Linear Algebra Done Right Solutions

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Forward

These solutions are for the 3rd edition *Linear Algebra Done Right* textbook by Sheldon Axler.

I will do my best to provide some guidance along the way in this book as I read it. I will highlight important theorems for each chapter.

Please note that I'm mostly going through this book for review[1], so I will likely skip over more introductory topics. I might also be lazy with proofs, so definitely feel free to submit feedback about problems that I could've done better. I still hope that whenever I say "intuition" or give sketches of solutions, that it helps.

[1] my third time doing linear algebra, although first two times were both introductory level

Chapter 1

Vector Spaces

1.1 $\mathbf{R}^n$ and $\mathbf{C}^n$

Exercise 1.1.1
Multiply both sides by $\frac{a-bi}{a-bi}$.

This is known as *rationalizing the denominator*.

Exercise 1.1.2
I think this is meant to be an arithmetic problem, but we can notice that

$$\frac{-1+\sqrt{3}i}{2} = \operatorname{cis}\left(\frac{2\pi}{3}\right) \implies \left(\frac{-1+\sqrt{3}i}{2}\right)^3 = \operatorname{cis}(2\pi) = 1$$

which is just 1 because we are at the rightmost part of the unit circle.

Exercise 1.1.3
$i = \operatorname{cis}\pi/2 + 2\pi k$ so if we take the square root we get solutions in the form of

$$\operatorname{cis}\pi/4 + \pi k$$

Exercise 1.1.4
Apply commutativity to real and imaginary parts.

Exercise 1.1.5
Separate real and imginary parts and use field properties.

Exercise 1.1.6
Separate real and imginary parts and use field properties.

Exercise 1.1.7
skip

Exercise 1.1.8
skip

Exercise 1.1.9
skip

Exercise 1.1.10
$x = (1/2, 6, -7/2, 1/2)$.

Exercise 1.1.11
We can equate two pairs of tuple values and show that no such λ exists after simplifications.

Skip the rest

1.2 Definition of Vector Space

The vector space definition in this chapter is very important!

Definition 1.2.1
A **vector space** is a set V along with an addition on V and a scalar multiplication on V such that the following properties hold:

- commutativity
$$u + v = v + v \quad \forall u, v \in V$$
- associativity
$$\begin{aligned}&\forall u, v, w \in V, a, b, \in \mathbf{F}\\&(u + v) + w = u + (v + w) \text{ and } (ab)v = a(bv)\end{aligned}$$
- additive identity
$$\exists 0 \in V \quad \forall v\,, v + 0 = v$$
- additive inverse
$$\forall v \in V, \exists w \in V \quad v + w = 0$$
- multiplicative identity
$$\forall v \in V, 1v = v$$
- distributive properties
$$\begin{aligned}&\forall a, b \in \mathbf{F} \text{ and } \forall u, v \in V\\&a(u + v) = au + av \text{ and } (a + b)v = av + bv\end{aligned}$$

Exercise 1.2.1
The additive inverse of v is $-v$. The additive inverse of $-v$ is $-(-v)$. We have
$$\begin{aligned}v + -v = 0 &= -v + -(-v)\\v &= -(-v)\end{aligned}$$

Exercise 1.2.2
Just do it by cases.

I think we have to prove it is not possible for $a \neq 0$ and $v \neq 0$ though... In this case you'd do an AFSOC, but showing $av \neq 0$ I'm not sure how to do without using information e.g. that two nonzero elements multiplied together is not zero (this is not trivial btw, e.g. you can have two non zero matrices that multiply to be 0).

Exercise 1.2.3
Because $x = \frac{1}{3}(w - v)$

Exercise 1.2.4
Since the empty set doesn't have elements, any statement with $\forall$ will be vacuously true. The only statement that doesn't look like that is the additive identity, which we can confirm is not true because $0 \notin \emptyset$.

Exercise 1.2.5
If we can replace the additive identity, then we should be able to derive it from our existing properties.

So let's start with some $v \in V$, we want to find some $w \in V$ such that $v + w = 0$.

I think we can choose $w = 0v - v$?

Exercise 1.2.6
skip

1.3 Subspaces

Subspace property here is very important

Definition 1.3.1
$U \subset V$ is a **subspace** of V if U is also a vector space.

