Bachelor thesis proposals in Applied Mathematics

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1 Hausdorff dimensions with `Google Map`

Keywords: Analysis, Approximations, Fractal dimension.

B.Mandelbrot [2] has highlighted the *coast paradox* which is the observation that the coastline of an island does not have a well-defined length. This results from the fractal-like properties of coastlines and this phenomenon was widely studied since Mandelbrot.

The aim of this project is to illustrate this fractal behaviour on large scales using pictures taken from `Google Map` (see [2] for a recent and related work).

The project could include:

- Reviewing definitions and first properties of *Hausdorff dimension*;
- How to approximate Hausdorff dimension? *divider dimension method*, *box-counting method*,...;
- Applications to your favourite island with `Google Map`.

References

[1] Ma, Jianhua, Dexin Liu, and Yanqiu Chen. Random fractal characters and length uncertainty of the continental coastline of China. *Journal of Earth System Science* 125.8 (2016): 1615-1621.
[2] B.Mandelbrot. How long is the coast of Britain? Statistical self-similarity and fractional dimension. *Science* vol.156 (1967) p.636-638.

2 Hand-made Fourier series

Keywords: Experimental Mathematics, Analysis.

In the early XIXth century J.Fourier realized that every periodic function can be approximated by sums of cos and sin functions. Fourier series are now a very powerful tool used in Theoretical physics, Mechanics, Thermodynamics, Number Theory, Probability,... Yet it looks abstract to most students, the idea of this project is to *hand-made* some Fourier decompositions.

In a marvelous `YouTube` video (What is a Fourier Series? [2], see also [3]) Fourier series are illustrated with wheels. The goal of this project is to reverse engineer this video and write a computer program which produces videos of Fourier decompositions with wheels.

The project could include:

- Reviewing definitions and first properties of Fourier series (see *e.g.*[1]);
- Programming nice videos which illustrate Fourier decompositions;
- Exploring decompositions that can be made with other wheel shapes (ovals, squares,...);
- (possibly) Designing and laser-cutting a gear mechanism (with $2-3$ gears) which allows to actually draw a simple function with a pen.
- (possibly) Discussing historical aspects: the Michelson's Harmonic Analyzer [4]

References

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[2] `Smarter Every Day`. What is a Fourier Series?
YouTube: `https://www.youtube.com/watch?v=ds0cmAV-Yek`.
[3] `Wikipedia` page of Fourier series.
[4] B.Hammack, P.Ryan, N.Ziech. *Albert Michelson's Harmonic Analyzer: A Visual Tour of a Nineteenth Century Machine That Performs Fourier Analysis*. Articulate Noise Books (2014). (Freely available on internet).

3 Modelling & optimisation on graphs for image processing and data analysis

Keywords: optimization, image processing.

Over the last years, the formulation of several classification, clustering and, in general, learning problems in the framework of weighted undirected graphs has become increasingly popular. Such framework is indeed very popular due to its intrinsic ability of comparing data not necessarily in close proximity by means of distinctive features and some similarity measure. The design of efficient optimisation strategies in this framework is very challenging, due to the large amount of data at hand and the number of features one wants to encode in the problem. Several modern approaches in the field of image analysis, machine learning and many more are in fact often formulated in a graph framework, see [3, 1] for some reviews.

The objective of this project is the understanding of the mathematical modeling of data (in particular, images) processing problems on graphs, the design of efficient algorithms solving that can take into account the presence of large data and the formulation of new models for some exemplar image processing task (e.g., image colorisation and image fusion).

Students interested in this project should have a solid background in linear algebra, analysis, computer science and good coding skills preferably in Python or MATLAB. Depending on the interest of the candidate, more theoretical questions (such as the mathematical analogy with Partial Differential Equation models) and/or specific applications such as community clustering for the study of criminal behaviours, see, e.g. [2].

References

[1] A. L. Bertozzi, E. Merkurjev, *Graph-based optimization approaches for machine learning, uncertainty quantification and networks*, Handbook of Numerical Analysis, Elsevier, 2019.

[2] Y. van Gennip , B. Hunter , R. Ahn , P. Elliott , K. Luh , M. Halvorson , S. Reid , M. Valasik , J. Wo , G. E. Tita , A. L. Bertozzi , P. J. Brantingham, *Community Detection Using Spectral Clustering on Sparse Geosocial Data*, SIAM J. Appl. Math., 73(1), 67–83, 2013.

[3] X. Zhu, *Semi-supervised learning with Graphs*, PhD thesis, School of Computer Science, Carnagie Mellon University, 2005.

4 An approximation algorithm for the MAX-CUT problem

Keywords: combinatorial optimization, probability, discrete mathematics.

