# 1 axiom1

Relation R has properties:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{x :: \Box(A \supset B) \vdash x :: \Box(A \supset B)}\,(Ass) \quad \dfrac{}{xRy \vdash xRy}\,(Ass)}{x :: \Box(A \supset B); xRy \vdash y :: A \supset B}\,(\Box E) \quad \dfrac{\dfrac{}{x :: \Box A \vdash x :: \Box A}\,(Ass) \quad \dfrac{}{xRy \vdash xRy}\,(Ass)}{x :: \Box A; xRy \vdash y :: A}\,(\Box E)}{x :: \Box(A \supset B); xRy; x :: \Box A \vdash y :: B}\,(\supset E)}{x :: \Box(A \supset B); x :: \Box A \vdash x :: \Box B}\,(\Box I)}{x :: \Box(A \supset B) \vdash x :: \Box A \supset \Box B}\,(\supset I)}{\vdash x :: \Box(A \supset B) \supset \Box A \supset \Box B}\,(\supset I)$$

# 2 axiom2

Relation R has properties:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{x :: \Box(A \supset B) \vdash x :: \Box(A \supset B)}\,(Ass) \quad \dfrac{}{xRy \vdash xRy}\,(Ass)}{x :: \Box(A \supset B); xRy \vdash y :: A \supset B}\,(\Box E) \quad \dfrac{}{y :: A \vdash y :: A}\,(Ass)}{x :: \Box(A \supset B); xRy; y :: A \vdash y :: B}\,(\supset E) \quad \dfrac{}{xRy \vdash xRy}\,(Ass)}{x :: \Box(A \supset B); xRy; y :: A \vdash x :: \Diamond B}\,(\Diamond I) \quad \dfrac{}{x :: \Diamond A \vdash x :: \Diamond A}\,(Ass)}{x :: \Diamond A; x :: \Box(A \supset B) \vdash x :: \Diamond B}\,(\Diamond E)}{x :: \Box(A \supset B) \vdash x :: \Diamond A \supset \Diamond B}\,(\supset I)}{\vdash x :: \Box(A \supset B) \supset \Diamond A \supset \Diamond B}\,(\supset I)$$

# 3 axiom3

Relation R has properties:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{x :: \Diamond\bot \vdash x :: \Diamond\bot}\,(Ass) \quad \dfrac{\dfrac{\dfrac{}{y :: \bot \vdash y :: \bot}\,(Ass)}{y :: \bot \vdash y :: \bot}\,(\bot E)}{xRy; y :: \bot \vdash x :: \bot}\,(W)}{x :: \Diamond\bot \vdash x :: \bot}\,(\Diamond E)}{\vdash x :: \Diamond\bot \supset \bot}\,(\supset I)$$

# 4 axiom4

Relation R has properties:

$$\dfrac{\dfrac{}{x :: \diamond(A \vee B); \vdash x :: \diamond(A \vee B)}\,(Ass) \qquad \dfrac{\dfrac{}{y :: A \vee B; \vdash y :: A \vee B}\,(Ass) \qquad \dfrac{\dfrac{\dfrac{}{y :: A; \vdash y :: A}\,(Ass) \quad \dfrac{}{xRy; \vdash xRy}\,(Ass)}{y :: A; xRy; \vdash x :: \diamond A}\,(\diamond I)}{y :: A; xRy; \vdash x :: \diamond A \vee \diamond B}\,(\vee I1) \qquad \dfrac{\dfrac{\dfrac{}{y :: B; \vdash y :: B}\,(Ass) \quad \dfrac{}{xRy; \vdash xRy}\,(Ass)}{y :: B; xRy; \vdash x :: \diamond B}\,(\diamond I)}{y :: B; xRy; \vdash x :: \diamond A \vee \diamond B}\,(\vee I2)}{y :: A \vee B; xRy; \vdash x :: \diamond A \vee \diamond B}\,(\vee E)}{x :: \diamond(A \vee B); \vdash x :: \diamond A \vee \diamond B}\,(\diamond E)}{\vdash x :: \diamond(A \vee B) \supset \diamond A \vee \diamond B}\,(\supset I)$$

# 5 axiom5

Relation R has properties:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{x :: \diamond A \supset \Box B; \vdash x :: \diamond A \supset \Box B}\,(Ass) \qquad \dfrac{\dfrac{}{y :: A; \vdash y :: A}\,(Ass) \quad \dfrac{}{xRy; \vdash xRy}\,(Ass)}{y :: A; xRy; \vdash x :: \diamond A}\,(\diamond I)}{x :: \diamond A \supset \Box B; y :: A; xRy; \vdash x :: \Box B}\,(\supset E) \qquad \dfrac{}{xRy; \vdash xRy}\,(Ass)}{x :: \diamond A \supset \Box B; y :: A; xRy; \vdash y :: B}\,(\Box E)}{x :: \diamond A \supset \Box B; xRy; \vdash y :: A \supset B}\,(\supset I)}{x :: \diamond A \supset \Box B; \vdash x :: \Box(A \supset B)}\,(\Box I)}{\vdash x :: (\diamond A \supset \Box B) \supset \Box(A \supset B)}\,(\supset I)$$

