Linear Algebra Done Right Solutions

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These solutions are for the 3rd edition *Linear Algebra Done Right* textbook by Sheldon Axler.

I will do my best to provide some guidance along the way in this book as I read it.

Please note that I'm mostly going through this book for review[1], so I will likely skip over more introductory topics.

[1] my third time doing linear algebra, although first two times were both introductory level

Chapter 1

Vector Spaces

1.1 $\mathbf{R}^n$ and $\mathbf{C}^n$

Exercise 1.1.1
Multiply both sides by $\frac{a-bi}{a-bi}$.

This is known as *rationalizing the denominator*.

Exercise 1.1.2
I think this is meant to be an arithmetic problem, but we can notice that

$$\frac{-1+\sqrt{3}i}{2} = \operatorname{cis}\left(\frac{2\pi}{3}\right) \implies \left(\frac{-1+\sqrt{3}i}{2}\right)^3 = \operatorname{cis}(2\pi) = 1$$

which is just 1 because we are at the rightmost part of the unit circle.

Exercise 1.1.3
$i = \operatorname{cis}\pi/2 + 2\pi k$ so if we take the square root we get solutions in the form of

$$\operatorname{cis}\pi/4 + \pi k$$

Exercise 1.1.4
Apply commutativity to real and imaginary parts.

Exercise 1.1.5
Separate real and imginary parts and use field properties.

Exercise 1.1.6
Separate real and imginary parts and use field properties.

Exercise 1.1.7
skip

Exercise 1.1.8
skip

Exercise 1.1.9
skip

Exercise 1.1.10
$x = (1/2, 6, -7/2, 1/2)$.

Exercise 1.1.11
We can equate two pairs of tuple values and show that no such λ exists after simplifications.

Skip the rest

1.2 Definition of Vector Space

The vector space definition in this chapter is very important!

Definition 1.2.1
A **vector space** is a set V along with an addition on V and a scalar multiplication on V such that the following properties hold:

- commutativity
$$u + v = v + v \quad \forall u, v \in V$$
- associativity
$$\begin{aligned} &\forall u, v, w \in V, a, b, \in \mathbf{F} \\ &(u + v) + w = u + (v + w) \text{ and } (ab)v = a(bv) \end{aligned}$$
- additive identity
$$\exists 0 \in V \quad \forall v\,, v + 0 = v$$
- additive inverse
$$\forall v \in V, \exists w \in V \quad v + w = 0$$
- multiplicative identity
$$\forall v \in V, 1v = v$$
- distributive properties
$$\begin{aligned} &\forall a, b \in \mathbf{F} \text{ and } \forall u, v \in V \\ &a(u + v) = au + av \text{ and } (a + b)v = av + bv \end{aligned}$$

Exercise 1.2.1
The additive inverse of v is $-v$. The additive inverse of $-v$ is $-(-v)$. We have
$$\begin{aligned} v + -v = 0 &= -v + -(-v) \\ v &= -(-v) \end{aligned}$$

Exercise 1.2.2
Just do it by cases.

I think we have to prove it is not possible for $a \neq 0$ and $v \neq 0$ though... In this case you'd do an AFSOC, but showing $av \neq 0$ I'm not sure how to do without using information e.g. that two nonzero elements multiplied together is not zero (this is not trivial btw, e.g. you can have two non zero matrices that multiply to be 0).

Exercise 1.2.3
Because $x = \frac{1}{3}(w - v)$

Exercise 1.2.4
Since the empty set doesn't have elements, any statement with $\forall$ will be vacuously true. The only statement that doesn't look like that is the additive identity, which we can confirm is not true because $0 \notin \emptyset$.

Exercise 1.2.5
If we can replace the additive identity, then we should be able to derive it from our existing properties.

So let's start with some $v \in V$, we want to find some $w \in V$ such that $v + w = 0$.

I think we can choose $w = 0v - v$?

Exercise 1.2.6
skip

1.3 Subspaces

Subspace property here is very important

Definition 1.3.1
$U \subset V$ is a **subspace** of V if U is also a vector space.

The conditions for a subspace $U \subset V$ are:

1. additive identity
$$0 \in U$$
2. closed under addition
$$u, w \in U \implies u + w \in U$$
3. closed under scalar multiplication
$$a \in \mathbf{F} \text{ and } u \in U \implies au \in U$$

Exercise 1.3.1

(a) yes

(b) no scalar fails

(c) Not closed under addition

(d) Yes

Exercise 1.3.2
We have to confirm, I'll just point out the ones that don't work.

(a) if $b \neq 0$ then scaling doesn't work

(b) yes

(c) yes

(d) not close dunder addition if $b \neq 0$.

(e) Limit 0 is key, another limit point and this won't work.

Exercise 1.3.15
$U + U = U$, since $\forall u \in U, 2u \in U$

Exercise 1.3.16
Commutativity should hold.

Exercise 1.3.17
Yes should be associative via field properties.

Exercise 1.3.18
I think they all should, since 0 is in the subspace and scalar is closed, so we have $v, -v$ for any $v \in V$.

Exercise 1.3.19
False, $U_1 = (u, 0), U_2 = (0, u), W = (x, y)$

Exercise 1.3.20
$W = (0, w_1, 0, w_2)$

Exercise 1.3.21
Can we not just choose a trivial W like $W = \mathbf{F}^5$?

