sineFit

sineFit is a function to detect the parameters of a noisy sine function.

1 Syntax:

SineParams=sineFit(x,y)

Input:

x and y values: y = offs + amp * sin(2pi * f * x + phi) + noise

Output:

SineParams(1):offset (offs)
SineParams(2): amplitude (amp)
SineParams(3): frequency (f)
SineParams(4): phaseshift (phi)

2 Minimum requirements:

- Length of input: tested for more than 0.1 periods
- Sampling rate: more than 2.0 samples per period, see also chapter 9
- Total number of samples: 4 or more

Required toolbox: statistics toolbox

3 Extra in the program:

- At the end of the first function in *sineFit.m* (approx. at line 134), you will find the statement "*PlotResults (x,y,…*". You may uncomment or delete it and its associated function.

4 Method:

This is a brief and not exact description of the program flow.

- Estimate the offset by the mean of all y values.
- Build the FFT with heavy zero padding.
- Take the frequency, amplitude and phase of the largest FFT peak.
  If the frequency is at the Nyquist limit, add an extra frequency for evaluation.
- Take those values as initial values for the *nlinfit* regressions.
- Take the resulting MSE as rating.
- Exclude results above Nyquist frequency.
- Depending on the number of samples and the MSE, set a limit for an accepted amplitude in relation to the FFT amplitude.
- If the amplitude from nlinfit is higher than the accepted amplitude, take the FFT parameters.

5 Processing time

The mean calculation time is on my PC 5 ms with a maximum of 260 ms.

The regression, nlinfit.m, uses about 80% of the total time, while the FFT requires only about 2% of the time.

6 Evaluations:

Sine curves tested:

100000 random sine curves were created with following properties:

- Offset: 0 to 10
- Amplitude: 0.1 to 10
- Frequency: 0.001 to 100 Hz
- Phase shift: 0 to 2π
- Number of samples per period: 2.01[*)] to 30
- Minimum number of all samples: 4
- Signal to noise ratio: 0 to 100 dB (not applied on offset).
  Low SNR are overrepresented, for 1 to 100 periods, see Fig. 2.
- x=0 within observed range +- ½ period
- Out of the 100 000 test patterns
  40000 have 0.1 to 1 periods
  30000 have 1 to 10 periods
  30000 have 10 to 100 periods

*) In order to detect the amplitude and phase, you need more than 2 samples per period, see chapter 9!

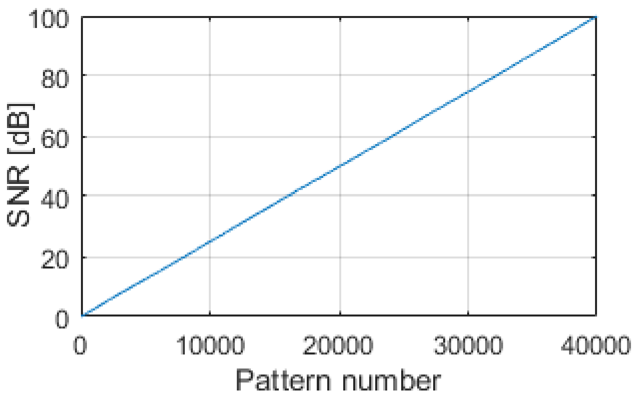

Fig. 1 For 0.1 to 1 period linear SNR distribution

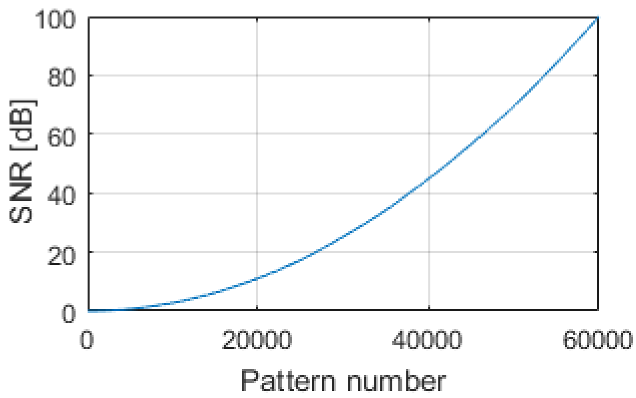

Fig. 2 For 1 to 100 periods Overrepresentation of low SNR (1/3 of all test patterns have a SNR below 11 dB)

6.1 Definition of 10% error:

- The amplitude or the frequency is faulty, if the deviation is more than 10% compared to the noise free sine.
- The offset can be zero and is considered to be wrong, if it deviates by more than 10% of the expected offset <u>and</u> more than 10% of the expected amplitude.
- The phase is wrong if it deviates by more than 10% of 2π.