The conditions for a subspace $U \subset V$ are:

1. additive identity
$$0 \in U$$
2. closed under addition
$$u, w \in U \implies u + w \in U$$
3. closed under scalar multiplication
$$a \in \mathbf{F} \text{ and } u \in U \implies au \in U$$

Exercise 1.3.1

(a) yes

(b) no scalar fails

(c) Not closed under addition

(d) Yes

Exercise 1.3.2
We have to confirm, I'll just point out the ones that don't work.

(a) if $b \neq 0$ then scaling doesn't work

(b) yes

(c) yes

(d) not close dunder addition if $b \neq 0$.

(e) Limit 0 is key, another limit point and this won't work.

Exercise 1.3.15
$U + U = U$, since $\forall u \in U, 2u \in U$

Exercise 1.3.16
Commutativity should hold.

Exercise 1.3.17
Yes should be associative via field properties.

Exercise 1.3.18
I think they all should, since 0 is in the subspace and scalar is closed, so we have $v, -v$ for any $v \in V$.

Exercise 1.3.19
False, $U_1 = (u, 0), U_2 = (0, u), W = (x, y)$

Exercise 1.3.20
$W = (0, w_1, 0, w_2)$

Exercise 1.3.21
Can we not just choose a trivial W like $W = \mathbf{F}^5$?

Otherwise, we notice that x, y are the only independent variables in the tuple, so we need 3 more tuples, so we just choose $W = (0, 0, x, y, z)$.

Exercise 1.3.22
As we saw in 1.3.21, we have 3 degrees of freedom, so we can choose

$$W_1 = (0, 0, w_1, 0, 0)$$
$$W_2 = (0, 0, 0, w_2, 0)$$
$$W_3 = (0, 0, 0, 0, w_3)$$

Exercise 1.3.23
False.

The intuition here is that the part W is "missing" from V is constant, but the trick is that we can have one of the U_1 "over contribute" and the other "under contribute".

So we can choose like $W = (x, y)$ and then $U_1 = (x, 0)$ and then $U_2 = (x, y)$ and the result is both sums are $\mathbb{R}^2$.

Exercise 1.3.24
skip

Chapter 2

Finite-Dimensional Vector Spaces

2.1 Span and Linear independence

Linear combination, span and linear independence (it's in the title duh) are the most definitions in this chapter.

Definition 2.1.1
A **linear combination** of a list $v_1, \ldots, v_m$ of vectors in V is a vector of the form

$$\sum_{i=1}^{m} a_i v_i,$$

where $a_1, \ldots, a_m \in \mathbf{F}$.

Definition 2.1.2
The set of all linear combinations of a list of vectors $v_1, \ldots, v_m$ in V is called the **span** of the list, denoted

$$\operatorname{span}(v_1, \ldots, v_m) = \left\{ \sum_{i=1}^{m} a_i v_i \ : \ a_1, \ldots, a_m \in \mathbf{F}. \right\}$$

The span of the empty list () is defined to be [0].

Definition 2.1.3

- A list $v_1, \ldots, v_m$ of vectors in V is called **linearly independent** if the only choice $a_1, \ldots, a_m \in \mathbf{F}$ that makes
$$\sum_{i=1}^{m} a_i v_i = 0$$
is $a_1 = \cdots = a_m = 0$.
- The empty list () is also declared to be linearly independent.

Exercise 2.1.1
We have

$$\begin{aligned} \operatorname{span}(v_4, v_3 - v_4) &= \operatorname{span}(v_4, v_3) \\ \operatorname{span}(v_4, v_3 - v_4, v_2 - v_3) = \operatorname{span}(v_4, v_3, v_2 - v_3) &= \operatorname{span}(v_4, v_3, v_2) \\ \operatorname{span}(v_4, v_3 - v_4, v_2 - v_3, v_1 - v_2) = \operatorname{span}(v_4, v_3, v_2, v_1 - v_2) &= \operatorname{span}(v_4, v_3, v_2, v_1) \end{aligned}$$

Exercise 2.1.2

(a) $v = 0$ does not work because we have $a_1 \neq 0, a_1 v = 0$. Otherwise, $v \neq 0 \implies av \neq 0$ for $a \neq 0$.

(b) If one is a scalar multiple of another, then you can write $v_1 = kv_2 \implies v_1 - kv_2 = 0$.

(c) Yes, tuple values are in separate coordinates.

(d) Yes.

