Derandomizing Semidefinite Programming Based Approximation Algorithms

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Abstract

Remarkable breakthroughs have been made recently in obtaining approximate solutions to some fundamental NP-Complete problems, namely Max-Cut, Max k-Cut, Max-Sat, Max-Dicut, Max-Bisection, k Vertex Coloring, Independent Set, etc. These breakthroughs all involve polynomial time randomized *algorithms based upon* Semidefinite Programming, *a technique pioneered by Goemans and Williamson[6].*

In this paper, we give techniques to derandomize the above class of randomized algorithms, thus obtaining polynomial time deterministic algorithms with the same approximation ratios for the above problems. Note that Goemans and Williamson also gave an elegant method to derandomize their Max-Cut algorithm[6]. We show here that their technique has a fatal flaw. The techniques we subsequently develop are very different from theirs. At the heart of our technique is the use of spherical symmetry to convert a nested sequence of n integrations, which cannot be approximated sufficiently well in polynomial time, to a nested sequence of just a constant number of integrations, which can be approximated sufficiently well in polynomial time.

1 Introduction

The application of *Semidefinite Programming* to obtaining approximation algorithms for NP-Complete problems was pioneered by Goemans and Williamson[6]. This technique involves relaxing an integer program (solving which is an NP-Complete problem) to a semidefinite program (which can be solved with a sufficiently small error in polynomial time). In a remarkable breakthrough, Goemans and Williamson showed how this technique could be used to give a *randomized* approximation algorithm for the Max-Cut problem with an approximation ratio of .878. This must be contrasted with the previously best known approximation ratio of .5 obtained by the simple random cut algorithm. Subsequently, semidefinite programming based techniques have led to randomized algorithms with substantially better approximation ratios for a number of fundamental problems.

Goemans and Williamson[6] obtained a .878 approximation algorithm for Max-2Sat and an .758 approximation algorithm for Max-Sat, improving upon the previously best known bound of 3/4 [16] for both. They also obtain a .796 approximation algorithm for Max-Dicut, improving up the previously best known ratio of .25 given by the random cut algorithm.

Karger, Motwani and Sudan obtained an algorithm for coloring any k-colorable graph with $O(n^{1-3/(k+1)}\log n)$ colors[9]; in particular, for 3-colorable graphs, this algorithm requires $O(n^{.25}\log n)$ colors. This improves upon the deterministic algorithm of Blum[2] which requires $O(n^{1-\frac{1}{k-4/3}}\log^{\frac{8}{5}} n)$ colors for k-colorable graphs.

Frieze and Jerrum[5] obtained a .65 approximation algorithm for Max-Bisection improving the previous best known bound of .5 given by the random bisection algorithm. They also obtained a $1-\frac{1}{k}+2\frac{\ln k}{k^2}$ approximation algorithm for the Max k-Cut problem, improving upon the previously best known ratio of $1-\frac{1}{k}$ given by a random k-Cut.

Alon and Kahale[1] obtained an approximation algorithm for the independent set problem. For any constant $k \geq 3$, if the given graph has an independent set of size $n/k+m$, where n is the number of vertices, they obtain an $\Omega(m^{\frac{3}{k+1}}\log m)$ sized independent set, improving the previously known bound of $\Omega(m^{\frac{1}{k-1}})$ due to Boppana and Halldorsson[3].

All the new developments mentioned above are *randomized* algorithms. All of them share the following common paradigm. First, a semidefinite program is solved to obtain a collection of n vectors in n dimensional space satisfying some properties dependent upon the particular problem in question. This step is deterministic. Second, a set of independent random vectors is generated, each vector being *spherically symmetric*, i.e., equally likely to pass through any point on the n dimensional unit sphere centered at the origin. Finally, the solution is obtained using some computation on the n given vectors and the random vectors.

It is not obvious how to derandomize the above randomized algorithms, i.e., to obtain a "good" set of random vectors deterministically. A natural way to derandomize is to use the method of Conditional Probabilities[12, 14]. The problem that occurs then is to compute the conditional probabilities in polynomial time.

- Goemans and Williamson[6] gave a very elegant way of organizing the sequence of conditional probability

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computations for their Max-Cut and Max-Sat algorithms. The first contribution of this paper is to exhibit a a fatal flaw in their technique. More specifically, we show that a random variable which they claim has a normal distribution is not normally distributed; this claim is central to their derandomization procedure. Thus, before this work, no correct derandomization procedure was known for any of the semidefinite programming based approximation algorithms listed above. Also note that even assuming the correctness of the Goemans and Williamson[6] derandomization procedure for Max-Cut, it is not clear how the Karger, Motwani, Sudan algorithm for coloring[9] can be derandomized. As stated in [13], standard methods do not seem to be applicable to the derandomization of this algorithm.

• The main contribution of this paper is a technique which enables derandomization of all the semidefinite programming based approximation algorithms listed above. This leads to deterministic approximation algorithms for Max-Cut, Max k-Cut, Max Bisection, Max-2Sat, Max-Sat, Max-Dicut, k Vertex Coloring, and Independent Set with the same approximation ratios as their randomized counterparts mentioned above.

Our derandomization also uses the conditional probability technique. We compute conditional probabilities as follows. First, we show how to express each conditional probability computation as a sequence of $O(n)$ nested integrals. Performing this sequence of integrations with a small enough error seems hard to do in polynomial time. The key observation which facilitates conditional probability computation in polynomial time is that, using spherical symmetry properties, the above sequence of $O(n)$ nested integrals can be reduced to evaluating an expression with just a constant number of nested integrals for each of the semidefinite based approximation algorithms mentioned above. This new sequence of integrations can be performed with a small enough error in polynomial time. A host of precision issues also crops up in the derandomization. Conditional probabilities must be computed only at a polynomial number of points. Further each conditional probability computation must be performed within a small error. We show how to handle these precision issues in polynomial time.

As mentioned above, our derandomization techniques apply to all the semidefinite programming based approximation algorithms mentioned above. Loosely speaking, we believe our techniques are even more general, i.e., applicable to any scheme which follows the above paradigm and in which the critical performance analysis boils down to an "elementary event" involving just a constant number of the n vectors at a time. For example, in the graph coloring algorithm, only two vectors, corresponding to the endpoints of some edge, need to be considered at a time. An example of an elementary event involving 3 vectors is the Max-Dicut algorithm of Goemans and Williamson.

