

1 axiom5

Relation R has properties:

$$\begin{array}{c}
\frac{\frac{\frac{}{x : \Diamond A \supset \Box B; \vdash x : \Diamond A \supset \Box B} (Hyp)}{x : \Diamond A \supset \Box B; y : A; xRy; \vdash x : \Box B} (Hyp)}{\frac{\frac{\frac{\frac{}{y : A; \vdash y : A} (Hyp)}{y : A; xRy; \vdash x : \Diamond A} (\Diamond I)}{x : \Diamond A \supset \Box B; y : A; xRy; \vdash x : \Box B} (\supset E)}{\frac{\frac{\frac{\frac{}{xRy; \vdash xRy} (Hyp)}{xRy; \vdash xRy} (\Diamond E)}{x : \Diamond A \supset \Box B; y : A; xRy; \vdash y : B} (\supset I)}{\frac{\frac{\frac{}{x : \Diamond A \supset \Box B; xRy; \vdash y : A \supset B} (\Box I)}{x : \Diamond A \supset \Box B; \vdash x : \Box(A \supset B)} (\Box I)}{\vdash x : (\Diamond A \supset \Box B) \supset \Box(A \supset B)} (\supset I)}
\end{array}$$

2 axiom11

Relation R2 has properties: Directedness,

$$\begin{array}{c}
\frac{\frac{\frac{}{x : \Diamond \Box A; \vdash x : \Diamond \Box A} (Hyp)}{xR2y; \vdash xR2y} (Hyp)}{\frac{\frac{\frac{\frac{\frac{}{z : \Box A; \vdash z : \Box A} (Hyp)}{zR2w; \vdash zR2w} (Hyp)}{z : \Box A; zR2w; \vdash w : A} (\Box E)}{\frac{\frac{\frac{}{yR2w; \vdash yR2w} (Hyp)}{yR2w; \vdash yR2w} (\Diamond I)}{z : \Box A; zR2w; yR2w; \vdash y : \Diamond A} (R_2)}{\frac{\frac{\frac{\frac{}{xR2z; \vdash xR2z} (Hyp)}{xR2y; xR2z; z : \Box A; \vdash y : \Diamond A} (\Diamond E)}{x : \Diamond \Box A; xR2y; \vdash y : \Diamond A} (\Box I)}{\frac{\frac{}{x : \Diamond \Box A; \vdash x : \Box \Diamond A} (\Box I)}{\vdash x : \Diamond \Box A \supset \Box \Diamond A} (\supset I)}
\end{array}$$

3 axiom4

Relation R has properties:

$$\begin{array}{c}
\frac{\frac{\frac{}{x : \Diamond(A \vee B); \vdash x : \Diamond(A \vee B)} (Hyp)}{y : A \vee B; \vdash y : A \vee B} (Hyp)}{\frac{\frac{\frac{\frac{\frac{}{y : A; \vdash y : A} (Hyp)}{xRy; \vdash xRy} (\Diamond I)}{y : A; xRy; \vdash x : \Diamond A} (\vee I1)}{\frac{\frac{\frac{\frac{}{y : B; \vdash y : B} (Hyp)}{xRy; \vdash xRy} (\Diamond I)}{y : B; xRy; \vdash x : \Diamond B} (\vee I2)}{y : A \vee B; xRy; \vdash x : \Diamond A \vee \Diamond B} (\vee E)}{\frac{\frac{\frac{}{x : \Diamond(A \vee B); \vdash x : \Diamond A \vee \Diamond B} (\Diamond E)}{\vdash x : \Diamond(A \vee B) \supset \Diamond A \vee \Diamond B} (\supset I)}
\end{array}$$

4 axiom2

Relation R has properties:

$$\begin{array}{c}
\frac{\frac{\frac{}{x : \Diamond A; \vdash x : \Diamond A}^{(Hyp)}}{x : \Box(A \supset B); \vdash x : \Box(A \supset B)}^{(Hyp)} \quad \frac{\frac{}{xRy; \vdash xRy}^{(Hyp)}}{x : \Box(A \supset B); xRy; \vdash y : A \supset B}^{(\Box E)} \quad \frac{\frac{}{y : A; \vdash y : A}^{(Hyp)}}{y : A; \vdash y : A}^{(\supset E)} \quad \frac{}{xRy; \vdash xRy}^{(Hyp)}}{x : \Box(A \supset B); xRy; y : A; \vdash y : B}^{(\supset E)} \quad \frac{}{xRy; \vdash xRy}^{(Hyp)}}{x : \Box(A \supset B); xRy; y : A; \vdash x : \Diamond B}^{(\Diamond I)} \\
\frac{}{x : \Diamond A; \vdash x : \Diamond A}^{(Hyp)} \quad \frac{}{x : \Box(A \supset B); xRy; y : A; \vdash x : \Diamond B}^{(\Diamond E)} \\
\frac{x : \Diamond A; x : \Box(A \supset B); \vdash x : \Diamond B}{x : \Box(A \supset B); \vdash x : \Diamond A \supset \Diamond B}^{(\supset I)} \\
\frac{x : \Box(A \supset B); \vdash x : \Diamond A \supset \Diamond B}{\vdash x : \Box(A \supset B) \supset \Diamond A \supset \Diamond B}^{(\supset I)}
\end{array}$$