Exercise 2.1.3
We want $(5, 9, t)$ to be a linear combination of the first two vectors, i.e.

$$x(3,1,4) + y(2,-3,5) = (5,9,t)$$

So we have $t = 4x + 5y$, and if we just need to solve

$$\begin{aligned} 3x + 2y &= 5 \\ x - 3y &= 9 \end{aligned}$$

I believe this is $y = -2, x = 3$.

Exercise 2.1.4
This is basically the same as the previous problem.

Exercise 2.1.5
(a) $\forall k \in \boldsymbol{R}, k(1+i) \neq 1 - i$

(b) $-i(1+i) = 1 - i$

Exercise 2.1.6
This is very similar to 2.1.1.

Exercise 2.1.7
True.

Exercise 2.1.17
If $\forall j, p_j(2) = 0$, then either $\exists p_j = 0$, in which case these polynomials are not linearly independent.

The other case is that $\forall p_j \neq 0$. In this case, we know they all have a zero at 2, which means we can divide all of them by $(x - 2)$.

Now we have $m + 1$ polynomials who degree is at most $m - 1$. We know that $\mathcal{P}_{m-1}(\mathbf{F})$ has rank m, so $m + 1$ elements in this space will definitely be linear dependent.

2.2 Bases

This chapter should be pretty familiar now that you already understand what span and linear independence are.

Exercise 2.2.1
These vector spaces are a scalar multiple of a single element.

Exercise 2.2.2
too lazy

Exercise 2.2.3

(a) $(3, 1, 0, 0, 0), (0, 0, 7, 1, 0), (0, 0, 0, 0, 1)$.

(b) add $(1, 0, 0, 0, 0), (0, 0, 1, 0, 0)$

(c) Take what we added above

Exercise 2.2.4
lazy

Exercise 2.2.5
False...I think.

Exercise 2.2.6
Just show fun span stuff.

2.3 Dimension

Dimension should be nothing new, it's just the number of vectors in a basis.

too lazy to do these exercises rip

Chapter 3

Linear Maps

3.1 The Vector Space of Linear Maps

The property of linearity is very crucial.

Definition 3.1.1
A **linear map** from $V \to W$ is a function $T : V \to W$ with the following properties:

- additivity
$$\forall u, v \in V \quad T(u+v) = Tu + Tv$$

- homogeneity
$$\forall \lambda \in \mathbf{F}, v \in V \quad T(\lambda v) = \lambda(Tv)$$

Exercise 3.1.1
You know $T(0) = 0$ so use that fact for b. For c just try any nonzero x, y, z.

Exercise 3.1.2
skip

Exercise 3.1.3
The general idea here is that we use the span of $x_1, \ldots, x_n$ to create each coordinate of the element in $\boldsymbol{F}^m$.

Exercise 3.1.4
Suppose $v_1, \ldots, v_m$ is linearly dependent, then for some v_k we can write it as a linear combination of the other v_i's, so we have

$$\begin{aligned} v_k &= \sum_{i \neq k} a_i v_i \qquad (a_i \text{ not all } 0) \\ T(v_k) &= T\left(\sum_{i \neq k} a_i v_i\right) \\ T(v_k) &= \sum_{i \neq k} a_i T(v_i) \\ \sum_{i \neq k} a_i T(v_i) - T(v_k) &= 0 \end{aligned}$$

the last part is a contradiction because we know Tv_i are linearly dependent.

Exercise 3.1.5
Tedious, just verify properties of a vector space in 1.2.1.

Exercise 3.1.6
skip

Exercise 3.1.7
If $\dim V = 1$, then $V = \operatorname{span}\{v\}$ for some $v \in V$. That means every element of V can be written as λv.

Exercise 3.1.8

$$\phi([v_1, v_2]) = \sqrt{v_1^2 + v_2^2}$$

Exercise 3.1.9
I'm stuck, I tried $\operatorname{cis}, \arg$ and no luck.

Exercise 3.1.10
AFSOC T is a linear map.

Suppose $0 \in U$. Then we have $T(0) = 0 = S(0)$ which is not true by $s \neq 0$ assumption.

So suppose $0 \in V \setminus U$. But then U is not a subspace since $0 \notin U$.

Exercise 3.1.11
I think we can use the construction from before and do $Tv = Sv$ if $v \in U$ and 0 otherwise. If there's no $S \neq 0$ restriction I think the construction works.

Exercise 3.1.12
We can trivially map the first $\dim V$ tuple elements of v to the first $\dim V$ tuple elements of w, and then pad with a zero, and then pad with two zeros and so on, for an infinite number of maps.

