Baby Rudin Solutions

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These solutions are for the 3rd edition *Principles of Mathematical Analysis* textbook by the legendary Walter Rudin.

I will do my best to provide some guidance along the way in this book.

Chapter 1

The Real and Complex Number Systems

The most important definition in this chapter is **1.10** about the *least-upper-bound property*, or the sup, and the corresponding inf. Without these concepts it's hard to prove anything else, so please pay attention to this definition!

Although you don't have to understand the R construction from Q perfectly, it is a good exercise. It is quite set-theory heavy, so if you haven't read *Naive Set Theory* by Paul Halmos, I would highly recommend give that book a skim.

I thought it was helpful to think the cuts in the construction as subsets of Q that kinda look like $(-\infty, r)$. This isn't super precise, but it should aid with the understanding quite a bit (the book should honestly include a diagram for this idea). I believe these cuts are known as Dedekind cuts in the real world.

1.1 Exercises

Exercise 1.1
Suppose they are trivial, then we have

(a) $r + x = q \in Q$ so $x = q - r \in Q$ which is a contradiction

(b) $rx = q \in Q$ so $x = \frac{q}{r} \in Q$ which is a contradiction

so we conclude both are irrational.

Exercise 1.2
This is the classic $\sqrt{2}$ is irrational proof, but you start with AFSOC $q \in Q$ such that

$$\begin{aligned} q^2 &= 12 \\ \left(\frac{m}{n}\right)^2 &= 12 && (m, n \in Z,\ m, n \text{ relatively prime}) \\ m^2 &= 12n^2 \end{aligned}$$

Now, m must be a multiple of 3, so let $m = 3k, k \in Z$.

$$m^2 = 9k^2 = 12n^2 3k^2 \qquad = 4n^2$$

and we conclude the same for n, which is a contradiction since we assume $\frac{m}{n}$ was represented as a fraction in simplest forms.

Exercise 1.3
These are pretty trivial.

Exercise 1.4
Pick an arbitrary $x \in E$, then we have $\alpha \leq x$ by lower bound definition and $x \leq \beta$ by upper bound definition. Putting these together, we get

$$\alpha \leq x \leq \beta \implies \alpha \leq \beta$$

Exercise 1.5
Let $\alpha = -\sup(-A) \implies -\alpha = \sup(-A)$. Then we know

$$\begin{aligned} -\alpha &> y && (y \in -A) \\ \alpha &< -y && (y \in -A) \\ \alpha &< y' && (y' = -y \in A) \end{aligned}$$

Therefore α is a lower bound for A. Now, because of the sup property of α on $-A$, we know $\nexists \beta$ such that

$$\beta > \alpha \text{ and } \beta < y \text{ for } y \in -A$$

which means α is the largest lower bound for A as well, so therefore we conclude $\alpha = \inf(A)$

Exercise 1.6

(a) I think the idea here is to show that $m/n = p/q$ AFSOC means $p = km, q = kn, k \in Q$, and then you say

$$\begin{aligned} (b^m)^{\frac{1}{n}} = A \implies b^m &= A^n && \text{(Theorem 1.21)} \\ b^{km} &= A^{kn} && \text{(Repeatedly multiply both sides } k \text{ times)} \\ (b^p)^{\frac{1}{q}} = \left(b^{km}\right)^{\frac{1}{kn}} &= A \end{aligned}$$

(b) Substitute fractions in for r, s and you can derive the rest with help from (a).

(c) Any $r' \leq r$ will have $b^{r'} \in B(r)$ by definition of $B(r)$. If $r' > r$, then $r' \notin B(r)$, hence we see that $b^r = \sup B(r)$.

(d) Add $B(x)$ and $B(y)$ and use (c).

Exercise 1.7
TODO

Exercise 1.8
$i^2 = -1 < 0$ so it is not ordered.

Exercise 1.9
TODO but you just run through the definition and verify.

Exercise 1.10
TODO

Exercise 1.11
We just have to set $r = |z|$ and divide z's coefficients by r to get w. These are unique because $|z|$ is unique.

Exercise 1.12
We can use induction to prove this, starting with the $n = 2$ case which is given by axioms.

Exercise 1.13
Start with triangle inequality and manipulate. I think you sould WLOG $|x| < |y|$ at some point when you try to get the $|x| - |y|$ term, so you can get your inequality.