Otherwise, we notice that x, y are the only independent variables in the tuple, so we need 3 more tuples, so we just choose $W = (0, 0, x, y, z)$.

Exercise 1.3.22
As we saw in 1.3.21, we have 3 degrees of freedom, so we can choose

$$
\begin{aligned}
W_1 &= (0, 0, w_1, 0, 0) \\
W_2 &= (0, 0, 0, w_2, 0) \\
W_3 &= (0, 0, 0, 0, w_3)
\end{aligned}
$$

Exercise 1.3.23
False.

The intuition here is that the part W is "missing" from V is constant, but the trick is that we can have one of the U_1 "over contribute" and the other "under contribute".

So we can choose like $W = (x, y)$ and then $U_1 = (x, 0)$ and then $U_2 = (x, y)$ and the result is both sums are $\mathbb{R}^2$.

Exercise 1.3.24
skip

Chapter 2

Finite-Dimensional Vector Spaces

2.1 Span and Linear independence

Linear combination, span and linear independence (it's in the title duh) are the most definitions in this chapter.

Definition 2.1.1
A **linear combination** of a list $v_1, \ldots, v_m$ of vectors in V is a vector of the form

$$\sum_{i=1}^{m} a_i v_i,$$

where $a_1, \ldots, a_m \in \mathbf{F}$.

Definition 2.1.2
The set of all linear combinations of a list of vectors $v_1, \ldots, v_m$ in V is called the **span** of the list, denoted

$$\operatorname{span}(v_1, \ldots, v_m) = \left\{ \sum_{i=1}^{m} a_i v_i \ : \ a_1, \ldots, a_m \in \mathbf{F}. \right\}$$

The span of the empty list () is defined to be [0].

Definition 2.1.3

- A list $v_1, \ldots, v_m$ of vectors in V is called **linearly independent** if the only choice $a_1, \ldots, a_m \in \mathbf{F}$ that makes
$$\sum_{i=1}^{m} a_i v_i = 0$$
is $a_1 = \cdots = a_m = 0$.
- The empty list () is also declared to be linearly independent.

Exercise 2.1.1
We have

$$\begin{aligned}
\operatorname{span}(v_4, v_3 - v_4) &= \operatorname{span}(v_4, v_3) \\
\operatorname{span}(v_4, v_3 - v_4, v_2 - v_3) &= \operatorname{span}(v_4, v_3, v_2 - v_3) = \operatorname{span}(v_4, v_3, v_2) \\
\operatorname{span}(v_4, v_3 - v_4, v_2 - v_3, v_1 - v_2) &= \operatorname{span}(v_4, v_3, v_2, v_1 - v_2) = \operatorname{span}(v_4, v_3, v_2, v_1)
\end{aligned}$$

Exercise 2.1.2

(a) $v = 0$ does not work because we have $a_1 \neq 0, a_1 v = 0$. Otherwise, $v \neq 0 \implies av \neq 0$ for $a \neq 0$.

(b) If one is a scalar multiple of another, then you can write $v_1 = kv_2 \implies v_1 - kv_2 = 0$.

(c) Yes, tuple values are in separate coordinates.

(d) Yes.

Exercise 2.1.3
We want $(5, 9, t)$ to be a linear combination of the first two vectors, i.e.

$$x(3, 1, 4) + y(2, -3, 5) = (5, 9, t)$$

So we have $t = 4x + 5y$, and if we just need to solve

$$\begin{aligned} 3x + 2y &= 5 \\ x - 3y &= 9 \end{aligned}$$

I believe this is $y = -2, x = 3$.

Exercise 2.1.4
This is basically the same as the previous problem.

Exercise 2.1.5
(a) $\forall k \in \boldsymbol{R}, k(1 + i) \neq 1 - i$

(b) $-i(1 + i) = 1 - i$

Exercise 2.1.6
This is very similar to 2.1.1.

Exercise 2.1.7
True.

Exercise 2.1.17
If $\forall j, p_j(2) = 0$, then either $\exists p_j = 0$, in which case these polynomials are not linearly independent.

The other case is that $\forall p_j \neq 0$. In this case, we know they all have a zero at 2, which means we can divide all of them by $(x - 2)$.

Now we have $m + 1$ polynomials who degree is at most $m - 1$. We know that $\mathcal{P}_{m-1}(\mathbf{F})$ has rank m, so $m + 1$ elements in this space will definitely be linear dependent.

2.2 Bases

This chapter should be pretty familiar now that you already understand what span and linear independence are.

Exercise 2.2.1
These vector spaces are a scalar multiple of a single element.

Exercise 2.2.2
too lazy

Exercise 2.2.3

(a) $(3, 1, 0, 0, 0), (0, 0, 7, 1, 0), (0, 0, 0, 0, 1)$.

(b) add $(1, 0, 0, 0, 0), (0, 0, 1, 0, 0)$

(c) Take what we added above

Exercise 2.2.4
lazy

Exercise 2.2.5
False...I think.

Exercise 2.2.6
Just show fun span stuff.

2.3 Dimension

Dimension should be nothing new, it's just the number of vectors in a basis.

too lazy to do these exercises rip