The paper is organized as follows. In Section 2, we describe the flaw in the Goemans and Williamson derandomization procedure. We then describe our derandomization scheme. Since the Karger, Motwani, Sudan coloring algorithm appears to be the hardest to derandomize amongst the algorithms mentioned above, our exposition concentrates on this algorithm. The derandomization of the other algorithms is similar. Section 3 describes the randomized scheme of Karger, Motwani and Sudan and outlines the derandomization procedure. The following sections describe the derandomization procedure in detail.

2 The Flaw in Goemans-Williamson

First we outline the randomized algorithm of Goemans and Williamson for the Max-Cut problem. A set of n unit vectors, one corresponding to each vertex of the graph in question, are obtained by solving a semidefinite program. We call these vectors *vertex vectors*. These are embedded in n dimensional space. Next, a spherically symmetric random hyperplane R through the center is chosen. This hyperplane divides the vertex vectors into 2 groups, which define a cut in the obvious manner. The expected number $E(W)$ of edges[1] across the cut is $\sum_{(v,w)\in E} \arccos(v.w)/\pi = \sum_{(v,w)\in E} Pr(sign(v\cdot R) \neq sign(w\cdot R))$, where E is the set of edges in the graph and v, w denote both vertices in the graph and the associated vertex vectors.

Note that the n random variables involved above are the n coordinates which define the random spherically symmetric hyperplane R. Let $R_1, R_2, \ldots, R_n$ be these random variables. For the hyperplane to be spherically symmetric, it suffices that the R_i's are independent and identically distributed with a mean 0 and variance 1 normal distribution, i.e., the density function is $\frac{1}{\sqrt{2\pi}}e^{-x^2/2}$.

The Goemans-Williamson procedure for derandomizing the above algorithm is as follows. From the initial set of unit length vertex vectors in n dimensions, a new set of vertex vectors in $n-1$ dimensions is obtained. These new vectors satisfy the property that $\sum_{(v,w)\in E} Pr(sign(v'\cdot R') \neq sign(w'\cdot R')) \geq E(W)$, where R' is a random spherically symmetric hyperplane in $n-1$ dimensions, v' is the new vertex vector for vertex v, and similarly for w. The above procedure is then repeated till the dimension drops to 1. At this point, the positive vertex vectors and the negative vertex vectors define a cut in the obvious way; further, the number of edges crossing this cut is at least $E(W)$.

It remains to show how the new vertex vectors in $n-1$ dimensions are obtained from the vertex vectors in n dimensions. Consider an edge (v, w). Let vertex vector $v = (v_1, \ldots, v_n)$ and $w = (w_1, \ldots, w_n)$. R_{n-1}, R_n are written in polar coordinates as $r\cos(\gamma)$ and $r\sin(\gamma)$, respectively, where $\gamma \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ and $r = sign(R_{n-1})\sqrt{R_{n-1}^2 + R_n^2}$ can be positive or negative. Similarly, v_{n-1}, v_n are written in polar coordinates as $x\cos(\alpha)$ and $x\sin(\alpha)$, respectively and w_{n-1}, w_n are written in polar coordinates as $y\cos(\beta)$ and $y\sin(\beta)$,

[1]For simplicity, we consider the unweighted Max-Cut problem.

respectively. Let $v' = (v_1, \ldots, v_{n-2}, x\cos(\alpha - \gamma))$ and $w' = (w_1, \ldots, w_{n-2}, y\cos(\beta - \gamma))$. Let $R' = (R_1, \ldots, R_{n-2}, r)$. Goemans and Williamson make the following key claim. Consequently, R' is a spherically symmetric random vector in $n-1$ dimensions.

Claim 2.1 *For any fixed γ, r is normally distributed.*

Clearly $v \cdot R = v' \cdot R'$ and $w \cdot R = w' \cdot R'$. Further, there exists a value for γ such that $\sum_{(v,w)\in E} Pr(sign(v' \cdot R') \neq sign(w' \cdot R')) \geq E(W)$; this is because $E(W) = \sum_{(v,w)\in E} Pr(sign(v \cdot R) \neq sign(w \cdot R)) = E_\gamma(\sum_{(v,w)\in E} Pr(sign(v \cdot R) \neq sign(w \cdot R)|\gamma)) =$
$E_\gamma(\sum_{(v,w)\in E} Pr(sign(v' \cdot R') \neq sign(w' \cdot R')|\gamma))$. This value of γ is found by computing $\sum_{(v,w)\in E} Pr(sign(v' \cdot R') \neq sign(w' \cdot R'))$ for a polynomial number of points in the suitably discretized interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$. Computing $Pr(sign(v' \cdot R') \neq sign(w' \cdot R'))$ is easy since this quantity is just $\arccos(\frac{v' \cdot w'}{|v'||w'|})/\pi$; note particularly that no integrations need to be performed in the above procedure.

The Flaw. The flaw is in Claim 2.1. We show that for fixed γ, r is not normally distributed. In fact, it is known that r and γ are independent [4]. Then, since $r = sign(R_{n-1})\sqrt{R_{n-1}^2 + R_n^2}$ and R_{n-1} and R_n are normally distributed, the density function for r given γ is $\frac{|r|}{2}e^{-r^2/2}$, which is clearly not normal. Therefore, the vector R' is no longer spherically symmetric and as a consequence, computing $Pr(sign(v' \cdot R') \neq sign(w' \cdot R'))$ becomes hard.

3 The Derandomization Scheme

For simplicity, we restrict our exposition here to the derandomization of the Karger, Motwani, Sudan algorithm for coloring 3-colorable graphs.

The Karger, Motwani, Sudan algorithm first obtains a set of vertex vectors such that $v \cdot w \leq -\frac{1}{2}$, for all edges (v, w). Subsequently, r vectors, $t_1, \ldots, t_r$, are chosen independently at random; each is spherically symmetric. Let the ith coordinate of t_j is denoted by $t_j[i]$, $1 \leq i \leq n$. Spherical symmetry is obtained by the following procedure, each $t_i[j]$, for each i, j, is chosen independently at random from a normal distribution with mean 0 and variance 1. The color that vertex v gets is simply c, where $t_c \cdot v = \max_{1\leq i\leq r} t_i \cdot v$.