Exercise 3.1.13
We just have to choose m linearly independent vectors in W, which wouldn't allow $Tv_k = w_k$, otherwise we have $v_1, \ldots, v_m$ linearly independent.

Exercise 3.1.14
skip

3.2 Null Spaces and Ranges

Definition 3.2.1
The **null space** is the subset of inputs that gets mapped to zero.
More formally, $T \in \mathcal{L}(V, W)$,

$$\operatorname{null} T = \{v \in V \,:\, Tv = 0\} \tag{3.1}$$

We start learning about relationships between dimension, linearity, null spaces, ranges and domains.

Theorem 1 *Fundamental Theorem of Linear Maps. Suppose V is finite dimensional and $T \in \mathcal{L}(V, W)$. Then* range T *is finite-dimensional and*

$$\dim V = \dim \operatorname{null} T + \dim \operatorname{range} T \tag{3.2}$$

Exercise 3.2.1
$T(v_1, v_2, v_3, v_4, v_5) = (v_1, v_2, 0, 0, 0)$.

Exercise 3.2.2

$$\begin{aligned}(ST)^2 v &= (ST)(ST)v \\ &= (ST)w && (w \in \operatorname{null} T) \\ &= S(Tw) \\ &= S(0) \\ &= 0\end{aligned}$$

Exercise 3.2.3
(a) $\dim \operatorname{null} T = 0$

(b) $z_i \neq 0$ for some set of $\forall i v_i$ are linearly independent

Exercise 3.2.4
$\dim \boldsymbol{R}^5 = 5 \neq \dim T + \dim \operatorname{range} T = 2 + 2$.

Exercise 3.2.5
Let

$$T(w, x, y, z) = (0, 0, w, x)$$

Then

$$\begin{aligned}\operatorname{range} T &= (0, 0, v_1, v_2) \\ \operatorname{null} T &= (0, 0, v_1, v_2)\end{aligned}$$

Exercise 3.2.6
$\dim \operatorname{range} T \neq \dim \operatorname{null} T$ since they have to add up to 5, so they cannot be equal.

gonna power through rest of the book

3.3 Matrices

Matrices are just a tool to represent linear maps, but they are no different or special.

Exercise 3.3.1
We can imagine $\dim \operatorname{range} T$ as the number of linearly independent columns in the matrix, which has to be at least the dimension, and those are nonzero.

Exercise 3.3.2
I think you just want to shift the constant term, so

$$\begin{matrix} a_1 \\ a_2 \\ a_3 \\ a_0 \end{matrix}$$

We didn't really learn much in this chapter except for matrix operations, so going to skip this.

3.4 Invertibility and Isomorphic Vector Spaces

The author says the definition of an operator is extremely important.

Definition 3.4.1
We define **operator** $\mathcal{L}(V)$ to be

- a linear map from a vector space to itself
- The notation $\mathcal{L}(V)$ denotes the set of all operators on V. In other words,

$$\mathcal{L}(V) = \mathcal{L}(V, V) \tag{3.3}$$

I'm going to be honest that I don't really understand this definition (as in how it works and why it's important), so I'm getting rekt in the problems already.

These problems are getting harder for sure.

Exercise 3.4.1
We can verify

$$(ST)^{-1}(ST) = T^{-1}S^{-1}ST = T^{-1}IT = T^{-1}T = I$$
$$(ST)(ST)^{-1} = STT^{-1}S^{-1} = SIS^{-1} = SS^{-1} = I$$

Exercise 3.4.2
The operators are noninvertible, so they cannot be surjective or injective.

A little stuck here but I think by using that fact you can show that they may not be closed under addition or something of that sort.

Exercise 3.4.3
If T invertible exists, then S must be injective, otherwise if $S(x) = S(x') = v$ then $T(v)$ cannot be defined.

If S is injective, then we know for every $v \in V$, there is at most one $u \in U$ such that $S(u) = v$, so we just define T to invert these v that have a corresponding u.

Exercise 3.4.4
If $T_1 = ST_2$, then suppose $v \in \operatorname{null} T_1$. Then we have

$$\begin{aligned} T_1 v = 0 &= ST_2 v \\ S^{-1}(0) &= S^{-1}ST_2 v && \text{(Since } S \text{ invertible.)} \\ 0 &= T_2 v \end{aligned}$$

so $v \in \operatorname{null} T_2$ as well

If $\operatorname{null} T_1 = \operatorname{null} T_2$...**TODO** My intuition is that since T_1, T_2 span the same space (I think...bc of the null spaces being the same), their basis vectors are similar enough where you can make them the same by using some linear map.