6.2 Definition of 1% error:

Like above, but replace 10% with 1%.

7 Results:

Table 1 shows the results of sineFit.

The four sections of the table are:

- 0.1 to 0.5 periods
- 0.5 to 1 period
- 1 to 10 periods
- 10 to 100periods

“0%” in the table 1 means no failure at all.

With a fraction of a period, first line of Table 1, 25% results of 40000 test pattern have a deviation of more than 10% compared to the noise free sine and 45% exceed the 1% limit.

Out of those test patterns, very low numbers of periods have the most errors. In line 2 we see that 39% of the evaluations have more than 10% deviation. We get only good results with nearly noise free sine curves (SNR>99.8 dB), see line 4.

If we have a little more periods (0.5 to 1) the failures with 10% error go down to 14%, see 5th line.

Finally, with many periods, line 12, we need only 28 dB SNR to detect all sine curve parameters with an error better than 1%.

The column “Failures best fit” is the percentage of wrong detected results, if the parameters of the noise free sine are used as initial values for the fitting process. I.e. if we have then a failure, another sine than the original sine fits the noisy sine better. Since all values of this column are equal to the values of the column “Failures with error > 1%”, the sineFit process cannot be improved further using this fitting method. The same holds for the 10% errors.

	Number of periods	Samples per periods	Total number of samples	Signal to noise [dB]	Failures with error > 10%	Failures with error > 1%	Failures best fit (>1%)	Number of tests
1	0.1 to 1		4 to 30	0 to 100	25%	44%	44%	40000
2	0.1 to **0.5**		4 to 30	0 to 100	38%	58%	58%	17808
3	0.1 to 0.5		4 to 30	**94** to 100	**0%**	1.7%	1.7%	1144
4	0.1 to 0.5		4 to 30	**99.8** to 100	0%	**0%**	0%	54
5	0.5 to 1		4 to 30	0 to 100	14%	33%	33%	22192
6	0.5 to 1		**6** to 30	**38** to 100	**0%**	2.3%	2.3%	12940
7	0.5 to 1		**4** to 30	**63** to 100	0%	**0%**	0%	6471
8	1 to 10	2.01 to 30		0 to 100	11%	44%	44%	30000
9	1 to 10	**2.5** to 30		**20** to 100	**0%**	5%	5%	16313
10	1 to 10	2.5 to 30		**41** to 100	0%	**0%**	0%	10606
11	10 to 100	2.01 to 30		0 to 100	0.9%	29%	29%	30000
12	10 to 100	**2.1** to 30		**7** to 100	**0%**	9%	9%	22193
13	10 to 100	2.1 to 30		**28** to 100	0%	**0%**	0%	14226

Table 1: Failures with 100000 noisy sine curves having 0.1 to 100 periods.

The number of failures depends mainly on the SNR, see Fig. 3 and also Fig. 11.

Fig. 3 gives an idea about the required SNR for reliable results. E.g. for around 10 periods you need an SNR of at least 26 dB and all results with more than 4 samples and more than 2.1 samples per period will very probably have an error less than 1%.

With another test pattern set, you will get similar curves.

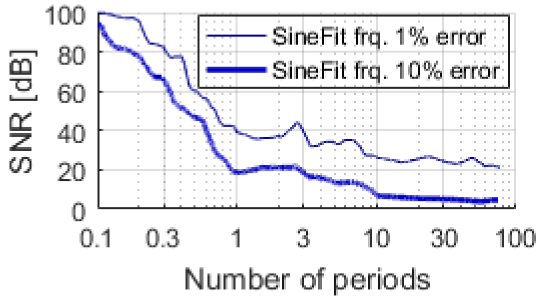

Fig. 3 Required SNR for zero failures

Number of periods: 0.1 to 100
Number of samples: >4
Samples per period > 2.1

The failure rate depends a little on the number of samples, see Fig. 4. With a number of periods from 0.1 to 1 and with 4 samples, 36% of all test pattern results deviate by more than 10% compared to the original sines. With 30 samples, the failures go down to about 22%.
With more than one period, the samples per periods are more meaningful. At 1 to 10 periods, Fig. 5, the 10% and 1%-error-failures decrease with the sampling rate by about 20%.
The number of periods has of course a big influence on the results, see Table 1.