Consider two vertex vectors v, w, such that (v, w) is an edge e in $G = (V, E)$. The probability that v and w get the same color in the algorithm is given by $Pr(E^e) = \sum_{k=1}^{r} Pr(E_k^e)$, where E_k^e is the following event.

E_k^e: $t_k \cdot v \geq t_1 \cdot v, \ldots, t_{k-1} \cdot v, t_{k+1} \cdot v, \ldots, t_r \cdot v \wedge t_k \cdot w \geq t_1 \cdot w, \ldots, t_{k-1} \cdot w, t_{k+1} \cdot w, \ldots, t_r \cdot w$

Karger, Motwani and Sudan[9] show that $\sum_{e\in E} Pr(E^e) < n/4$, for $r = d^{1/3}\log^{4/3} d$, where d is the maximum degree of the graph. Thus, at the end of the above procedure, the expected number of edges whose endpoints have the same color is less than $n/4$. All vertices except those upon which these "bad" edges are incident are discarded (the colors assigned to them are final). The remaining vertices, which are at most $n/2$ in number, are recolored by repeating the above procedure $O(\log n)$ times. This gives an $O(d^{1/3}\log^{4/3} d \log n)$ coloring of a 3-colorable graph This combined with Wigderson's trick [15] gives an $O(n^{1/4}\log n)$ coloring of a 3-colorable graph.

Notations. For a vector u, we denote by $u[1 \ldots m]$ the vector formed by the first m coordinates of u.

The Derandomization Scheme. The scheme is essentially to use the method of conditional expectations to deterministically find values for the vectors $t_1, \ldots, t_r$ so that the number of "bad" edges is just $n/4 + \tau$, for some constant τ. Note that this suffices to get a deterministic coloring algorithm.

We order the conditional variables: $t_1[1] \ldots t_1[n], t_2[1] \ldots t_2[n], \ldots, t_r[1] \ldots t_r[n]$. The values of these are fixed one by one, in order. So suppose that the values
$t_1[1 \ldots n], t_2[1 \ldots n], \ldots, t_i[1 \ldots j-1]$ have been determined. We will fix $t_i[j]$ to that value which minimizes $\sum_{e\in E}\sum_{k=1}^{r} Pr(E_k^e|t_1[1] \ldots t_i[j])$. The question now is how to compute this value of $t_i[j]$.

Let $p(\delta)$ be the function $\sum_{e\in E}\sum_{k=1}^{r} Pr(E_k^e|t_1[1] \cdots t_i[j-1], t_i[j] = \delta)$. Let $f(\delta)$ be the function $Pr(E_k^e|t_1[1] \cdots t_i[j-1], t_i[j] = \delta)$ for some fixed k and $e = (v, w)$. Strictly speaking, f should be indexed by both e and k, but this only makes the notation more cumbersome.

Let δ_{min} be a value δ which minimizes $p(\delta)$, $-\infty \leq \delta \leq \infty$. We will show the following.

Theorem 3.1 *A value κ satisfying the following property can be computed in polynomial time: $|p(\kappa) - p(\delta_{min})| = O(1/n^2)$.*

From the above theorem, we derive the following corollary.

Corollary 3.2 *After all $t_i[j]$'s have been fixed and colors assigned to vertices as in the randomized algorithm, the number of bad edges is at most $\frac{n}{4} + O(1)$.*

Proof. The number of conditional variables $t_i[j]$ is $nr \leq n^2$ (actually for 3-colorable graphs r is much smaller, namely $d^{1/3}\log^{4/3} d$, where d is the maximum degree). Therefore by Theorem 3.1, the total error over all conditional variables is $O(1)$. □

To show Theorem 3.1, we will do the following,

Step 1. First we discretize the vertex vectors. This ensures that derivatives of the functions we need to integrate are bounded by polynomials in n, thus facilitating discrete evaluation. This discretization is given in Section 4.

Step 2. In Section 5, we show how to express $Pr(E_k^e|t_1[1] \cdots t_i[j-1], t_i[j] = \delta)$ in terms of a function

I defined as follows. This enables the probability to be computed using integrals with just constant nesting depth.

Definition. Let b, b' be vectors of the same dimension, which is at least 2. Let a be another vector of the same dimension whose entries are independent and normally distributed. Let $x \leq y$ and $x' \leq y'$ be in the range $-\infty \ldots \infty$. Then $I(b, b', x, y, x', y')$ denotes $Pr((x \leq a \cdot b \leq y) \wedge (x' \leq a \cdot b' \leq y'))$.

Step 3. Computing the integral corresponding to I is the key question. As mentioned in the introduction, naive computation would require a nested sequence of $O(n)$ integrals. Using spherical symmetry properties we show how to perform this integration using just integrals with constant nesting. This is described in Section 6.

Step 4. In order to compute κ, we can afford to evaluate $p(\delta)$ only for a polynomial number of points. So we have to discretize the range $-\infty \ldots \infty$ for δ. This is done in Section 7.

Step 5. Finally we need to show how the integrations in Section 5 and 6 can be evaluated within an additive inverse polynomial error in polynomial time. This is again done by discretizing the range between the limits of the integration and is described in Section 8.

4 Discretizing the Input Vectors

For simplicity, we assume that $v \cdot w = -1/2$. Our algorithm can be easily generalized as long as $v \cdot w$ is at least some constant greater than -1.

We first round off each of the components of vectors v, w to non-zero multiples of $\epsilon = 1/(24n^3)$ (not necessarily the nearest) so as to satisfy the following property: for each h, $1 \leq h < n$, $||w'|^2 - \frac{(v' \cdot w')^2}{|v'|^2}| \geq \epsilon^4$ where $v' = v[h \ldots n]$ and $w' = w[h \ldots n]$. This property will be necessary in our proofs later, in Lemma 6.2, in particular.