5 axiom10

Relation R5 has properties: Euclideaness,

$$\begin{array}{c}
\frac{\frac{}{x : \Diamond A; \vdash x : \Diamond A}^{(Hyp)}}{x : \Diamond A; \vdash x : \Diamond A}^{(Hyp)} \quad \frac{\frac{}{xR5y; \vdash xR5y}^{(Hyp)}}{xR5y; \vdash xR5y}^{(Hyp)} \quad \frac{\frac{}{xR5z; \vdash xR5z}^{(Hyp)}}{xR5z; \vdash xR5z}^{(Hyp)} \quad \frac{\frac{}{z : A; \vdash z : A}^{(Hyp)} \quad \frac{}{yR5z; \vdash yR5z}^{(Hyp)}}{z : A; yR5z; \vdash y : \Diamond A}^{(\Diamond I)} \\
\frac{}{x : \Diamond A; \vdash x : \Diamond A}^{(Hyp)} \quad \frac{}{xR5y; xR5z; z : A; \vdash y : \Diamond A}^{(R_5)} \\
\frac{}{x : \Diamond A; xR5y; \vdash y : \Diamond A}^{(\Diamond I)} \\
\frac{x : \Diamond A; \vdash x : \Box \Diamond A}{\vdash x : \Diamond A \supset \Box \Diamond A}^{(\supset I)}
\end{array}$$

6 axiom1

Relation R has properties:

$$\begin{array}{c}
\frac{\frac{}{x : \Box(A \supset B); \vdash x : \Box(A \supset B)}^{(Hyp)}}{x : \Box(A \supset B); \vdash x : \Box(A \supset B)}^{(Hyp)} \quad \frac{\frac{}{xRy; \vdash xRy}^{(Hyp)}}{xRy; \vdash xRy}^{(\Box E)} \quad \frac{\frac{}{x : \Box A; \vdash x : \Box A}^{(Hyp)}}{x : \Box A; \vdash x : \Box A}^{(Hyp)} \quad \frac{\frac{}{xRy; \vdash xRy}^{(Hyp)}}{xRy; \vdash xRy}^{(\Box E)}}{x : \Box(A \supset B); xRy; \vdash y : A \supset B}^{(\supset E)} \\
\frac{}{x : \Box(A \supset B); xRy; x : \Box A; \vdash y : B}^{(\Box I)} \\
\frac{x : \Box(A \supset B); x : \Box A; \vdash x : \Box B}{x : \Box(A \supset B); \vdash x : \Box A \supset \Box B}^{(\supset I)} \\
\frac{x : \Box(A \supset B); \vdash x : \Box A \supset \Box B}{\vdash x : \Box(A \supset B) \supset \Box A \supset \Box B}^{(\supset I)}
\end{array}$$

7 axiom7

Relation RT has properties: Reflexivity,

$$\begin{array}{c}
\frac{\frac{}{x : \Box A; \vdash x : \Box A}^{(Hyp)}}{x : \Box A; \vdash x : \Box A}^{(Hyp)} \quad \frac{\frac{}{xRTx; \vdash xRTx}^{(Hyp)}}{xRTx; \vdash xRTx}^{(\Box E)} \\
\frac{}{x : \Box A; xRTx; \vdash x : A}^{(R_T)} \\
\frac{x : \Box A; \vdash x : A}{\vdash x : \Box A \supset A}^{(\supset I)}
\end{array}$$

8 axiom8

Relation RB has properties: Symmetry,

$$\frac{\frac{\frac{}{xRBy; \vdash xRBy}^{(Hyp)} \quad \frac{\frac{\frac{}{x:A; \vdash x:A}^{(Hyp)} \quad \frac{}{yRBx; \vdash yRBx}^{(Hyp)}}{x:A; yRBx; \vdash y:\Diamond A}^{(\Diamond I)}}{xRBy; x:A; \vdash y:\Diamond A}^{(\Diamond I)}}{\frac{x:A; \vdash x:\Box \Diamond A}{}^{(\Box I)}}^{(\supset I)} \quad \frac{}{\vdash x:A \supset \Box \Diamond A}^{(\supset I)}$$

9 axiom9

Relation R4 has properties: Transitivity,

$$\frac{\frac{\frac{}{xR4y; \vdash xR4y}^{(Hyp)} \quad \frac{}{yR4z; \vdash yR4z}^{(Hyp)} \quad \frac{\frac{\frac{}{x:\Box A; \vdash x:\Box A}^{(Hyp)} \quad \frac{}{xR4z; \vdash xR4z}^{(Hyp)}}{x:\Box A; xR4z; \vdash z:A}^{(\Box E)}}{xR4y; yR4z; x:\Box A; \vdash z:A}^{(R_4)}}{\frac{xR4y; x:\Box A; \vdash y:\Box A}{}^{(\Box I)}}^{(\supset I)} \quad \frac{}{\vdash x:\Box A \supset \Box \Box A}^{(\supset I)}$$

10 axiom6

Relation RD has properties: Seriality,

$$\frac{\frac{\frac{}{x:\Box A; \vdash x:\Box A}^{(Hyp)} \quad \frac{}{xRDy; \vdash xRDy}^{(Hyp)}}{x:\Box A; xRDy; \vdash y:A}^{(\Box E)} \quad \frac{}{xRDy; \vdash xRDy}^{(Hyp)}}{\frac{x:\Box A; xRDy; \vdash x:\Diamond A}{}^{(R_D)}}^{(\supset I)} \quad \frac{}{\vdash x:\Box A \supset \Diamond A}^{(\supset I)}$$

11 axiom3

Relation R has properties:

$$\frac{\frac{\frac{}{x:\Diamond \perp; \vdash x:\Diamond \perp}^{(Hyp)} \quad \frac{\frac{\frac{}{y:\perp; \vdash y:\perp}^{(Hyp)}}{y:\perp; \vdash x:\perp}^{(\perp E)}}{xRy; y:\perp; \vdash x:\perp}^{(W)}}{\frac{x:\Diamond \perp; \vdash x:\perp}{}^{(\Diamond E)}}^{(\supset I)} \quad \frac{}{\vdash x:\Diamond \perp \supset \perp}^{(\supset I)}$$