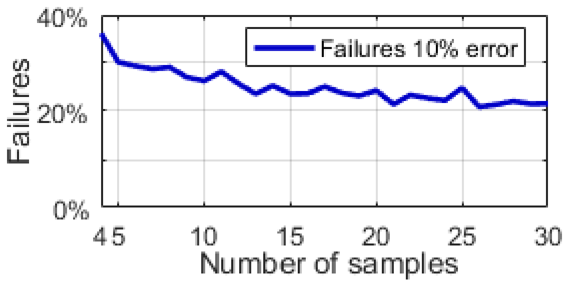

Fig. 4 10%-error-failures for 0.1 to 1 period depending on number of samples

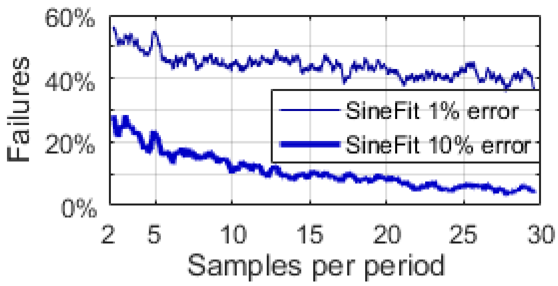

Fig. 5 Failures for 1 to 10 periods depending on samples per period

7.1 Failure details

The top line in Fig. 6 are the total failures of sineFit for 10% error for 1 to 10 periods.
The main cause for the failures are by far the faulty detected amplitudes, followed by the offset.
The frequency and phase contribute only at low number of periods to the failures.

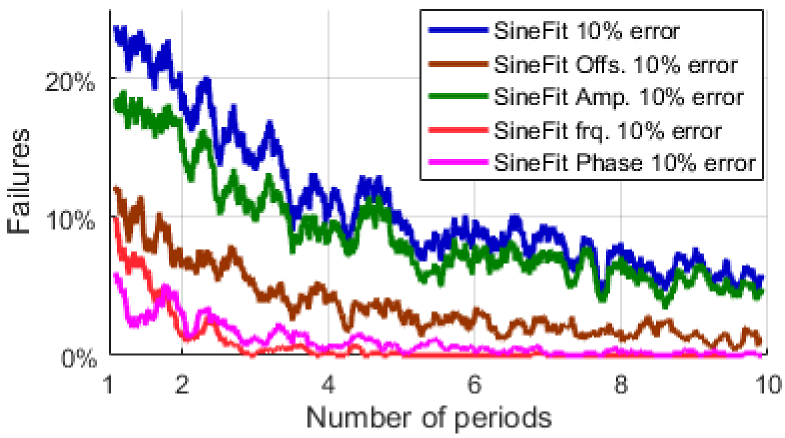

Fig. 6 Failures due to offset, amplitude, frequency and phase

8 sineFit compared to FFT

In most cases (more than 1 period), the position of the peak of the FFT is close to the expected frequency. However, due to the added noise, other FFT peaks or frequencies may represent the noise free input.
In the following example, we demonstrate such a case.
The involved functions are:

$$
\begin{aligned}
&\text{a) Expected sine: } y = 0.90 + 1.8 * \sin(2 * pi * 81 + 2.4) \\
&\text{b) SineFit Result: } y = 0.77 + 1.7 * \sin(2 * pi * 85 + 1.6) \\
&\text{c) From FFT peak: } y = 0.82 + 2.1 * \sin(2 * pi * 222 + 3.3)
\end{aligned}
\tag{1}
$$

The green line Fig. 7 is the expected sine, see also equation (1) a.
The red line is this function with noise added, sampled with 5.5 samples per period and limited to 1.8 periods.
This pattern (red line in Fig. 7) is the input for the Fourier transformation, see Fig. 8.
The maximum of the FFT is at the nyquist limit (red "+" at 222 Hz). However, the expected result (green "x" at 80.9 Hz) is close to another peak.
The result of sineFit (blue line) is not perfect, but close to the expected function (green line).
In many similar cases, also sineFit fails often, since another sine curve better fits the noisy sine curve.

