematically formal description of your problem formulation and challenges using equations, with all variables defined explicitly.

   (c) *Methodology*: A mathematically formal description of the approaches to be taken in addressing the problem, **with a theoretical/empirical justification for why your solution is expected to work**.

   (d) *Preliminary/Expected Results*: A summary of progress to date, which could include initial findings, anticipated outcomes, and any unsuccessful/incomplete attempts. **Note that the problem does not need to be fully solved to receive full credit**.

2. The report should be written in LaTeX using the IEEE conference template with the style files `ieeeconf.cls` and `IEEEtran.bst` on Canvas. If you are targeting a specific conference/journal, you may use its template instead, provided the page requirement below is satisfied.

3. **The report must be at least P pages in length** when formatted using the IEEE conference template, including figures, tables, and references, where P is given by

$$P = 2 + 1 \times (\# \text{ team members} - 1).$$

   For example, if your team has 3 members, the minimum page requirement is 4 pages. There is no maximum page limit.

4. If $\#$ team members ≥ 2, please **clearly indicate the individual contributions of each team member** in the report.

4 Final Project Presentation

This problem is intended to help track the progress of your project and will not be graded.

1. The final project presentation, **which will take place on May 5 and 7**, should cover all the four sections outlined above for the report.

2. Each team member must present for exactly 5 minutes, so the total presentation time should be ($5 \times \#$ team members) minutes.

3. The presentation is intended primarily for feedback, and full credit will be given as long as you participate.