The above rounding is done as follows. First, all the entries in v, w are rounded upwards (in absolute value) to the nearest non-zero multiples of ϵ. Next, upto 2ϵ is added (in absolute value) to v_{n-1} so that $|v_{n-1}w_n - w_{n-1}v_n| > 2\epsilon^2$. Note that in this process $|v|^2, |w|^2 \leq 2$, for small enough ϵ, i.e., for large enough n.

Lemma 4.1 *For each h, $1 \leq h < n$, $||w'|^2 - \frac{(v' \cdot w')^2}{|v'|^2}| \geq \epsilon^4$ where $v' = v[h \ldots n]$ and $w' = w[h \ldots n]$.*

Proof. First consider $h = n-1$. $(v' \cdot w')^2 = (v_{n-1}w_{n-1} + v_n w_n)^2 = (v_{n-1}^2 + v_n^2)(w_{n-1}^2 + w_n^2) - (v_{n-1}w_n - w_{n-1}v_n)^2 \leq |v'|^2|w'|^2 - 4\epsilon^4$. Therefore, $||w'|^2 - \frac{(v' \cdot w')^2}{|v'|^2}| \geq \frac{4\epsilon^4}{|v'|^2} \geq 2\epsilon^4 \geq \epsilon^4$.

Next consider $h < n-1$. Let $l = v[h, \ldots, n-2]$ and $m = w[h, \ldots, n-2]$. Let $l' = v[n-1, n]$ and $m' = w[n-1, n]$. $(v' \cdot w')^2 = (l \cdot m + l' \cdot m')^2 = (l \cdot m)^2 + (l' \cdot m')^2 + 2(l' \cdot m')(l \cdot m) \leq |l|^2|m|^2 + (l' \cdot m')^2 + 2|l'||m'||l||m|$. By the previous paragraph, $(l' \cdot m')^2 \leq |l'|^2|m'|^2 - 2\epsilon^4$. Therefore, $(v' \cdot w')^2 \leq |l|^2|m|^2 + |l'|^2|m'|^2 + |l'|^2|m|^2 + |l|^2|m'|^2 - \epsilon^4 \leq (|l|^2 + |l'|^2)(|m|^2 + |m'|^2) - \epsilon^4 = |v'|^2|w'|^2 - 4\epsilon^4$. Therefore, $||w'|^2 - \frac{(v' \cdot w')^2}{|v'|^2}| \geq \frac{4\epsilon^4}{|v'|^2} \geq 2\epsilon^4 \geq \epsilon^4$. □

Lemma 4.2 shows that the above rounding doesn't change $p(\gamma_{min})$ by much.

Lemma 4.2 *$p(\gamma_{min})$ changes by at most $\frac{24n\epsilon}{1+12n\epsilon}|E| \leq 1$ due to the above rounding.*

Proof. Originally $v \cdot w = -1/2$ and v, w were unit vectors. The above rounding changes $\frac{v \cdot w}{|v||w|}$ by at most $|\epsilon'| \leq 6n\epsilon$. $Pr(E^e) = \sum_{i=0}^{\infty} a_i (\frac{v \cdot w}{|v||w|})^i$, where $\sum_i a_i = 1$, $a_i \geq 0$ (see [5], Lemma 3). Then, after the rounding,
$|\Delta Pr(E^e)| = \sum_{i=0}^{\infty} a_i |(-\frac{1}{2} + \epsilon')^i - (-\frac{1}{2})^i| = \sum_{i=0}^{\infty} a_i |(\frac{1}{2} - \epsilon')^i - (\frac{1}{2})^i|$

First, suppose $\epsilon' \geq 0$. Then $|\Delta Pr(E^e)| = \sum_{i=0}^{\infty} a_i ((\frac{1}{2})^i - (\frac{1}{2} - \epsilon')^i)$
$\leq \sum_{i=0}^{\infty} ((\frac{1}{2})^i - (\frac{1}{2} - \epsilon')^i)$
$\leq 4\epsilon'/(1 + 2\epsilon')$. Similarly for $\epsilon' < 0$, $|\Delta Pr(E^e)| \leq -4\epsilon'/(1 + 2\epsilon')$. □

5 Computing Conditional Probabilities

We are required to compute $p(\delta)$, i.e., $\sum_{e \in E} \sum_{k=1}^{r} Pr(E_k^e | t_1[1], \ldots, t_i[j-1], t_i[j] = \delta)$. We sometimes write the probability in the sum as $Pr(E_k^e | t_1[1], \ldots, t_i[j])$ when it is clear that $t_i[j]$ is being assigned δ. For $e = (v, w)$ and some fixed k, we show how to express $Pr(E_k^e | t_1[1], \ldots, t_i[j])$ in terms of the integrals $I()$ defined earlier. Let $\alpha' = t_i[1 \ldots j-1] \cdot v[1 \ldots j-1]$. Let $\beta' = t_i[1 \ldots j-1] \cdot w[1 \ldots j-1]$.

There are 3 cases.

Case 1. $k < i$. Let $t_k \cdot v = \alpha$ and $t_k \cdot w = \beta$. If one of $t_1 \cdot v, \ldots, t_{i-1} \cdot v$ is greater than α or if one of $t_1 \cdot w, \ldots, t_{i-1} \cdot w$ is greater than β then the above probability is 0. Otherwise, it is:

$Pr(E_k^e | t_1[1] \cdots t_i[j]) =$
$Pr(\wedge_{l=i}^{r} (t_l \cdot v \leq \alpha \wedge t_l \cdot w \leq \beta) | t_i[1 \ldots j])$

Note that the events $t_l \cdot v \leq \alpha \wedge t_l \cdot w \leq \beta$, $i \leq l \leq r$, are all independent. Then, by symmetry:

Case 1.1. $j < n-1$.