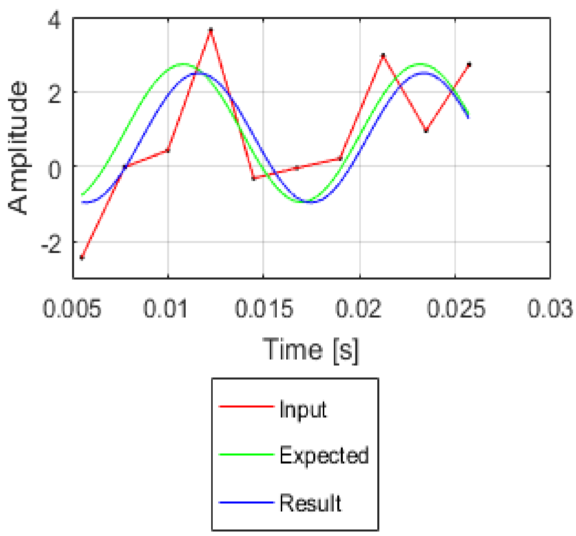

Fig. 7 Pattern to evaluate (in red)

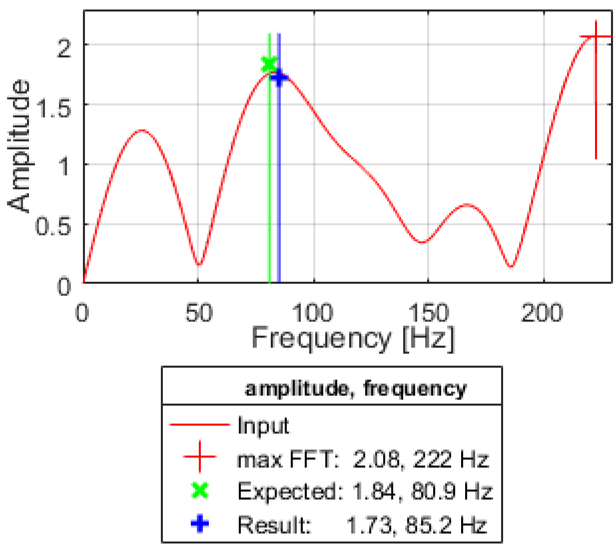

Fig. 8 Corresponding FFT

Many failures of the FFT in respect to the expected frequency happens with samples rates below 3 per period. The following example (*Fig. 9*) has 2.6 samples per period and 2.3 periods. The FFT (*Fig. 10*) has its maximum at the nyquist frequency (106 Hz) while the expected frequency is 83 Hz. The result, blue line in

