1 axiom1

Relation R has properties:

$$\dfrac{\dfrac{\overline{x :: \Box(A \supset B); \vdash x :: \Box(A \supset B)} \ (Ass) \quad \overline{xRy; \vdash xRy} \ (Ass)}{x :: \Box(A \supset B); xRy; \vdash y :: A \supset B} \ (\Box E) \quad \dfrac{\overline{x :: \Box A; \vdash x :: \Box A} \ (Ass) \quad \overline{xRy; \vdash xRy} \ (Ass)}{x :: \Box A; xRy; \vdash y :: A} \ (\Box E)}{\dfrac{\dfrac{x :: \Box(A \supset B); xRy; x :: \Box A; \vdash y :: B}{x :: \Box(A \supset B); x :: \Box A; \vdash x :: \Box B} \ (\Box I)}{\dfrac{x :: \Box(A \supset B); \vdash x :: \Box A \supset \Box B}{\vdash x :: \Box(A \supset B) \supset \Box A \supset \Box B} \ (\supset I)} \ (\supset I)} \ (\supset E)}$$

2 axiom2

Relation R has properties:

$$\dfrac{\overline{x :: \Diamond A; \vdash x :: \Diamond A} \ (Ass) \quad \dfrac{\dfrac{\dfrac{\overline{x :: \Box(A \supset B); \vdash x :: \Box(A \supset B)} \ (Ass) \quad \overline{xRy; \vdash xRy} \ (Ass)}{x :: \Box(A \supset B); xRy; \vdash y :: A \supset B} \ (\Box E) \quad \overline{y :: A; \vdash y :: A} \ (Ass)}{x :: \Box(A \supset B); xRy; y :: A; \vdash y :: B} \ (\supset E) \quad \overline{xRy; \vdash xRy} \ (Ass)}{x :: \Box(A \supset B); xRy; y :: A; \vdash x :: \Diamond B} \ (\Diamond I)}{\dfrac{\dfrac{x :: \Diamond A; x :: \Box(A \supset B); \vdash x :: \Diamond B}{x :: \Box(A \supset B); \vdash x :: \Diamond A \supset \Diamond B} \ (\supset I)}{\vdash x :: \Box(A \supset B) \supset \Diamond A \supset \Diamond B} \ (\supset I)} \ (\Diamond E)}$$

3 axiom3

Relation R has properties:

$$\dfrac{\overline{x :: \Diamond\bot; \vdash x :: \Diamond\bot} \ (Ass) \quad \dfrac{\dfrac{\overline{y :: \bot; \vdash y :: \bot} \ (Ass)}{y :: \bot; \vdash x :: \bot} \ (\bot E)}{xRy; y :: \bot; \vdash x :: \bot} \ (W)}{\dfrac{x :: \Diamond\bot; \vdash x :: \bot}{\vdash x :: \Diamond\bot \supset \bot} \ (\supset I)} \ (\Diamond E)}$$

4 axiom4

Relation R has properties:

1

$$\cfrac{\cfrac{}{x :: \Diamond(A \lor B); \vdash x :: \Diamond(A \lor B)} \ (Ass) \quad \cfrac{\cfrac{}{y :: A \lor B; \vdash y :: A \lor B} \ (Ass) \quad \cfrac{\cfrac{\cfrac{}{y :: A; \vdash y :: A} \ (Ass) \quad \cfrac{}{xRy; \vdash xRy} \ (Ass)}{y :: A; xRy; \vdash x :: \Diamond A} \ (\Diamond I)}{y :: A; xRy; \vdash x :: \Diamond A \lor \Diamond B} \ (\lor I1) \quad \cfrac{\cfrac{\cfrac{}{y :: B; \vdash y :: B} \ (Ass) \quad \cfrac{}{xRy; \vdash xRy} \ (Ass)}{y :: B; xRy; \vdash x :: \Diamond B} \ (\Diamond I)}{y :: B; xRy; \vdash x :: \Diamond A \lor \Diamond B} \ (\lor I2)}{y :: A \lor B; xRy; \vdash x :: \Diamond A \lor \Diamond B} \ (\Diamond E)}{\cfrac{x :: \Diamond(A \lor B); \vdash x :: \Diamond A \lor \Diamond B}{\vdash x :: \Diamond(A \lor B) \supset \Diamond A \lor \Diamond B} \ (\supset I)}$$

5 axiom5

Relation R has properties:

$$\cfrac{\cfrac{\cfrac{}{x :: \Diamond A \supset \Box B; \vdash x :: \Diamond A \supset \Box B} \ (Ass) \quad \cfrac{\cfrac{\cfrac{}{y :: A; \vdash y :: A} \ (Ass) \quad \cfrac{}{xRy; \vdash xRy} \ (Ass)}{y :: A; xRy; \vdash x :: \Diamond A} \ (\Diamond I)}{x :: \Diamond A \supset \Box B; y :: A; xRy; \vdash x :: \Box B} \ (\supset E)}{\cfrac{\cfrac{x :: \Diamond A \supset \Box B; y :: A; xRy; \vdash y :: B}{x :: \Diamond A \supset \Box B; xRy; \vdash y :: A \supset B} \ (\supset I)}{\cfrac{x :: \Diamond A \supset \Box B; \vdash x :: \Box(A \supset B)}{\vdash x :: (\Diamond A \supset \Box B) \supset \Box(A \supset B)} \ (\supset I)} \ (\Box I)} \quad \cfrac{}{xRy; \vdash xRy} \ (Ass)}{} \ (\Box E)$$