Exercise 1.14

$$\begin{aligned}|1+z|^2+|1-z|^2 &= (1+z)(1+\bar{z})+(1-z)(1-\bar{z}) \\ &= 1+z+\bar{z}+z\bar{z}+1-z-\bar{z}+z\bar{z} \\ &= 4 && (z\bar{z}=1)\end{aligned}$$

You can also do this problem geometrically, realizing that $|z|=1$ means z is on the unit circle in the complex plane, and that

$$|1+z|^2+|1-z|^2 = |1-(-z)|^2+|1-z|^2 \tag{1.1}$$

which means we are finding the sum of square distances of $z,-z$ from 1 on the complex plane, which, since $z,-z$ are endpoints of a diameter, their sum distances are the sums of squares of two legs of a right triangle with the diameter as the hypotenuse, which has diameter length 2. We then conclude their square sums $=$ $2^2=4$.

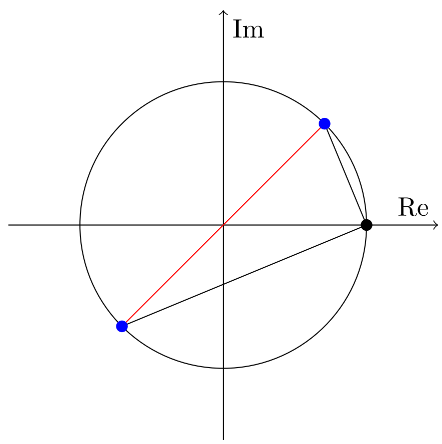


Figure 1.1: $z,\bar{z}$ and their square sum distance from 1 is the diameter

Exercise 1.15
Taking a look at our derivation, equality would happen when every $|Ba_j - Cb_j| = 0$.

Exercise 1.16

(a) For some reason I'm only getting k solutions here, since you have the locus of points that are r away from $\boldsymbol{x},\boldsymbol{y}$ respectively and you find their intersection.

(b) There is only one point that is equidistant from $\boldsymbol{x}$ and $\boldsymbol{y}$ and sums up to d, which is the midpoint of the two points.

(c) This is impossible because there is no point that can be $< r$ to both points but also cover the d distance.

Exercise 1.17
Proving equality is just an algebra exercise, use the conjugate definition.

Geometrically, we see that $|\boldsymbol{x}+\boldsymbol{y}|^2+|\boldsymbol{x}-\boldsymbol{y}|^2$ are the sum of squares of the diagonals, and the rest are the sums of squares of the side lengths. And we've proved they are equal.

Exercise 1.18
In $k=1$ this is not possible because $xy=0$ implies one of them is 0.

To some up with a general $\boldsymbol{y}$ for any k, observe that

$$\begin{aligned}
\boldsymbol{x} \cdot \boldsymbol{y} &= \sum_{i=1}^{k} x_i y_i \\
&= x_1 \cdot \frac{1}{x_1} + x_2 \cdot \frac{1}{x_2} + \cdots + x_k \cdot \frac{-(k-1)}{x_k} \\
&= k - 1 - (k-1) = 0
\end{aligned}$$

Geometrically, any perpendicular vector would have a dot product of 0.

Exercise 1.19
TODO

Exercise 1.20
TODO

Chapter 2

Basic Topology

The definitions in 2.18 are crucial to understanding the whole chapter, so it's important that you pay attention to those. Of course, first time reading through, it's ok to not entirely digest them and look back for reference.

I thought some of the proofs were pretty hard to understand, so I was hoping that doing some of the problems would help me solidify the concept of open, closed and compact sets.

Some gotchas:

- It sounds obvious now, but if a set has no limit points, then it is closed! This is because it's vacuously true that it contains all of its limit points.

- If you have to prove a set is closed, you can also try to instead prove that the complement is open; it may be much easier the other way. This also applies to open sets.

2.1 Compactness

Compactness is a definition that I thoroughly struggled with, so here is how I understood it:

- A set is compact if every open cover contains a finite subcover.

- This means given *any* open cover, so some $\bigcup_\alpha G_\alpha$ where G_α are all open and we can have an infinite cover here, we can find some finite subset of these G_α such that their union completely contains K.