$Pr(E_k^e | t_1[1] \cdots t_i[j]) =$
$Pr(t_i \cdot v \leq \alpha \wedge t_i \cdot w \leq \beta)$
$Pr(t_r \cdot v \leq \alpha \wedge t_r \cdot w \leq \beta)^{r-i} =$
$I(v[j+1 \ldots n], w[j+1 \ldots n], -\infty, \alpha - \alpha' - v[j]\delta, -\infty, \beta - \beta' - w[j]\delta)$
$\times I^{r-i}(v, w, -\infty, \alpha, -\infty, \beta)$

Case 1.2. $j = n-1$. Assume that both $v[n]$ and $w[n]$ are positive. The other cases are similar.

$$Pr(E_k^e|t_1[1]\cdots t_i[j]) = Pr(t_i \cdot v \leq \alpha \wedge t_i \cdot w \leq \beta) Pr(t_r \cdot v \leq \alpha \wedge t_r \cdot w \leq \beta)^{r-i} = (\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\min\{\frac{\alpha-\alpha'-v[n-1]\delta}{v[n]}, \frac{\beta-\beta'-w[n-1]\delta}{w[n]}\}} e^{-z^2/2} dz) I^{r-i}(v, w, -\infty, \alpha, -\infty, \beta)$$

Note that the derivative of the above probability with respect to δ is undefined at only one point, namely, the value of δ for which $\frac{\alpha-\alpha'-v[n-1]\delta}{v[n]} = \frac{\beta-\beta'-w[n-1]\delta}{w[n]}$.

Case 1.3. $j = n$. If $\alpha' + v[n]\delta > \alpha$ or $\beta' + w[n]\delta > \beta$, then $Pr(E_k^e|t_1[1]\cdots t_i[j]) = 0$. Otherwise,

$$Pr(E_k^e|t_1[1]\cdots t_i[j]) = Pr(t_i \cdot v \leq \alpha \wedge t_i \cdot w \leq \beta) Pr(t_r \cdot v \leq \alpha \wedge t_r \cdot w \leq \beta)^{r-i} = I^{r-i}(v, w, -\infty, \alpha, -\infty, \beta).$$

Note that the derivative of the above probability with respect to δ is undefined only for two values, namely, when $\alpha = \alpha' + v[n]\delta$ and $\beta = \beta' + w[n]\delta$.

Case 2. $k > i$. Let $\max\{t_1 \cdot v, \ldots, t_{i-1} \cdot v\} = \alpha$ and $\max\{t_1 \cdot w, \ldots, t_{i-1} \cdot w\} = \beta$. Then, let A be the event $t_k \cdot v \geq \alpha \wedge t_k \cdot w \geq \beta$ and B_l be the event $t_l \cdot v \leq t_k \cdot v \wedge t_l \cdot w \leq t_k \cdot w$.

Note that the events B_l in this case are not independent. However, they are independent for fixed values of $t_k \cdot v$ and $t_k \cdot w$. Then:

Case 2.1. $j < n - 1$.

$$Pr(E_k^e|t_1[1]\cdots t_i[j]) = Pr(A \wedge B_i \wedge \ldots B_{k-1} \wedge B_{k+1} \wedge \ldots B_r | t_i[1 \ldots j-1], t_i[j] = \delta) = \int_{x=\alpha}^{\infty} \int_{y=\beta}^{\infty} Pr((x \leq t_k.v \leq x + dx) \wedge (y \leq t_k.w \leq y + dy)) Pr(t_i.v \leq x \wedge t_i.w \leq y | t_i[1 \ldots j]) Pr(t_{i+1}.v \leq x \wedge t_{i+1}.w \leq y) \ldots Pr(t_{k-1}.v \leq x \wedge t_{k-1}.w \leq y) Pr(t_{k+1}.v \leq x \wedge t_{k+1}.w \leq y) \ldots Pr(t_r.v \leq x \wedge t_r.w \leq y) = \int_{x=\alpha}^{\infty} \int_{y=\beta}^{\infty} I(v, w, x, x + dx, y, y + dy) I^{r-i-1}(v, w, -\infty, x, -\infty, y) I(v[j+1 \ldots n], w[j+1 \ldots n], -\infty, x - \alpha' - v[j]\delta, -\infty, y - \beta' - w[j]\delta)$$

Case 2.2. $j = n - 1$. Assume that $v[n]$ and $w[n]$ are positive. The remaining cases are similar.

$$Pr(E_k^e|t_1[1]\cdots t_i[j]) = Pr(A \wedge B_i \wedge \ldots B_{k-1} \wedge B_{k+1} \wedge \ldots B_r | t_i[1 \ldots j]) = \int_{x=\alpha}^{\infty} \int_{y=\beta}^{\infty} Pr((x \leq t_k.v \leq x + dx) \wedge (y \leq t_k.w \leq y + dy)) Pr(t_i.v \leq x \wedge t_i.w \leq y | t_i[1 \ldots j]) Pr(t_{i+1}.v \leq x \wedge t_{i+1}.w \leq y) \ldots Pr(t_{k-1}.v \leq x \wedge t_{k-1}.w \leq y) Pr(t_{k+1}.v \leq x \wedge t_{k+1}.w \leq y) \ldots Pr(t_r.v \leq x \wedge t_r.w \leq y) = \int_{x=\alpha}^{\infty} \int_{y=\beta}^{\infty} I(v, w, x, x + dx, y, y + dy) I^{r-i-1}(v, w, -\infty, x, -\infty, y) (\frac{1}{\sqrt{2\pi}} \int_{z=-\infty}^{\min\{\frac{x-\alpha'-v[n-1]\delta}{v[n]}, \frac{y-\beta'-w[n-1]\delta}{w[n]}} e^{-z^2/2} dz) = \frac{1}{\sqrt{2\pi}} \int_{z=-\infty}^{\infty} \int_{x=\max\{\alpha, \alpha'+v[n]z+v[n-1]\delta\}}^{\infty} \int_{y=\max\{\beta, \beta'+w[n]z+w[n-1]\delta\}}^{\infty} I(v, w, x, x + dx, y, y + dy) I^{r-i-1}(v, w, -\infty, x, -\infty, y) e^{-z^2/2} dz.$$

Note that the derivative of the above expression with respect to δ is undefined only when $\frac{\alpha-\alpha'-v[n-1]\delta}{v[n]} = \frac{\beta-\beta'-w[n-1]\delta}{w[n]}$. We see this by the following argument. Consider the values of δ for which $\frac{\alpha-\alpha'-v[n-1]\delta}{v[n]} < \frac{\beta-\beta'-w[n-1]\delta}{w[n]}$. For other values, a similar expression will hold. The above expression then becomes as follows. This expression is differentiable for all values of δ for which it is defined. The same holds for the other expression.