6 axiom6

Relation RD has properties: Seriality,

$$\cfrac{\cfrac{\cfrac{\cfrac{}{x :: \Box A; \vdash x :: \Box A} \ (Ass) \quad \cfrac{}{xRDy; \vdash xRDy} \ (Ass)}{x :: \Box A; xRDy; \vdash y :: A} \ (\Box E) \quad \cfrac{}{xRDy; \vdash xRDy} \ (Ass)}{\cfrac{x :: \Box A; xRDy; \vdash x :: \Diamond A}{x :: \Box A; \vdash x :: \Diamond A} \ (R_D)} \ (\Diamond I)}{\vdash x :: \Box A \supset \Diamond A} \ (\supset I)$$

7 axiom7

Relation RT has properties: Reflexivity,

$$\cfrac{\cfrac{\cfrac{\cfrac{}{x :: \Box A; \vdash x :: \Box A} \ (Ass) \quad \cfrac{}{xRTx; \vdash xRTx} \ (Ass)}{x :: \Box A; xRTx; \vdash x :: A} \ (\Box E)}{x :: \Box A; \vdash x :: A} \ (R_T)}{\vdash x :: \Box A \supset A} \ (\supset I)$$

8   axiom8

Relation RB has properties: Symmetry,

$$
\cfrac{\cfrac{}{xRBy; \vdash xRBy}\ (Ass) \qquad \cfrac{\cfrac{\cfrac{}{x::A; \vdash x::A}\ (Ass) \qquad \cfrac{}{yRBx; \vdash yRBx}\ (Ass)}{x::A; yRBx; \vdash y::\Diamond A}\ (\Diamond I)}{x::A; yRBx; \vdash y::\Diamond A}\ (R_B)}{\cfrac{\cfrac{xRBy; x::A; \vdash y::\Diamond A}{x::A; \vdash x::\Box\Diamond A}\ (\Box I)}{\vdash x::A \supset \Box\Diamond A}\ (\supset I)}
$$

9   axiom9

Relation R4 has properties: Transitivity,

$$
\cfrac{\cfrac{}{xR4y; \vdash xR4y}\ (Ass) \qquad \cfrac{}{yR4z; \vdash yR4z}\ (Ass) \qquad \cfrac{\cfrac{}{x::\Box A; \vdash x::\Box A}\ (Ass) \qquad \cfrac{}{xR4z; \vdash xR4z}\ (Ass)}{x::\Box A; xR4z; \vdash z::A}\ (\Box E)}{\cfrac{\cfrac{\cfrac{xR4y; yR4z; x::\Box A; \vdash z::A}{xR4y; x::\Box A; \vdash y::\Box A}\ (\Box I)}{x::\Box A; \vdash x::\Box\Box A}\ (\Box I)}{\vdash x::\Box A \supset \Box\Box A}\ (\supset I)}\ (R_4)
$$

10   axiom10

Relation R5 has properties: Euclideanness,

$$
\cfrac{\cfrac{}{x::\Diamond A; \vdash x::\Diamond A}\ (Ass) \qquad \cfrac{\cfrac{}{xR5y; \vdash xR5y}\ (Ass) \qquad \cfrac{}{xR5z; \vdash xR5z}\ (Ass) \qquad \cfrac{\cfrac{\cfrac{}{z::A; \vdash z::A}\ (Ass) \qquad \cfrac{}{yR5z; \vdash yR5z}\ (Ass)}{z::A; yR5z; \vdash y::\Diamond A}\ (\Diamond I)}{xR5y; xR5z; z::A; \vdash y::\Diamond A}\ (R_5)}{xR5y; xR5z; z::A; \vdash y::\Diamond A}\ (\Diamond E)}{\cfrac{\cfrac{x::\Diamond A; xR5y; \vdash y::\Diamond A}{x::\Diamond A; \vdash x::\Box\Diamond A}\ (\Box I)}{\vdash x::\Diamond A \supset \Box\Diamond A}\ (\supset I)}
$$

11   axiom11

Relation R2 has properties: Directedness,

<div align="center">3</div>

$$
\dfrac{
  \dfrac{
    x :: \Diamond\Box A;\vdash x :: \Diamond\Box A
  }{} \, (Ass)
  \qquad
  \dfrac{
    \dfrac{
      \dfrac{xR2y;\vdash xR2y}{} \,(Ass)
      \qquad
      \dfrac{xR2z;\vdash xR2z}{} \,(Ass)
      \qquad
      \dfrac{
        \dfrac{
          \dfrac{z :: \Box A;\vdash z :: \Box A}{} \,(Ass)
          \quad
          \dfrac{zR2w;\vdash zR2w}{} \,(Ass)
        }{z :: \Box A;\, zR2w;\vdash w :: A} \,(\Box E)
        \qquad
        \dfrac{yR2w;\vdash yR2w}{} \,(Ass)
      }{\dfrac{z :: \Box A;\, zR2w;\, yR2w;\vdash y :: \Diamond A}{}} \,(\Diamond I)
    }{\dfrac{xR2y;\, xR2z;\, z :: \Box A;\vdash y :: \Diamond A}{}} \,(R_2)
  }{x :: \Diamond\Box A;\, xR2y;\vdash y :: \Diamond A} \,(\Diamond E)
}{
  \dfrac{
    \dfrac{x :: \Diamond\Box A;\vdash x :: \Box\Diamond A}{\vdash x :: \Diamond\Box A \supset \Box\Diamond A}\,(\supset I)
  }{}\,(\Box I)
}
$$