1 axiom1

Relation R has properties:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{x:\Box(A\supset B);\vdash x:\Box(A\supset B)}\,(Hyp) \quad \dfrac{}{xRy;\vdash xRy}\,(Hyp)}{x:\Box(A\supset B);xRy;\vdash y:A\supset B}\,(\Box E) \quad \dfrac{\dfrac{}{x:\Box A;\vdash x:\Box A}\,(Hyp) \quad \dfrac{}{xRy;\vdash xRy}\,(Hyp)}{x:\Box A;xRy;\vdash y:A}\,(\Box E)}{x:\Box(A\supset B);xRy;x:\Box A;\vdash y:B}\,(\supset E)}{x:\Box(A\supset B);x:\Box A;\vdash x:\Box B}\,(\Box I)}{x:\Box(A\supset B);\vdash x:\Box A\supset\Box B}\,(\supset I)}{\vdash x:\Box(A\supset B)\supset\Box A\supset\Box B}\,(\supset I)$$

2 axiom2

Relation R has properties:

$$\dfrac{\dfrac{\dfrac{}{x:\Diamond A;\vdash x:\Diamond A}\,(Hyp) \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{x:\Box(A\supset B);\vdash x:\Box(A\supset B)}\,(Hyp) \quad \dfrac{}{xRy;\vdash xRy}\,(Hyp)}{x:\Box(A\supset B);xRy;\vdash y:A\supset B}\,(\Box E) \quad \dfrac{}{y:A;\vdash y:A}\,(Hyp)}{x:\Box(A\supset B);xRy;y:A;\vdash y:B}\,(\supset E) \quad \dfrac{}{xRy;\vdash xRy}\,(Hyp)}{x:\Box(A\supset B);xRy;y:A;\vdash x:\Diamond B}\,(\Diamond I)}{x:\Diamond A;x:\Box(A\supset B);\vdash x:\Diamond B}\,(\Diamond E)}{x:\Box(A\supset B);\vdash x:\Diamond A\supset\Diamond B}\,(\supset I)}{\vdash x:\Box(A\supset B)\supset\Diamond A\supset\Diamond B}\,(\supset I)$$

3 axiom3

Relation R has properties:

$$\dfrac{\dfrac{}{x:\Diamond\bot;\vdash x:\Diamond\bot}\,(Hyp) \quad \dfrac{\dfrac{\dfrac{}{y:\bot;\vdash y:\bot}\,(Hyp)}{y:\bot;\vdash x:\bot}\,(\bot E)}{xRy;y:\bot;\vdash x:\bot}\,(W)}{x:\Diamond\bot;\vdash x:\bot}\,(\Diamond E)}{\vdash x:\Diamond\bot\supset\bot}\,(\supset I)$$

4 axiom4

Relation R has properties:

$$\dfrac{\dfrac{}{x:\diamond(A\vee B);\vdash x:\diamond(A\vee B)}\,(Hyp)\qquad \dfrac{\dfrac{}{y:A\vee B;\vdash y:A\vee B}\,(Hyp)\qquad \dfrac{\dfrac{\dfrac{}{y:A;\vdash y:A}\,(Hyp)\qquad \dfrac{}{xRy;\vdash xRy}\,(Hyp)}{y:A;xRy;\vdash x:\diamond A}\,(\diamond I)}{y:A;xRy;\vdash x:\diamond A\vee\diamond B}\,(\vee I1)\qquad \dfrac{\dfrac{\dfrac{}{y:B;\vdash y:B}\,(Hyp)\qquad \dfrac{}{xRy;\vdash xRy}\,(Hyp)}{y:B;xRy;\vdash x:\diamond B}\,(\diamond I)}{y:B;xRy;\vdash x:\diamond A\vee\diamond B}\,(\vee I2)}{y:A\vee B;xRy;\vdash x:\diamond A\vee\diamond B}\,(\vee E)}{x:\diamond(A\vee B);\vdash x:\diamond A\vee\diamond B}\,(\diamond E)}{\vdash x:\diamond(A\vee B)\supset\diamond A\vee\diamond B}\,(\supset I)$$

5 axiom5

Relation R has properties:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{x:\diamond A\supset\Box B;\vdash x:\diamond A\supset\Box B}\,(Hyp)\qquad \dfrac{\dfrac{}{y:A;\vdash y:A}\,(Hyp)\qquad \dfrac{}{xRy;\vdash xRy}\,(Hyp)}{y:A;xRy;\vdash x:\diamond A}\,(\diamond I)}{x:\diamond A\supset\Box B;y:A;xRy;\vdash x:\Box B}\,(\supset E)\qquad \dfrac{}{xRy;\vdash xRy}\,(Hyp)}{x:\diamond A\supset\Box B;y:A;xRy;\vdash y:B}\,(\Box E)}{x:\diamond A\supset\Box B;xRy;\vdash y:A\supset B}\,(\supset I)}{x:\diamond A\supset\Box B;\vdash x:\Box(A\supset B)}\,(\Box I)}{\vdash x:(\diamond A\supset\Box B)\supset\Box(A\supset B)}\,(\supset I)$$