$$Pr(E_k^e|t_1[1]\cdots t_i[j]) = (\frac{1}{\sqrt{2\pi}} \int_{z=-\infty}^{\frac{\alpha-\alpha'-v[n-1]\delta}{v[n]}} \int_{x=\alpha}^{\infty} \int_{y=\beta}^{\infty} I(v, w, x, x + dx, y, y + dy) I^{r-i-1}(v, w, -\infty, x, -\infty, y) e^{-z^2/2} dz) + (\frac{1}{\sqrt{2\pi}} \int_{z=\frac{\alpha-\alpha'-v[n-1]\delta}{v[n]}}^{\frac{\beta-\beta'-w[n-1]\delta}{w[n]}} \int_{x=\alpha'+v[n]z+v[n-1]\delta}^{\infty} \int_{y=\beta}^{\infty} I(v, w, x, x + dx, y, y + dy) I^{r-i-1}(v, w, -\infty, x, -\infty, y) e^{-\frac{z^2}{2}} dz) + (\frac{1}{\sqrt{2\pi}} \int_{z=\frac{\beta-\beta'-w[n-1]\delta}{w[n]}}^{\infty} \int_{x=\alpha'+v[n]z+v[n-1]\delta}^{\infty} \int_{y=\beta'+w[n]z+w[n-1]\delta}^{\infty} I(v, w, x, x + dx, y, y + dy) I^{r-i-1}(v, w, -\infty, x, -\infty, y) e^{-\frac{z^2}{2}} dz)$$

Case 2.3. $j = n$.

$$Pr(E_k^e|t_1[1]\cdots t_i[j]) = Pr(A \wedge B_i \wedge \ldots B_{k-1} \wedge B_{k+1} \wedge \ldots B_r | t_i[1 \ldots j]) = \int_{y=\max\{\beta, \beta'+w[n]\delta\}}^{\infty} \int_{x=\max\{\alpha, \alpha'+v[n]\delta\}}^{\infty} Pr((x \leq t_k.v \leq x + dx) \wedge (y \leq t_k.w \leq y + dy)) Pr(t_{i+1}.v \leq x \wedge t_{i+1}.w \leq y) \ldots Pr(t_{k-1}.v \leq x \wedge t_{k-1}.w \leq y) Pr(t_{k+1}.v \leq x \wedge t_{k+1}.w \leq y) \ldots Pr(t_r.v \leq x \wedge t_r.w \leq y) = \int_{\max\{\beta, \beta'+w[n]\delta\}}^{\infty} \int_{\max\{\alpha, \alpha'+v[n]\delta\}}^{\infty} I(v, w, x, x + dx, y, y + dy) I^{r-i-1}(v, w, -\infty, x, -\infty, y)$$

Note that the derivative of the above expression with respect to δ is undefined only for two values, namely, when $\alpha = \alpha' + v[n]\delta$ and $\beta = \beta' + w[n]\delta$.

Case 3. $k = i$. Let $\max\{t_1 \cdot v, \ldots, t_{i-1} \cdot v\} = \alpha$ and $\max\{t_1 \cdot w, \ldots, t_{i-1} \cdot w\} = \beta$. Then, let A be the event $t_i \cdot v \geq \alpha \wedge t_i \cdot w \geq \beta$ and B_l be the event $t_l \cdot v \leq t_i \cdot v \wedge t_l \cdot w \leq t_i \cdot w$.

Again, note that the events B_l in this case are not independent. However, they are independent for fixed values of $t_i \cdot v$ and $t_i \cdot w$. Then, as in Case 2:

Case 3.1. $j < n-1$.

$$Pr(E_k^e|t_1[1]\cdots t_i[j]) =$$
$$Pr(A \wedge B_{i+1} \wedge \ldots \wedge B_r|t_i[1\ldots j]) =$$
$$\int_{x=\alpha}^{\infty}\int_{y=\beta}^{\infty}$$
$$I(v[j+1\ldots n], w[j+1\ldots n], x-\alpha'-v[j]\delta, x+\mathrm{d}x-\alpha'-v[j]\delta, y-\beta'-w[j]\delta, y+\mathrm{d}y-\beta'-w[j]\delta)$$
$$I^{r-i}(v, w, -\infty, x, -\infty, y)$$

Case 3.2. $j = n-1$. Assume that $v[n]$ and $w[n]$ are positive. The other cases are similar.

$$Pr(E_k^e|t_1[1]\cdots t_i[j]) =$$
$$Pr(A \wedge B_{i+1} \wedge \ldots \wedge B_r|t_i[1\ldots j]) =$$
$$\frac{1}{\sqrt{2\pi}}\int_{z=\max\{\frac{\alpha-\alpha'-v[n-1]\delta}{v[n]}, \frac{\beta-\beta'-w[n-1]\delta}{w[n]}\}}^{\infty}$$
$$I^{r-i}(v, w, -\infty, \alpha'+v[n-1]\delta+v[n]z, -\infty, \beta'+w[n-1]\delta+w[n]z)$$
$$e^{-z^2/2}\mathrm{d}z.$$

Note that the derivative of the above expression with respect to δ is undefined only when $\frac{\alpha-\alpha'-v[n-1]\delta}{v[n]} = \frac{\beta-\beta'-w[n-1]\delta}{w[n]}$.

Case 3.3. $j = n$.

$$Pr(E_k^e|t_1[1]\cdots t_i[j]) =$$
$$Pr(A \wedge B_{i+1} \wedge \ldots \wedge B_r|t_i[1\ldots j]) =$$
$$I^{r-i}(v, w, -\infty, \alpha'+v[n]\delta, -\infty, \beta'+w[n]\delta)$$

6 Computing $I(b, b', x, y, x', y')$

Recall that $I(b, b', x, y, x', y')$ denotes $Pr((x \le a\cdot b \le y) \wedge (x' \le a\cdot b' \le y'))$, where a is a spherically symmetric random vector. We show how to compute this probability approximately.