- A good way to illustrate how this work, is consider $(0,1)$. We can come up with an open cover as

$$(0,1) = \bigcup_{i=1}^{\infty} \left(\frac{1}{i}, 1 \right) \tag{2.1}$$

  No matter what finite subcover you choose here, you will have some maximum N such that $\left(\frac{1}{N}, 1\right)$ is the widest interval you chose, and you will be missing the elements $0 < x < \frac{1}{N}$, so any finite subcover will not cover K.

- Another thing to notice is that the open cover and subcover are covers, meaning they can also "over"-cover the set K, and include elements that are not in K. This is important!

- I think this definition is hard to understand because you have to consider a "nested statement". In this case, we have to consider for all open covers, and then within each open cover, we have to show that there exists a finite subcover.

2.2 Exercises

Exercise 2.1
It is vacuously true that every element of the empty set is in any other set.

Exercise 2.2
I'm not actually sure how to use the hint given in the text, but my thinking was that for every n, we know the set of polynomials in the form

$$a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n = 0 \tag{2.2}$$

is Z^n, which we know is countable [1]

Now, what we need to show is if we have a countable set of countable sets, is their union countable? As we saw in Theorem 2.12, the answer is yes.

Just to add another explanation, in case the book explanation isn't convincing, imagine you have $f(j,k)$ which tells you the k^{th} element of the countable set Z^j. Then we now have a function for retrieving any element from our countable union of countable sets. We can visualize this as

$$\begin{array}{ccccc} f(1,j): & x_{11} & x_{12} & x_{13} & \cdots \\ f(2,j): & x_{21} & x_{22} & x_{23} & \cdots \\ f(3,j): & x_{31} & x_{32} & x_{33} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{array} \tag{2.3}$$

Now, you might ask, doesn't this seem like we can make a diagonalization argument and prove this is uncountable? And this is a great question, because it should be natural to ask that.

However, if you make the diagonalization argument, by taking each x_{ii} and trying to construct some polynomial that doesn't exist, you realize that the polynomial you are constructing is an infinitely long polynomial, which of course is not in our set, and also isn't a polynomial we aren't considering. It's because of the fact that we are only considering finite polynomials that makes this countable. Hence, the set of algebraic numbers is countable.

Exercise 2.3
The set of real numbers is uncountable, so if there are countable algebraic numbers, there must exist some real numbers which are not in the algebraic number set, otw we reach a contradiction that real numbers are countable.

Exercise 2.4
No, because the rational numbers are countable.

Exercise 2.5
The set defined by:

$$S = \bigcap_{i=1}^{\infty} \left[0, \frac{1}{i}\right) \cup \bigcap_{i=1}^{\infty} \left[1, 1+\frac{1}{i}\right) \cup \bigcap_{i=1}^{\infty} \left[2, 2+\frac{1}{i}\right) \tag{2.4}$$

The limit points being $0, 1, 2$ in this case.

Exercise 2.6
E' is closed because AFSOC there is a limit point p of E' that is $\notin E'$. Since p is a limit point of E', we know for every neighborhood N_p around p, there $\exists q \neq p'$ such that $q \in E'$. We also know that these q are limit points of E, so we can find $r \neq q$ in every neighborhood N_q around q such that $r \in E$. Then, for every N_p, we can choose some N_q that is entirely contained within N_p, such that $r \in N_p$. In this case, we have found for every N_p around p that there is some $r \in N_p$ but also $r \in E$, which means p is a limit point of E, and therefore $p \in E'$, which is a contradiction. Therefore, E' contains all of its limit points and it is closed.

The proof we have just done above shows that E' doesn't have any limit points that are $\notin E'$, meaning E' and E have the same limit points. Therefore, $\bar{E}$ has the union of limit points of E and E which is just limit points of E, so therefore $\bar{E}$ and E' have the same set of limit points.

[1] I think in the text we proved that Z^2 was countable, so we can use induction and show that Z^k in general is countable.

Careful here, we haven't proved that E, E' have the *same* limit points, only that every limit point of E' is also a limit point of E. In order to show they are the same, we'd have to prove that every limit point of E is also a limit point for E'. We can try to prove this by saying every limit point of E is some $p \in E'$, and this p has for every neighborhood some $q \in E$ that exists. But the argument breaks down here, because how in the world do we prove that $q \in E'$ as well? The answer is we can't, because we can take some inspiration from our answer in 2.4 and see that if

$$E = \bigcap_{i=1}^{\infty} \left[0, \frac{1}{i}\right) \tag{2.5}$$

Then we have $E' = [0]$, but E' has no limit points. So we have found a counterexample where E and E' have different limit points.