Exercise 3.4.5
I think the $T_1 = T_2 S$ direction is pretty much the same as 3.4.4.

The other direction...also **TODO**

Exercise 3.4.6
TODO

Exercise 3.4.7

(a) $0 \in E$, $(T_1 + T_2)v = T_1 v + T_2 v = 0 + 0 = 0$, so $T_1 + T_2 \in E$, and $\lambda T v = T(\lambda v) = 0$.

(b) I think the dimension is 1??? I want to say something about $\operatorname{null} V$, but we are only taking a single vector.

Exercise 3.4.8
skip

Exercise 3.4.9
We know $(ST)^{-1} = T^{-1}S^{-1}$, so if ST is invertible, both T, S need to be invertible[1].

Alternatively, if T, S are invertible, then we can define $(ST)^{-1} = T^{-1}S^{-1}$.

Exercise 3.4.10
Forward direction,

$$\begin{aligned} ST &= I \\ STS &= IS = S \\ S^{-1}STS &= TS = S^{-1}S = I \end{aligned}$$

Backwards direction,

$$\begin{aligned} TS &= I \\ STS &= SI = S \\ STSS^{-1}TS &= SS^{-1} = I \end{aligned}$$

Exercise 3.4.11

$$\begin{aligned} STU &= I \\ T &= S^{-1}U^{-1} \\ T^{-}1 &= US \end{aligned}$$

Sorry did a lackluster job, on we go.

[1] sorry this is probably not an actual proof

3.5 Products and Quotients of Vector Spaces

Quotients are something we don't usually get in our intro linear algebra class, so that's cool.

Exercise 3.5.1
If T is a linear map, then we get that the graph of T is a subspace pretty easily with linear properties.

If the graph of T is a subspace, then we have that

$$(v_1, Tv_1) + (v_2, Tv_2) = (v_1 + v_2, Tv_1 + Tv_2) \in \text{graph of } T$$

here we see that $Tv_1 + Tv_2 = Tv'$ for some $v' \in V$. Then we must have (v', Tv') in the graph as well, which means

$$(v_1 + v_2, Tv_1 + Tv_2) = (v', Tv')$$

and we have $v' = v_1 + v_2$, so we conclude that $Tv_1 + Tv_2 = T(v_1 + v_2)$.

The proof for scalar is pretty similar.

Exercise 3.5.2
If $\exists V_j$ infinite-dimensional, then the cross product could be infinite dimensional.

Exercise 3.5.3
Let $U_1, U_2 = (u_1, u_2, u_3, \dots)$, then

$$U_1 \times U_2 = \boldsymbol{F}^\infty = U_1 + U_2,$$

but we clearly have multiple choices for $u \in U_1, u' \in U_2$ such that their sum is the same.

Exercise 3.5.4
Just intuition here, one is mapping an m tuple's elements each to W, the other is a m-tuple of $V_i \to W$ mappings.

Exercise 3.5.5
Very similar intuition to Exercise 3.5.4.

Exercise 3.5.6
Let's take advantage of the last two exercises, so we have

$$\begin{aligned}\mathcal{L}(\boldsymbol{F}^n, V) &= \prod_{i=1}^{n} \mathcal{L}(\boldsymbol{F}, V) \\ &= \mathcal{L}(\boldsymbol{F}, V^n) = V^n\end{aligned}$$

Exercise 3.5.7
AFSOC $U \neq W$, then WLOG $\exists w \in W, \notin U$. Then we have that

$$v \in V, \forall u \in U \quad v + w \neq v + u$$

which violates our assumption.

Exercise 3.5.14

(a) We can check a few things here

- $0 \in U$
- Closed by addition since finite $x_j \neq 0$ will still result in that property
- Closed by scalar since the finite $x_j \neq 0$ still holds

Thus we can conclude the subspace

(b) Consider the mapping $f : U \to \boldsymbol{F}^\infty$ where for $u \in U$, say j is the largest $x_j \neq 0$, then

$$f(u) = (\dots, x_j, x_j, x_j, \dots)$$

This $f(u)$ has infinitely many $x_i \neq 0$, so $f(u) \notin u$, but $f(u) \in \boldsymbol{F}^\infty$.