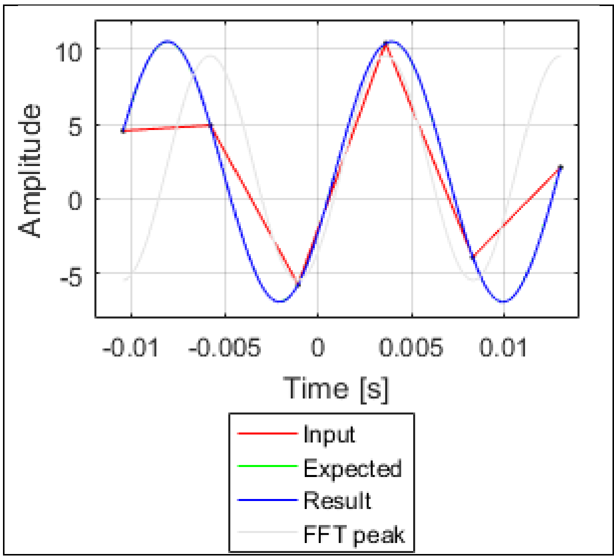

Fig. 9 Sine with low sample rate

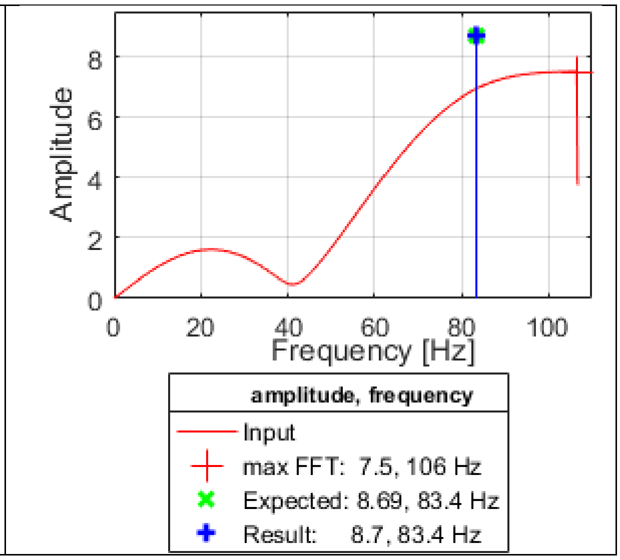

Fig. 10 Corresponding FFT

Table 2 compares sineFit with FFT.
The first line of the table is the global result for all 100000 test patterns. The total number of failures with 10% (1%) errors on all 100000 test patterns is with sineFit: 13% (40%). Taking only the FFT peak parameters, the failures are much higher: 41% (74%).
With fractions of a period, 2nd line, the FTT results are very bad.
With 1 to 10 periods, only the 1% error failures are significantly worse with the FFT.
From 10 to 100 the results of sineFit and FFT are similar.
Of course, you may improve the FFT result, if you know you have more than 4 or 5 periods. E.g., use a Hann window.
Remark: With the test set of this paper, we at up to 4.6 periods the FFT frequency is often more than 3 times higher than the expected frequency, also with more than 5 samples per period..

	sineFit results			**FFT results**			
Number of periods	**Failures with error > 10%**	**Failures with error > 1%**	**Frequency failures with error > 10% (>1%)**	**Failures with error > 10%**	**Failures with error > 1%**	**Frequency failures with error > 10% (>1%)**	**Number of tests**
0.1 to 100	13%	40%	9%	41%	74%	34%	100000
0.1 to 1	25%	44%	21%	92%	100%	84%	40000
1 to 10	12%	44%	0.9% (12%)	14%	82%	1.7% (24%)	30000
10 to 100	0.9%	29%	0% (100ppm)	1%	31%	0% (630ppm)	30000

Table 2: sineFit compared to FFT, SNR 0 to 100 dB

Fig. 11 shows the dependency of the failures on the noise ratio for 1 to 10 periods. While we have no 10% errors with sineFit from 20 dB on, the FFT failures remain at around 3%. For the 1% errors sineFit has no failures from 41 dB on and the FFT failures remain at around 70%.

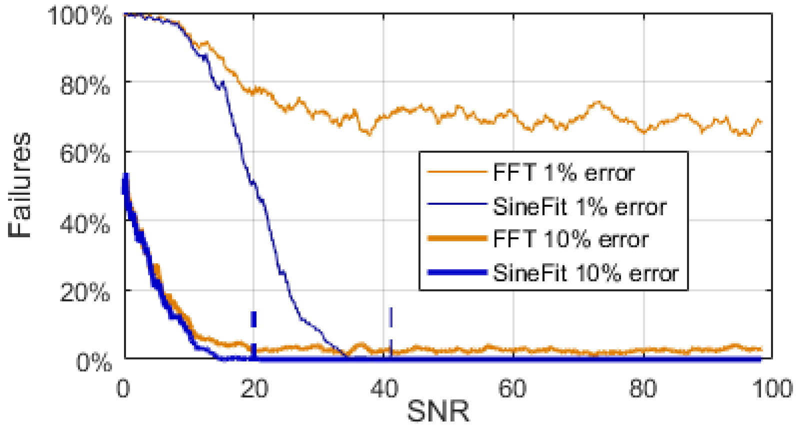

Fig. 11 Failures depending on SNR, 1 to 10 periods more than 2.5 samples per period

If you look for the reasons for the FFT failures, Fig. 12, you find that the amplitude failures are dominant. The frequency failures, starting from 30%, become very small from 1.5 periods on. Then the offset, calculated outside of the FFT, becomes the main failure. The phase failures are always small. The curves are similar to the failures of sineFit, Fig. 6