We consider an unoriented graph $G = (V, E)$ where V is the set of vertices and E the set of edges. The MAXCUT problem is the problem of finding the subset of vertices $W \subseteq V$ with the maximum number of outgoing edge, i.e. with the maximum number of edges having one end in W and one end in $V \setminus W$. This problem, although very simple to state, it is computationally very hard to solve. Indeed, the MAXCUT problem is NP-complete. Quite surprisingly, there exists an *approximation algorithm*, due to Goemans and Williamson, that runs in *polynomial time* and has a very good guaranteed performance. The expected value of the solution produced by this algorithm is at least 87% of the optimal value! The goal of this thesis is to learn the basics of the theory of approximation algorithms and how this clever algorithm works and implement in Python. If time allows, the question of how to adapt this algorithm to solve other combinatorial optimization problems can be considered.

References

[1] David Shmoys and David Williamson. The design of Approximation Algorithms. *Cambridge University Press*, 2011.

5 The turnpike property in deterministic optimal control

Keywords: Analysis, mathematical physics, mathematical biology, dynamical systems .

When planning a trip between two distant cities, most drivers adopt the strategy of finding the quickest combination of local routes leading her/him to the highway, drive along the highway as much as possible and use the closest highway exit in order to reach the final destination. The problem of planning a car trip may be seen as a particular instance of a optimal control problem, i.e. an optimization problem where the goal is to find the cheapest trajectory according to some criteria (in this example, time) among a set of admissible trajectories (in this example, all routes joining the two cities). The word turnpike is a synonymous for highway and by turnpike property we mean the fact that optimal solutions of optimal control problems consist approximately of three pieces. The first and third pieces are rapid transitions from the initial state to a steady state (equilibrium), and from the steady state to the final state respectively. The second piece is a long time interval in which solutions stay exponentially close to the steady state. In this thesis we will derive rigorously the turnpike property for some classical control problems and see how this property may be used to design efficient numerical algorithms. Depending on your interest, specific applications in mathematical physics or mathematical biology can be further studied.

References

[1] Emmanuel Trélat and Enrique Zuazua. The turnpike property in finite-dimensional nonlinear optimal control. *Journal of Differential Equations*, 258(1):81–114, 2015.

6 Modelling the price of a risky asset: a comparative approach between Bachelier and Black and Scholes models.

(it is recommended to attend the course on stochastic process)

Keywords: Mathematical finance.

The modelling of financial markets is one of the most challenging problems in finance. The volatility of the market is indeed a parameter that practitioners and researchers aims at calibrating to fit the most accurately the market data. In 1900 Louis Bachelier addressed this question by introducing the Brownian motion to model the dynamic of a risky asset. Later, Black, Scholes and Merton have introduced a slightly more complex model to describe the price of a risky asset allowing to derive explicit formula for call and put options. In this Bachelor thesis, we will compare the prices induced by these two models together with the implied volatilities associated with them.

References:

- *Options, Futures and Other Derivatives,* John Hull, 2011.
- *How close are the option pricing formulas of Bachelier and Black-Merton-Scholes?* Walter Schachermayer and Josef Teichmann.

7 Pricing and hedging of (exotic) options

(it is recommended to attend the course on stochastic process)

Keywords: Mathematical finance, market options options.

Most investors aim at limiting their risks about possible high variation of prices on the market. An option is a very used financial product dealing with this question. It allows the owner to buy/sell a financial (risky) product at a determined price fixed by the contract, by paying a price to the seller. When the maturity of the contract is fixed, the buyer exercises it if the relative payoff is positive, otherwise, she does not use it. However, this kind of contract does not prevent from high trend reversal on the market. A barrier option is a financial product where the payoff depends on whether or not the underlying asset has reached or exceeded a predetermined price named the 'limit price'. In this Bachelor thesis, we will study the pricing and the hedging of such options and we will illustrated it with numerical simulations.

References:

- *A continuity correction for discrete barrier options.* Glasserman and Kou. Math. Fin. 1997
- *On pricing of discrete barrier options,* S. Kou, 2003.
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8 Exploitation of renewable natural resources

Keywords: Contract theory, optimal control.

The exploitation of natural resources is fundamental for the survival and development of the growing human population. However, natural resources are limited since they are either non renewable (e.g. minerals, oil, gas and coal) so that the available quantity is limited, or renewable (e.g. food, water and forests) and in this case the natural resource is limited by its ability to renew itself. In particular, an excessive exploitation of such resources might lead to their extinctions and therefore affect the depending economies with, for instance, high increases of prices and higher uncertainty on the future. Thus, the natural resource manager faces a dilemma: either harvesting intensively the resource to increase her incomes, or taking into account the potential externalities induced by an overexploitation of the resource and impacting her future ability to harvest the resource. In this Bachelor thesis, we will study the different mechanisms related to this problem and we will illustrate it with numerical simulations.

References:

- Edward B. Barbier. *The role of economic incentives for natural resource economic progress and environmental concerns.* pages 153–178. A Publications of the Egon-SohmenFoundation. Springer, Berlin, Heidelberg, 1993.
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- R. May, JR Beddington, JW Horwood and JG Sheered, *Exploiting natural Populations in an uncertain world.* Mathematical Biosciences 1978

- Rodin and Adelani. *Optimal Harvesting of a Renewable Economic Resource in a Model with Bertalanffy Growth Law II.* Apply. Math Lett. 1989

9 Granular media modeling

Keywords: granular media, simulations, numerical analysis.