Since there are infinite number of coordinates that are nonzero, we must have infinite dimension.

3.6 Duality

Duality has been hard for me to understand, but it seems to be some corresponding linear operator that can be constructed from any basis of a vector space.

Definition 3.6.1
The **dual space** of V, denoted V', is the vector space of all linear functionals on V. In other words, $V' = \mathcal{L}(V, \boldsymbol{F})$.

Definition 3.6.2
For $U \subset V$, the **annihilator** of U, denoted U^0, is defined by

$$U^0 = \{\phi \in V' : \phi(u) = 0, \forall u \in U\} \tag{3.4}$$

Yeah I tried reading through the section and 80% made sense, but I would still myself rekt by not really understanding what a dual it and how to use it.

At least the matrix definitions are pretty trivial.

If you want to prove the row and column rank are the same, you can use Gaussian elimination instead, although we haven't learned that in this book yet. But most intro courses would definitely introduce Gaussian elimination, since it's an easy way to solve linear equations.

Too lazy to do exercises, hope I can get by this course without them...

Chapter 4

Polynomials

Short chapter let's go...

Not a huge fan of using the Complex Analysis Liouville's Theorem to prove the Fundamental Theorem of Algebra, but it works.

Okay yeah this chapter was truly short and I would say anyone who has taken Algebra II can probably skip this chapter.

I'm going to omit the exercises since there wasn't any new content introduced in this chapter.

Chapter 5

Eigenvalues, Eigenvectors, and Invariant Subspaces

I think the first time learning about Eigenvalues was super lit. Just the name sounds legendary, and when you see the results from it, you not only get cool results, but applicable properties that you find in so many fields, especially in Machine Learning and Stats (e.g. PCA).

5.1 Invariant Subspaces

Definition 5.1.1
Suppose $T \in \mathcal{L}(V)$. A subspace U of V is called **invariant** under T if $u \in U$ implies $Tu \in U$.

I think having this definition of an invariant subspace is cool, as usually we think of eigenvalues and eigenvectors as magical $Tv = \lambda v$ property, but not as the invariant subspace of $\operatorname{span} v$.

Definition 5.1.2
Suppose $T \in \mathcal{L}(V)$. A number $\lambda \in \boldsymbol{F}$ is called an **eigenvalue** of T if there exists $v \in V$ such that $v \neq 0$ and $Tv = \lambda v$. This v is called an **eigenvector** of T corresponding to λ.

I think the cool part of linear algebra is that the linear independence, span, dimension and other properties we learned earlier all apply to linearly-behaving entities. See the next Theorem.

Theorem 2 *Let $T \in \mathcal{L}(V)$. Suppose $\lambda_1, \ldots, \lambda_m$ are distinct eigenvalues of T and $v_1, \ldots, v_m$ are corresponding eigenvectors. Then $v_1, \ldots, v_m$ is linearly independent.*

I think this chapter is lacking practical techniques to solve for eigenvalues and eigenvectors, but it is a second course on Linear Algebra after all...and it's also more pure math anyway.

Exercise 5.1.1

(a) $u \in U$ means $Tu = 0 \in U$.

(b) $u \in U$ means $Tu \in \operatorname{range} T \implies Tu \in U$.

Exercise 5.1.2
Choose $v \in \operatorname{null} S$,

$$\begin{aligned}
Sv &= 0 \\
TSv &= T(0) = 0 \\
STv &= 0 && \text{(assumption of problem)} \\
S(Tv) &= 0
\end{aligned}$$

Therefore $Tv \in \operatorname{null} S$.

Exercise 5.1.3
I think this is essentially a repeat of 5.1.2.

Exercise 5.1.4
We can use the distributive property of T since it is linear, and case on each U_i.

Exercise 5.1.5
If we let this collection be U_i, then if we have some $u \in \bigcap_i U_i$, then $Tu \in U_i, \forall i \implies \in \bigcap_i U_i$.

Exercise 5.1.6
Seems true.

Exercise 5.1.7
Our system of equations is

$$\begin{aligned} -3y &= \lambda x \\ x &= \lambda y \end{aligned}$$

so we have

$$-3y = \lambda^2 y \implies \lambda^2 = -3$$

and we find $\lambda = \pm\sqrt{3}i$.

Exercise 5.1.8
Our system of equations is

$$\begin{aligned} w &= \lambda z \\ z &= \lambda w \end{aligned}$$

so we have

$$w = \lambda^2 w \implies \lambda^2 = 1$$

and we find $\lambda = \pm 1$.