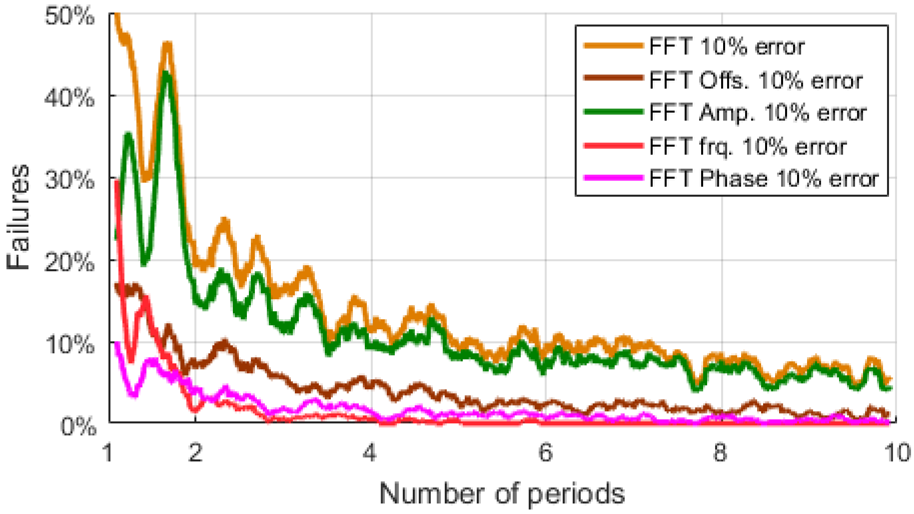

Fig. 12 FFT, Failures due to offset, amplitude, frequency and phase

Again, with other Fourier transformations, esp. window technics, you may get better results. In addition, the FFT by itself is about 6 times faster than sineFit. If speed is crucial and 10% errors are good enough, you may consider using only the FFT. In this case, you need to know, that you have more than about five periods.

9 About samples per period

With exactly 2 samples per period the amplitude or the phase cannot be determined. With a phase=0 and 2 samples per period, only the DC-level and frequency is known. The amplitude seems to be zero, see Fig. 13, red dots.

9.1 Time limited sine function

Since the number of samples is an integer number, the sample rate, the number of periods are dependent on each other.

Ns: Number of samples, <u>integer number!</u>
SpP: Number of samples per period, more than 2.0
Np: Number of periods.
Following equations hold:

$$Ns = Np * SpP \quad \text{with SpP} > 2 \text{ and Ns} = 4, 5, 6, \ldots \tag{2}$$

The constraints are:
More than 2 samples per period with a minimum of 4 samples in total.

This means SpP is restricted by the duration of the sine and the integer value of Ns.
For example, with a sine limited to 2.5 periods, following sampling SpP are allowed:

$$SpP = \frac{Ns}{Np} = \frac{6}{2.5}, \frac{7}{2.5}, \cdots \tag{3}$$

Fig. 13 shows the first two sample rates acc. Equation (3) and in addition the not allowed 2 samples per period. The first sample of the next period would be for all sample rates at y,x=0, 2.5, see black circle in Fig. 13.

If you want to determine the amplitude, you must have more than two samples per period. In the case of Equation (3) this is at least 6/2.5=2.4 samples per period. All patterns used to test sineFit meet this constraint.

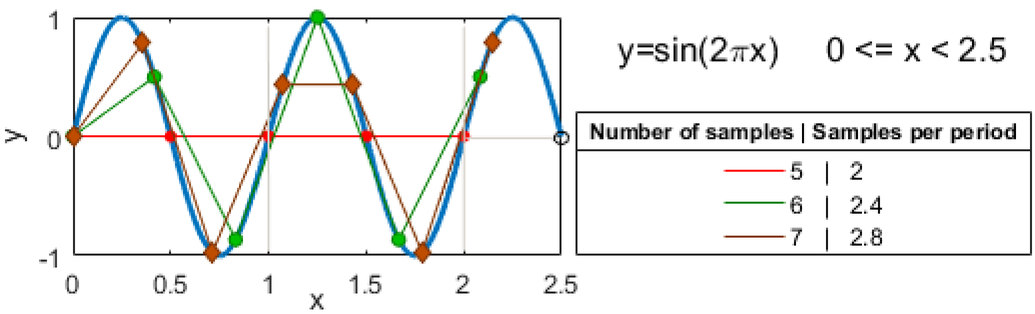

Fig. 13 Sine curve with 2.5 periods, sampled with different sample rates

In Fig. 14 we see what happens if we have only 3 samples in a sine (green and red line). E.g. for any amplitude larger than the initial amplitude we can find a sine curve that fit the three samples. Both sine curves, green and blue line in Fig. 14, go through the three points.