# 6 axiom6

Relation RD has properties: Seriality,

$$\dfrac{\dfrac{\dfrac{\dfrac{}{x :: \Box A; \vdash x :: \Box A}\,(Ass) \quad \dfrac{}{xRDy; \vdash xRDy}\,(Ass)}{x :: \Box A; xRDy; \vdash y :: A}\,(\Box E) \qquad \dfrac{}{xRDy; \vdash xRDy}\,(Ass)}{x :: \Box A; xRDy; \vdash x :: \diamond A}\,(\diamond I)}{x :: \Box A; \vdash x :: \diamond A}\,(R_D)}{\vdash x :: \Box A \supset \diamond A}\,(\supset I)$$

# 7 axiom7

Relation RT has properties: Reflexivity,

$$\dfrac{\dfrac{\dfrac{}{x :: \Box A; \vdash x :: \Box A}\,(Ass) \quad \dfrac{}{xRTx; \vdash xRTx}\,(Ass)}{x :: \Box A; xRTx; \vdash x :: A}\,(\Box E)}{x :: \Box A; \vdash x :: A}\,(R_T)}{\vdash x :: \Box A \supset A}\,(\supset I)$$

## 8 axiom8

Relation RB has properties: Symmetry,

$$\dfrac{\dfrac{\dfrac{}{xRBy;\vdash xRBy}\,(Ass) \quad \dfrac{\dfrac{}{x :: A;\vdash x :: A}\,(Ass) \quad \dfrac{}{yRBx;\vdash yRBx}\,(Ass)}{x :: A; yRBx;\vdash y :: \diamond A}\,(\diamond I)}{xRBy; x :: A;\vdash y :: \diamond A}\,(R_B)}{x :: A;\vdash x :: \Box \diamond A}\,(\Box I)}{\vdash x :: A \supset \Box \diamond A}\,(\supset I)$$

## 9 axiom9

Relation R4 has properties: Transitivity,

$$\dfrac{\dfrac{\dfrac{\dfrac{}{xR4y;\vdash xR4y}\,(Ass) \quad \dfrac{}{yR4z;\vdash yR4z}\,(Ass) \quad \dfrac{\dfrac{}{x :: \Box A;\vdash x :: \Box A}\,(Ass) \quad \dfrac{}{xR4z;\vdash xR4z}\,(Ass)}{x :: \Box A; xR4z;\vdash z :: A}\,(\Box E)}{xR4y; yR4z; x :: \Box A;\vdash z :: A}\,(R_4)}{xR4y; x :: \Box A;\vdash y :: \Box A}\,(\Box I)}{x :: \Box A;\vdash x :: \Box\Box A}\,(\Box I)}{\vdash x :: \Box A \supset \Box\Box A}\,(\supset I)$$

## 10 axiom10

Relation R5 has properties: Euclideanness,

$$\dfrac{\dfrac{\dfrac{\dfrac{}{x :: \diamond A;\vdash x :: \diamond A}\,(Ass) \quad \dfrac{\dfrac{}{xR5y;\vdash xR5y}\,(Ass) \quad \dfrac{}{xR5z;\vdash xR5z}\,(Ass) \quad \dfrac{\dfrac{}{z :: A;\vdash z :: A}\,(Ass) \quad \dfrac{}{yR5z;\vdash yR5z}\,(Ass)}{z :: A; yR5z;\vdash y :: \diamond A}\,(\diamond I)}{xR5y; xR5z; z :: A;\vdash y :: \diamond A}\,(R_5)}{x :: \diamond A; xR5y;\vdash y :: \diamond A}\,(\diamond E)}{x :: \diamond A;\vdash x :: \Box \diamond A}\,(\Box I)}{\vdash x :: \diamond A \supset \Box \diamond A}\,(\supset I)$$

## 11 axiom11

Relation R2 has properties: Directedness,

$$\dfrac{\dfrac{}{x :: \diamond\Box A; \vdash x :: \diamond\Box A}\,(Ass) \qquad \dfrac{\dfrac{}{xR2y; \vdash xR2y}\,(Ass) \qquad \dfrac{}{xR2z; \vdash xR2z}\,(Ass) \qquad \dfrac{\dfrac{\dfrac{}{z :: \Box A; \vdash z :: \Box A}\,(Ass) \qquad \dfrac{}{zR2w; \vdash zR2w}\,(Ass)}{z :: \Box A; zR2w; \vdash w :: A}\,(\Box E) \qquad \dfrac{}{yR2w; \vdash yR2w}\,(Ass)}{z :: \Box A; zR2w; yR2w; \vdash y :: \diamond A}\,(\diamond I)}{xR2y; xR2z; z :: \Box A; \vdash y :: \diamond A}\,(R_2)}{x :: \diamond\Box A; xR2y; \vdash y :: \diamond A}\,(\diamond E)}{x :: \diamond\Box A; \vdash x :: \Box \diamond A}\,(\Box I)}{\vdash x :: \diamond\Box A \supset \Box \diamond A}\,(\supset I)$$