Exercise 5.1.9
Our system of equations is

$$\begin{aligned} 2z_2 &= \lambda z_1 \\ 0 &= \lambda z_2 \\ 5z_3 &= \lambda z_3 \end{aligned}$$

I think here we have to conclude that $z_2 = 0$, otherwise if $\lambda = 0$ then all $z_i = 0$, and that wouldn't be an eigenvector.

Then we have $0 = \lambda z_1$, and we conclude $z_1 = 0$. We are just left with $\lambda = 5$, and our eigenvector is $(0, 0, 1)$.

Exercise 5.1.10

(a) Eigenvectors are the standard basis, with $\lambda = i$ for e_i.

(b) Invariant subspaces are just the spans of e_i individually.

5.2 Eigenvectors and Upper-Triangular Matrices

Exercise 5.2.1

(a) Not actually sure how to prove this is invertible...**TODO**

(b) It's like adding all the parts of T back in.

Exercise 5.2.2
Fundamental Theorem of Algebra.

Exercise 5.2.3
$T^2 - I = (T - I)(T + I) = 0$. If $\lambda \neq -1$, then $T \neq -I$, so $T = I$ in this case.

Exercise 5.2.4
$P(P - I) = 0$ means that $\operatorname{null} P = \operatorname{range}(P - I)$. Stuck here...

Exercise 5.2.5
The key here is that $(STS^{-1})^n$ expands in a way that cancels out all S, S^{-1} except for the outer ones. E.g.

$$\left(STS^{-1}\right)^3 = (STS^{-1})(STS^{-1})(STS^{-1}) = ST\left(S^{-1}S\right)T\left(S^{-1}S\right)TS^{-1} = ST^3S^{-1}$$

Exercise 5.2.6
Applying T^n to U will still be invariant in U.

Exercise 5.2.7
$T^2 = 9 \implies (T - 3)(T + 3) = 0$.

Exercise 5.2.8
Not sure if this exists.

Exercise 5.2.9
By FTA we can write $p(T) = \prod_i (T - \lambda_i I)$ So for any $(T - \lambda_i I)v = 0$, this implies λ_i is an eigenvalue, which is a zero of p.

Exercise 5.2.10

$$\begin{aligned}
p(T)v &= \left(\sum_{i=0}^{n} a_i T^i\right)v \\
&= \sum_{i=0}^{n} a_i T^i v \\
&= \sum_{i=0}^{n} a_i \lambda_i^i v \\
&= \left(\sum_{i=0}^{n} a_i \lambda_i^i\right)v \\
&= p(\lambda)v
\end{aligned}$$

5.3 Eigenspaces and Diagonal Matrices

Theorem 3 *Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Let $\lambda_1, \ldots, \lambda_m$ denote the distinct eigenvalues of T. Then the following are equivalent:*

(a) *T is diagonalizable*

(b) *V has a basis consisting of eigenvectors of T*

(c) *There exist 1-dimensional subspaces $U_1, \ldots, U_n$ of V, each invariant under T, such that*

$$V = \bigoplus_{i=1}^{n} U_i \tag{5.1}$$

(d) $V = \bigoplus_{i=1}^{m} E(\lambda_i, T)$

(e) $\dim V = \sum_{i=1}^{m} \dim E(\lambda_i, T)$

Definition 5.3.1
Suppose $T \in \mathcal{L}(V)$ and $\lambda \in \boldsymbol{F}$. The **eigenspace** of T corresponding to λ, denoted $E(\lambda, T)$, is defined by

$$E(\lambda, T) = \operatorname{null}(T - \lambda I). \tag{5.2}$$

So $E(\lambda, T)$ is the span of all eigenvectors corresponding to λ plus the 0 vector.

Exercise 5.3.1
$V = \operatorname{range} T$ since T is diagonalizable, and $\operatorname{null} T = \{0\}$.

Exercise 5.3.2
I don't think it is always true, we just need a $\operatorname{null} T$ that is nontrivial.

Exercise 5.3.3
If we have (a) then (b) follows directly. I think (b) also gives (a) pretty easily because we know $\operatorname{null} T, \operatorname{range} T$ cannot have common elements besides 0.

I think (b) gives us (c) directly from the discussion earlier. I think we still need to show (c) $\implies$ (a) or (b)...

Exercise 5.3.4
TODO

I should do more of these problems :)