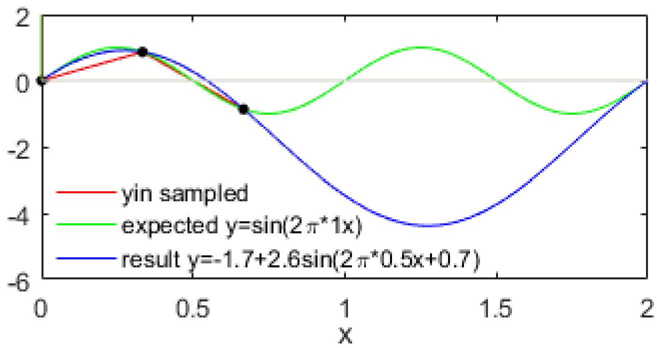

Fig. 14 Two sine curves with the same 3 samples

To determine the four unknown parameters of a sine you need at least four samples

10 About offset

The offset of an ordinary unlimited sine is zero and equal to its mean value. This holds also for full periods. With other time-limited sine curves, the mean value does not represent the offset.

The first estimate of the mean value (*meanSamples*) is the average of all y-samples:

$$meanSamples = \frac{1}{N_s}\sum_{1}^{N_s} y_i \qquad (4)$$

y_i: y-values of samples
N_s: Number of samples

The mean value of a time limited sine curve without a DC-value (*meanSine*) is:

$$meanSine = \frac{A}{(2\pi \mathrm{f}(\mathrm{x}_{end}\text{-}\mathrm{x}_1))}\int_{x_1}^{x_{end}} \sin(2\pi \mathrm{f}\mathrm{x}+\varphi)dx \qquad (5)$$

A: Amplitude of sine
x_1: Start of sine (time of first sample)
x_{end}: End of sine (time of last sample)
f: Frequency
φ: Phase shift

We get the values *A, f* and φ from the FFT.
With *meanSine* and *meanSamples* we calculate a better estimation for the offset (*offsetEstimate*):

$$offsetEstimate = meanSamples - meanSine \qquad (6)$$

Example, Fig. 15:

$$y_{sine} = 1 + 2\sin(2\pi 1x + 3.1) \qquad (7)$$

Offset of sine =1
meanSamples = 0.62 (eq. (4))

With parameters from y_{sine}:
- meanSine = -0.35 (eq. (5))
- offsetEstimate = 0.97 (eq. (6))

With parameters from FFT:
- meanSine = -0.30
- offsetEstimate = 0.93

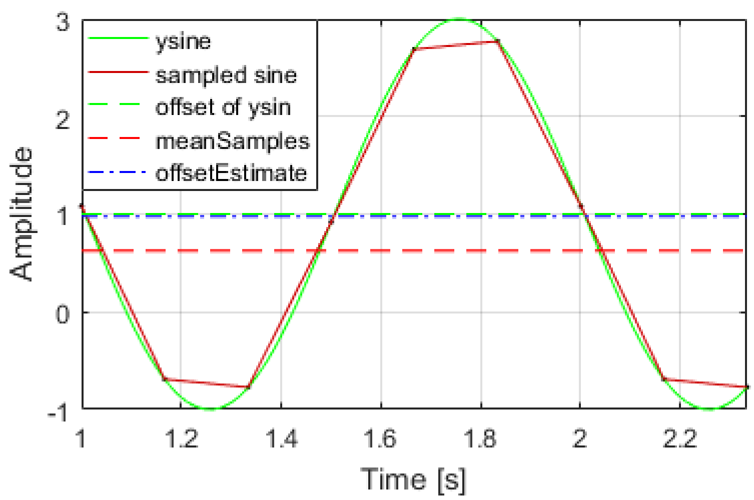

Fig. 15 Offset values for time limited sine

If you set the number of samples to 100, you get:
meanSamples = 0.57

With parameters from y_{sine}:
- meanSine = -0.43
- offsetEstimate = 1.0

With parameters from FFT:
- meanSine = -0.39
- offsetEstimate = 0.97

This offset estimation improves very little the sineFit results. However, it improves a lot the results taken from the FFT and the separate calculated offset.