Exercise 2.7

(a) I'm too lazy to do this but I think it's pretty similar to 2.6.

(b) I'm not going to do the main proof, but will do the followup question. This inclusion can be proper because imagine that

$$A_i = \left[\sum_{i=0}^{i} \frac{1}{2^i}, \sum_{i=0}^{i+1} \frac{1}{2^i}\right] \tag{2.6}$$

Then 1 is a limit point of B, but it is not a limit point of any A_i.

Exercise 2.8

For open sets, yes, since for every point, we have some neighborhood N that is entirely contained in the set. So either we have some larger N' around the point that contains N as a subset, in which we can take some element q in N which is also $\in N'$ and q is in our set, so therefore our point is a limit point. Otherwise, if we have a smaller neighborhood, then it will be a subset of N, which we know is already a subset of the set, and therefore any points in those neighborhoods will be in the set. It remains to prove that such a point exists, but it will because for any p and neighborhood with radius r, we can find a point $(p_1 + r/3, p_2 + r/3)$ which is contained in the neighborhood and exists in R^2. This argument exists in general for R^n, so we know every interior point in R^n is a limit point as well.

For closed sets, this is not true in general because we can try to choose a set with no limit points, which is closed by the set vacuously having all of its limit points, e.g. a singleton like [1]. In this case, there are no limit points, so 1 is not a limit point but it is in the set, which is closed.

Exercise 2.9

(a) If every point of E° is an interior point of E, then they have a neighborhood N entirely contained within E. Thus, if we can find a neighborhood entirely contained within E°, we can prove that these points are also interior points of E° and therefore the set is open. We can show that this N must exist in E°, by showing that every point in N is an interior point. This is true, because suppose we have some interior point of E with a neighborhood N of radius r. Suppose we have some point $p' \in N, p' \neq p$. Suppose this point is r' distance from p, then we can find the neighborhood N' with radius $\min(r', r - r')$ around p', which is entirely contained within N and thus E, and thus is an interior point. Therefore, every point of E° is an interior point, and thus the set is open.

(b)
   (a) If E is open, then we know that every point of E is an interior point so therefore $E = E^{\circ}$.
   (b) If $E = E^{\circ}$, that implies that every point of E is an interior point, so therefore E is open.

(c) G is open, which means every point is an interior point of G. $G \subset E$, the neighborhoods which makes these points interior points are also in G, so these points are also interior points of E, and therefore are in E°.

(d) We are trying to prove that $(E^{\circ})^{c} = \overline{E^{c}}$.

   - Suppose $x \in (E^{\circ})^{c}$, then we know that $x \notin E^{\circ}$, meaning it is not an interior point of E. Then we have two cases

(a) $x \in E \setminus E^{\circ}$, then x is not an interior point of E. This implies that every neighborhood N around x has the property that $N \not\subset E$, which means $\exists q \neq x$ such that $q \notin E$, but $q \in N$. If we consider this x, q from the perspective of E^{c}, this means x is a limit point of E^{c} so therefore $x \in \overline{complement E}$.

(b) $x \in E^{c}$: By the definition of closure, this implies $x \in \overline{E^{c}}$

- Suppose $x \in \overline{E^{c}}$, then we know that $x \in E^{c}$ or $x \in (E^{c})'$. We can use a pretty similar argument from above to prove this direction.

Exercise 2.10
This is a metric because

(a) $d(p,q) = 1 > 0$ if $p \neq q$, and $d(p,p) = 0$ by def.

(b) $d(p,q) = 1 = d(q,p)$

(c) $d(p,q) = 1 \leq d(p,r) + d(r,q) = 2$

Intuition runthrough:

- Every subset is open? Because you can just draw a neighborhood with $r = 1/2$ and that will only contain the point which is a subset of the set itself.
- Every set has no limit points, because the $r = 1/2$ neighborhood only includes a single point, which cannot be a limit point, and therefore since every set has no limit points, every set is vacuously closed.
- Finite sets are compact.

Exercise 2.11
Too lazy, but you just check the 3 rules.