Let b and b' be h dimensional. Note that $h \ge 2$. Consider the h dimensional coordinate system with respect to which b, b' are specified. Note that a naive way to compute I is to perform a sequence of h nested integrals. This seems hard to do in polynomial time with the required error. We use the following method instead.

Note that since each coordinate of a is normally distributed with 0 mean and variance 1, a has a spherically symmetric distribution. We rotate the coordinate system so that $b = (b_1, 0, \ldots, 0)$ and $b' = (b'_1, b'_2, 0\ldots, 0)$, where $b_1, b'_2 \ge 0$. As we will show shortly, both will be strictly positive for all our calls to I. Let $a' = (a_1, a_2, \ldots, a_h)$, each of which is independently chosen from a mean-0 variance-1 normal distribution; here the co-ordinates are in the new coordinate system. The following lemma is key.

Lemma 6.1 *The probability distribution of a' is identical to that of a.*

Proof. Follows from the fact that a is spherically symmetric as is a'. □

Note that $a'\cdot b = a_1b_1$ and $a'\cdot b' = a_1b'_1 + a_2b'_2$. Now $I(b, b', x, y, x', y')$ denotes $Pr((x \le a_1b_1 \le y) \wedge (x' \le a_1b'_1 + a_2b'_2 \le y'))$. This equals

$$Pr((\tfrac{x}{b_1} \le a_1 \le \tfrac{y}{b_1}) \wedge (\tfrac{x'-a_1b'_1}{b'_2} \le a_2 \le \tfrac{y'-a_1b'_1}{b'_2})) = \frac{1}{2\pi}\int_{x/b_1}^{y/b_1} e^{-\frac{z^2}{2}}\left(\int_{(x'-zb'_1)/b'_2}^{(y'-zb'_1)/b'_2} e^{-\frac{z'^2}{2}}\ \mathrm{d}z'\right)\ \mathrm{d}z$$

Lemma 6.2 $|b_1| \ge \frac{1}{24n^3}$ *and* $|b'_2| \ge (\frac{1}{24n^3})^2$.

Proof. Recall that by the rounding in Section 4 and Lemma 4.1. $|b_1| = |b| \ge \frac{1}{24n^3}$. $|b'_2| = \sqrt{||b'|^2 - \frac{(b\cdot b')^2}{|b'|^2}|} \ge (\frac{1}{24n^3})^2$. □

7 Discretizing $t_i[j]$

For the purpose of this section, assume that integrations can be performed exactly. This will be dealt with in Section 8.

The values δ we choose for $t_i[j]$ will be multiples of $O(1/n^{11})$ in the range $-n^3\ldots n^3$. We need to show that discretizing the range of δ causes at most $\frac{1}{n^2}$ error for each of the conditionality variables. We consider three cases, $j = n$, $j = n-1$ and $j < n-1$. Lemma 7.1 proves the above when $j < n-1$. Note that for $j < n-1$, $f'(\delta)$ is always defined. For $j = n-1$ and $j = n$, a similar proof holds with the following difference. As shown in Section 5, $f'(\delta)$ is undefined at at most two values of δ when $j = n-1$ or $j = n$; So when $j = n-1, n$, $p'(\delta)$ is undefined only at $O(r|E|) = O(n^3)$ values of δ. We add these points in our discretization. These divide the range $-n^3\ldots n^3$ into $O(n^3)$ subranges, in each of which $f'(\delta)$ and $p'(\delta)$ are defined. In each of these ranges, a proof similar to that of Lemma 7.1 shows the needful.

Lemma 7.1 $|f'(\delta)| = O(n^6)$ *Therefore* $|f(\delta \pm O(\frac{1}{n^{11}})) - f(\delta)| \le O(\frac{1}{n^{11}})O(n^6)$ *and* $|p(\delta \pm O(\frac{1}{n^{11}})) - p(\delta)| \le O(\frac{1}{n^{11}})O(n^6)r|E| = O(\frac{1}{n^2})$.

Proof. Note that the function f depends upon which of Cases 1.1,2.1,3.1 hold in Section 5. We show the lemma only for Case 2.1. The other cases can be shown similarly.

For Case 2.1: $f(\delta) = \int_\alpha^\infty\int_\beta^\infty g(x,y)h(x,y,\delta)\mathrm{d}y\mathrm{d}x$,
where $g(x,y)\mathrm{d}y\mathrm{d}x = I(v, w, x, x+\mathrm{d}x, y, y+\mathrm{d}y)I^{r-i-1}(v, w, -\infty, x, -\infty, y)$ and
$h(x,y,\delta) = I(v[j+1\ldots n], w[j+1\ldots n], -\infty, x - t_i[1\ldots j]\cdot v[1\ldots j], -\infty, y - t_i[1\ldots j]\cdot w[1\ldots j])$.

$|f'(\delta)| \le \int_\alpha^\infty\int_\beta^\infty |g(x,y)||\frac{\partial h(x,y,\delta)}{\partial\delta}|\mathrm{d}y\mathrm{d}x \le \max_{x,y}|\frac{\partial h(x,y,\delta)}{\partial\delta}|$, since $\int_\alpha^\infty\int_\beta^\infty g(x,y)\mathrm{d}y\mathrm{d}x \le 1$.

We show that $|f'(\delta)| = O(n^6)$ by estimating $\max_{x,y}|\frac{\partial h(x,y,\delta)}{\partial\delta}|$.

Let $c(x,\delta) = x - t_i[1\ldots j-1]\cdot v[1\ldots j-1] - v[j]\delta = c' - v[j]\delta$ and
$d(y,\delta) = y - t_i[1\ldots j-1]\cdot w[1\ldots j-1] - w[j]\delta =$

$d' - w[j]\delta$.
From Section 6,
$h(x,y,\delta) = \frac{1}{2\pi}\int_{-\infty}^{c(x,\delta)/b_1} e^{-\frac{z^2}{2}}(\int_{-\infty}^{(d(y,\delta)-zb'_1)/b'_2} e^{-\frac{z'^2}{2}}\ dz')\ dz$

Let $G(y,\delta,l) = \frac{1}{\sqrt{2\pi}}\int e^{-\frac{l^2}{2}}H(y,\delta,l)dl$, where $H(y,\delta,l) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{(d(y,\delta)-lb'_1)/b'_2} e^{-\frac{z'^2}{2}}\ dz'$.