For Fig. 16 we take noise free sine curves . All have an offset of zero and an amplitude of one.
The blue curve represents the mean values of the samples.
The green curve is built up according eq.(6) with the parameters of the clean sine.
The red curve is more realistic in our case, since we gather amplitude, frequency and phase from the FFT .
At up to 0.4 periods the values we gain from the FFT are very bad and the improvement is marginal. Mainly the amplitude is detected much too low. However, from 0.4 periods on the red line is closer to the expected zero line.

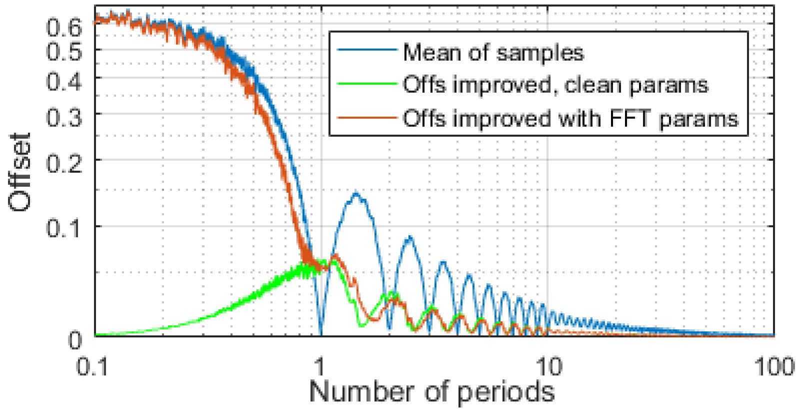

Fig. 16 Offset of clean sine curves

The most realistic experiment is with the test set of 100000 noisy sine curves. Fig. 17 demonstrates the improvement (green line). Below 0.5 periods the improvement is low, since the detected amplitudes, frequencies and phases from the FFT are far off from the expected values. In a small region around 0.9 periods, the pure mean values are better. Most improvement is between 0.6 and 3.5 periods.

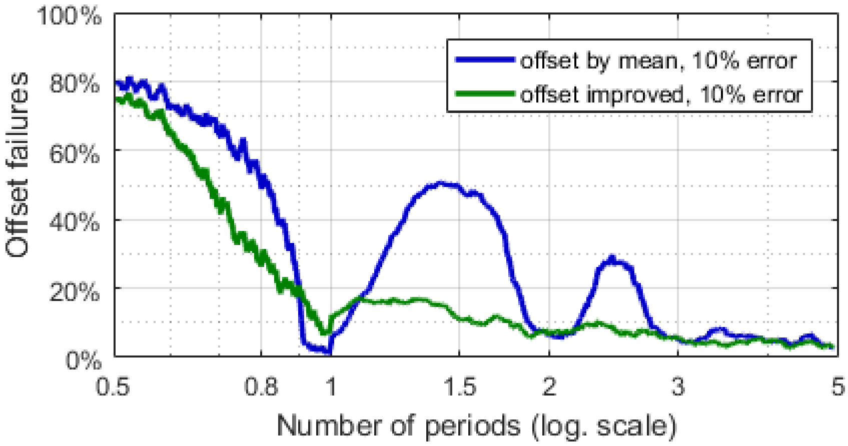

Fig. 17 Offset failures for max. 10% error, 0.5 to 5 periods, SNR 0 to 100

An attempt to exclude the offset correction at around 0.9 periods fails. The variation of the determined number of periods is too large.
The number of periods (Np) is calculated by this formula:

$$Np = (\underbrace{x_{end} - x_1}_{\text{total duration}} + \underbrace{x_2 - x_1}_{\text{time within 2 samples}})f \tag{8}$$

x_1: Start of sine (time of first sample)
x_2: time of second sample
x_{end}: End of sine (time of last sample)
f: Frequency of FFT peak

In the observed range of 0.9 to 1.1 periods, the frequency of the noisy sine curves detected by the FFT is up to +- 25% off from the expected frequency (all values nearly linear distributed, some extreme values already excluded,). Therefore, the number of periods, eq.(8), becomes too uncertain.