6 axiom6

Relation RD has properties: Seriality,

$$\dfrac{\dfrac{\dfrac{\dfrac{}{x:\Box A;\vdash x:\Box A}\,(Hyp)\qquad \dfrac{}{xRDy;\vdash xRDy}\,(Hyp)}{x:\Box A;xRDy;\vdash y:A}\,(\Box E)\qquad \dfrac{}{xRDy;\vdash xRDy}\,(Hyp)}{x:\Box A;xRDy;\vdash x:\diamond A}\,(\diamond I)}{x:\Box A;\vdash x:\diamond A}\,(R_D)}{\vdash x:\Box A\supset\diamond A}\,(\supset I)$$

7 axiom7

Relation RT has properties: Reflexivity,

$$\dfrac{\dfrac{\dfrac{}{x:\Box A;\vdash x:\Box A}\,(Hyp)\qquad \dfrac{}{xRTx;\vdash xRTx}\,(Hyp)}{x:\Box A;xRTx;\vdash x:A}\,(\Box E)}{x:\Box A;\vdash x:A}\,(R_T)}{\vdash x:\Box A\supset A}\,(\supset I)$$

8 axiom8

Relation RB has properties: Symmetry,

$$\dfrac{\dfrac{\dfrac{\dfrac{}{xRBy;\vdash xRBy}\,(Hyp) \quad \dfrac{\dfrac{}{x:A;\vdash x:A}\,(Hyp) \quad \dfrac{}{yRBx;\vdash yRBx}\,(Hyp)}{x:A;yRBx;\vdash y:\diamond A}\,(\diamond I)}{xRBy;x:A;\vdash y:\diamond A}\,(R_B)}{x:A;\vdash x:\Box\diamond A}\,(\Box I)}{\vdash x:A\supset\Box\diamond A}\,(\supset I)$$

9 axiom9

Relation R4 has properties: Transitivity,

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{xR4y;\vdash xR4y}\,(Hyp) \quad \dfrac{}{yR4z;\vdash yR4z}\,(Hyp) \quad \dfrac{\dfrac{}{x:\Box A;\vdash x:\Box A}\,(Hyp) \quad \dfrac{}{xR4z;\vdash xR4z}\,(Hyp)}{x:\Box A;xR4z;\vdash z:A}\,(\Box E)}{xR4y;yR4z;x:\Box A;\vdash z:A}\,(R_4)}{xR4y;x:\Box A;\vdash y:\Box A}\,(\Box I)}{x:\Box A;\vdash x:\Box\Box A}\,(\Box I)}{\vdash x:\Box A\supset\Box\Box A}\,(\supset I)$$

10 axiom10

Relation R5 has properties: Euclideanness,

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{x:\diamond A;\vdash x:\diamond A}\,(Hyp) \quad \dfrac{\dfrac{}{xR5y;\vdash xR5y}\,(Hyp) \quad \dfrac{}{xR5z;\vdash xR5z}\,(Hyp) \quad \dfrac{\dfrac{}{z:A;\vdash z:A}\,(Hyp) \quad \dfrac{}{yR5z;\vdash yR5z}\,(Hyp)}{z:A;yR5z;\vdash y:\diamond A}\,(\diamond I)}{xR5y;xR5z;z:A;\vdash y:\diamond A}\,(R_5)}{x:\diamond A;xR5y;\vdash y:\diamond A}\,(\diamond E)}{x:\diamond A;\vdash x:\Box\diamond A}\,(\Box I)}{\vdash x:\diamond A\supset\Box\diamond A}\,(\supset I)$$

11 axiom11

Relation R2 has properties: Directedness,

$$\dfrac{\dfrac{}{x : \diamond\Box A; \vdash x : \diamond\Box A}\,(Hyp) \qquad \dfrac{\dfrac{}{xR2y; \vdash xR2y}\,(Hyp) \qquad \dfrac{}{xR2z; \vdash xR2z}\,(Hyp) \qquad \dfrac{\dfrac{\dfrac{}{z : \Box A; \vdash z : \Box A}\,(Hyp) \qquad \dfrac{}{zR2w; \vdash zR2w}\,(Hyp)}{z : \Box A; zR2w; \vdash w : A}\,(\Box E) \qquad \dfrac{}{yR2w; \vdash yR2w}\,(Hyp)}{z : \Box A; zR2w; yR2w; \vdash y : \diamond A}\,(\diamond I)}{xR2y; xR2z; z : \Box A; \vdash y : \diamond A}\,(R_2)}{x : \diamond\Box A; xR2y; \vdash y : \diamond A}\,(\diamond E)}{x : \diamond\Box A; \vdash x : \Box \diamond A}\,(\Box I)}{\vdash x : \diamond\Box A \supset \Box \diamond A}\,(\supset I)$$