Then $|\frac{\partial h}{\partial \delta}| = A + B$, where $A = |\frac{\partial G}{\partial \delta}_{l=c(x,\delta)/b_1} - \frac{\partial G}{\partial \delta}_{l=-\infty}|$
and $B = |\frac{\partial G}{\partial l}_{l=c(x,\delta)/b_1}||\frac{\partial l}{\partial \delta}_{l=c(x,\delta)/b_1}|$.

Note that $|\frac{\partial G}{\partial l}| \leq 1$ for all l since $H(y,\delta,l) \leq 1$.
Further, $|\frac{\partial l}{\partial \delta}_{l=c(x,\delta)/b_1}| = |v[j]/b_1| = O(n^3)$, by Lemma 6.2 and the discretization of $t_i[j]$.
Therefore, $B = O(n^3)$.

A is bounded as follows.
$A = |\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{c(x,\delta)/b_1} e^{-\frac{l^2}{2}}\frac{\partial H(y,\delta,l)}{\partial \delta}dl| \leq \max_{y,\delta,l}|\frac{\partial H(y,\delta,l)}{\partial \delta}|$.

It remains to bound $\max_{y,\delta,l}|\frac{\partial H(y,\delta,l)}{\partial \delta}|$. This is done below using the same technique as above. Recall that

$H(y,\delta,l) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{(d(y,\delta)-lb'_1)/b'_2} e^{-\frac{z'^2}{2}}\ dz'$.

Let $J(m) = \frac{1}{\sqrt{2\pi}}\int e^{-\frac{m^2}{2}}\ dm$. Then

$|\frac{\partial H}{\partial \delta}| = |\frac{dJ}{dm}|_{m=(d(y,\delta)-lb'_1)/b'_2}|\frac{\partial m}{\partial \delta}_{m=(d(y,\delta)-lb'_1)/b'_2}| \leq |w[j]/b'_2| = O(n^6)$

Therefore, $|f'(\delta)| \leq A + B = O(n^6)$. The lemma follows. □

The next lemma shows that considering values between $-n^3 \ldots n^3$ only causes an error of $\frac{1}{n^2}$ for each of the conditional variables.

Lemma 7.2 *$\min_\delta p(\delta)$ can be estimated within an additive $|r|E|\frac{1}{n^6}| \leq |\frac{1}{n^3}|$ while only considering values between $-n^3 \ldots n^3$ for δ.*

Proof. Note that $Pr(E_k^e|t_1[1]\cdots t_i[j-1]) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(\delta)e^{-\frac{\delta^2}{2}}d\delta$
$\leq \frac{1}{\sqrt{2\pi}}(\int_{-n^3}^{n^3} f(\delta)e^{-\frac{\delta^2}{2}}d\delta + 2\int_{n^3}^{\infty} e^{-\frac{\delta^2}{2}}d\delta)$ since $f(\delta) \leq 1$. Now,
$2\int_{n^3}^{\infty} e^{-\frac{\delta^2}{2}}d\delta \leq \frac{1}{n^6}$, by Chebyschev's inequality. □

From the above 2 lemmas,

Corollary 7.3 *$\min_\delta p(\delta)$ can be estimated within an additive $|r|E|O(\frac{1}{n^5})| = |O(\frac{1}{n^2})|$ while considering only multiples of $O(\frac{1}{n^{11}})$ between $-n^3 \ldots n^3$ for δ, provided integrations can be exactly performed.*

8 Performing Integrations

Note that each of the conditional variables is in the range $-n^3 \ldots n^3$ and each of the components of v, w is in the range $-2 \ldots 2$ (liberally speaking, due to the rounding in Section 4). Therefore, $-2n^4 \leq \alpha, \beta \leq 2n^4$ (recall α, β are defined in Section 5). For the same reason, in Section 5, ∞ can be replaced by $2n^4$ and $-\infty$ by $-2n^4$ at all places where they occur. So, in calls to $I(b, b', x, y, x', y')$, x, x', y, y' are all in the range $-2n^4 \ldots 2n^4$. By Lemma 6.2, the limits of all integrations are in the range $-O(n^{13}) \ldots O(n^{13})$.

We perform integrations by converting them into summations using a step size of ρ where $1/\rho = O(n^{64})$ [2]. Consider an integral $\int_a^{a+\rho}\lambda(x)dx$.

The following lemma is classical.

Lemma 8.1 *$|\int_a^{a+\rho}\lambda(x)dx - \lambda(a)\rho| \leq M\rho^2$, where M upper bounds $\lambda'(x)$.*

The following lemma can be obtained by just differentiating all the functions we deal with.

Lemma 8.2 *For each $\lambda(x)$ we integrate, $|\lambda'(x)|$ is bounded by $O(n|\frac{1}{b_1}|)$ or $O(n|\frac{1}{b'_2}|)$, which is $O(n^7)$. Therefore, each integral can be performed with an error of at most $O(n^{13}n^7\rho)$. Four nested integrals can be performed with an error of $O(n^{52}n^7\rho) = O(1/n^5)$.*

Since we have at most 4 nested integrations to compute $Pr(E_k^e|t_1[1], \ldots, t_i[j-1], t_i[j] = \delta)$, the error incurred in computing $p(\delta)$, i.e., $\sum_{e\in E}\sum_{k=1}^{r} Pr(E_k^e|t_1[1], \ldots, t_i[j-1], t_i[j] = \delta)$, on account on the integrations is $O(r|E|/n^5) = O(\frac{1}{n^2})$. Therefore Lemma 8.3 and Theorem 3.1 follow.

Lemma 8.3 *$\min_\delta p(\delta)$ can be estimated to an additive $O(1/n^2)$.*

9 Conclusions

We believe that the techniques used here can be used to derandomize a general class of randomized algorithms based on Semidefinite Programming. Loosely speaking, this class would comprise of those whose expected value calculations involve just a constant number of vectors in each "elementary" event. This class contains all randomized Semidefinite Programming based algorithms known so far. It would be nice to obtain a general theorem to this effect.

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[2] The estimates here are very liberal. Tighter estimates can reduce the exponent substantially.

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