Granular materials cover a broad area of research at the intersection of different scientific fields including physics, mechanics, mathematical modeling, numerical simulations...

The discrete granular structure of these materials leads to a complex behavior which plays a key role in a large area of applications such that civil engineering, resistance of soils, coast erosion, biophysics... Despite this large area of applications, these complex systems are still not yet well understood and provide a huge amount of research.

A way to better understand granular material is to use numerical simulations. One of the main difficulty in such simulations is to model the contact between grains.

A simple mathematical way to describe the contact is to consider an explicit expression for the contact force and to make the system evolve through Newton's law. This leads to very stiff systems of ODEs that has to be solved numerically using efficient algorithm (see for example [1, 2]). This method is called the Discrete Element Method (DEM).

Then, depending on the interest/background of the student, the following items could be studied:

- understand the DEM model for non frictional spherical particles in 2 dimensions, to study the corresponding numerical scheme and to implement it for a few particles.

- Consider a high number of particles and try to implement an efficient code. Here, the key step is to implement an efficient way to find the neighbours.

- Develop the model adding for example capillarity forces or lubrication forces to model wet granular media or adding an aggregation force.

- Consider non spherical grains. Here one need to find the "contact point" between the two particles and to deal with rotation. This can be achieved using a newton algorithm for example [3].

This ODE model leads to stiff equations which impose to use very small time steps and lead to heavy simulations. A way to solve this problem is to change of point of view and consider the contact force as an unknown: the contact is written as the constraint "the particles should not overlap". This method is called the "contact dynamics" method (CD) and leads to constrained optimization problems [3, section 3]. An interesting direction could also be to extract from [3] the informations allowing to understand the numerical procedure proposed and to implement the corresponding contact dynamics scheme. This algorithm allows to consider large assemblies of particles and to run long time simulations of granular flows. New constraints of "aggregation" can easily be added to model assemblies of spherical particles (pairs of particles, red blood cells...) or to study aggregation of particles. Here again, the detection of neighbours has to be achieved efficiently and several external forces can be taken into account to enrich the model.

References:

[1] LUDING, Stefan. Introduction to discrete element methods: basic of contact force models and how to perform the micro-macro transition to continuum theory. European Journal of Environmental and Civil Engineering, 2008, vol. 12, no 7-8, p. 785-826.

[2] RADJAÏ, Farhang et DUBOIS, Frédéric. Discrete-element modeling of granular materials. Wiley-Iste, 2011.

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[4] MAURY, Bertrand. A time-stepping scheme for inelastic collisions. Numerische Mathematik, 2006, vol. 102, no 4, p. 649-679.

10 Crowd motion modeling

Keywords: Crowd motions, simulations, numerical analysis.

Understanding the behavior of crowds is of high importance for example to design towns, airports, public places. In particular, being able to understand emergency situations is essential.

Several models of crowds has been developed in the last decades. A microscopic model called the social force model has been proposed in 1995. In this model, each person is considered as a particle and its behavior is modeled using a system of differential equations [1]. The acceleration of the individual is obtained according to Newton's law and several forces are introduced to take into account a desired velocity or a repulsion force to avoid other peoples or obstacles.

In the previous model, and ODE of order 2 is used to model the pedestians. Passing through the limit, this model can provide a new model made of an ODE of arder one [2]. This new model can also be numerically tested.

These two ODE models has the main disadvantage to involve stiff ODEs so that small time steps has to be used to ensure stability of the scheme.

Another model, also introduced in [2] can be derived from the 1rst order ODE model. This new model is based on the following constraint: "two persons can not overlap"! From a numerical point of view, it leads to constrained minimization problems to be solved at each time step [3].

Depending on the interest/background of the student, the two schemes can be studied and implemented during the project or the student can focus on one of them.

One can first try to implement this model and. Since it leads to stiff systems, one may need to implement implicit schemes.

Several tests can be done once the code is implemented and tested to try to recover well known situations in crowd motion [4, 5]:

- integrate various strategies in the model (such as different mean velocities depending on the persons, avoiding the more crowded zones...)
- recover the lanes appearing in case of two populations walking in opposite direction
- study the time needed to empty a room, depending on the form/place of various obstacles in this room.
- study the static jams appearing at the exits of a room
- ...

References

[1] HELBING, Dirk et MOLNAR, Peter. Social force model for pedestrian dynamics. Physical review E, 1995, vol. 51, no 5, p. 4282.

[2] FAURE, Sylvain et MAURY, Bertrand. Crowd motion from the granular standpoint. Mathematical Models and Methods in Applied Sciences, 2015, vol. 25, no 03, p. 463-493.

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