Exercise 2.12
The idea here is that some set in the open cover will contain 0, which means it will contain some neighborhood around 0, which will help us cover all $\frac{1}{n}$ that are small enough to fit in this neighborhood. Then, we are only left with finitely many $\frac{1}{n}$ that aren't in this cover, and we can just take the union of those sets that contain those elements, which will give us a finite subcover.

Exercise 2.13
It doesn't say countably infinite[2] so you can just do a finite number of limit points...$[0, 1]$ will do.

Exercise 2.14
Hey check out my explanation of compactness in 2.1!

Exercise 2.15

- Open counterexample in R

$$\bigcap_{i=0}^{\infty}(()0, \frac{1}{2^i}) = \emptyset$$

but every finite subcollection intersection is not empty.

- Closed counterexample in R (not extended)

$$\bigcap_{i=0}^{\infty}(()i, \infty) = \emptyset$$

this is because for any n you choose, I can find an i such that $i > n$ and (i, ∞) is one of the sets in this intersection, so the set is empty. If you take any finite intersection, you can find the max i in these sets, and $i < x < \infty$ will exist in this intersection so it is not empty.

[2] I know I'm being cheap here

Exercise 2.16
Since there are "holes" in rationals, we can intuitively figure out this is not compact. But more formally, you can construct an infinite covering where we get $1/2^i$ closer to $\sqrt{2}$, and you won't be able to take any finite subcovering, or else you miss out on some rationals that are even closer to $\sqrt{2}$ that you inevitably cannot include with a finite subcovering. Yes E is open, you should be able to draw a neighborhood around every rational where the entire neighborhood is in E.

Exercise 2.17
E is not countable because of infinite $4, 7$ and we can do diagonalization proof.

E is not dense since 1.4 is at least 0.04 away from any other element, so we can choose something like 1.2 which is not a limit point of E and is not in E.

I originally tried to prove this by showing that E contains all its limit points, but it's honestly really hard, because you don't really know where certain values are converging towards. Instead, an easier approach here is to try to prove the complement of E within the interval, namely $[0, 1] \setminus E$ is open, in which case we can see that given any x without $4, 7$ in the decimal expansion, we can find where it doesn't have a $4, 7$, and that will produce some radius for a neighborhood that is small enough such that x is entirely contained within numbers that are not $4, 7$, making x and interior point and this complement open.

We want to prove $E = E'$ if E is perfect. We know E is closed so $E' \subset E$. It remains to show that $E \subset E'$. Ok so I think my intution was originally wrong about this problem, and since we are considering decimal expensions only including $4, 7$, that means each decimal expension is infinite. I think this makes the problem make *way more sense* and I wish the author would've mentioned it. But yeah, in that case, it's pretty easy to show every point is a limit point because given any $x \in E$, you can just find a y that matches up with x long enough until the ϵ is small enough, therefore showing that x is a limit point.

I'm going to stop here for now and then continue reading the rest of the book

Chapter 3

Numerical Sequences and Series

It is absolutely paramount that you understand $\delta - \epsilon$ proofs before reading this chapter, since Rudin doesn't explain them at all. Checking out *Understanding Analysis by Abbott* Solutions that I wrote has some cheatsheets in the back fro review, and reading the book is a good idea as well.

Some dumbed-down student interpretations of definitions:

- **Diameter**: The furthest 2 points are from each other in a sequence.
- $limsup_{n\to\infty} s_n = s^*$: the largest value that some subsequence in E converges to. There is an analogous definition for $\liminf$.

3.1 Exercises

Exercise 3.1
Triangle inequality says that

$$||s_n| - |s|| \leq |s_n - s| < \epsilon \tag{3.1}$$

so we conclude that $[|s_n|]$ converges.

The converse is not true, because you can take $s_n = (-1)^n$ which converges in the absolute value to 1, but diverges without the absolute value.

Exercise 3.2
There are some algebra tricks you need to solve this problem.

Although this is kinda "beyond the scope of what we've learned so far," the first solution that came to mind to me was the expand the power series for the square root, which is really just a binomial to the 1/2 power.

As a reminder, the binomial expansion is

$$(1+x)^n = 1 + nx + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \cdots = \sum_{i=0}^{\infty}\binom{n}{i}x^i \tag{3.2}$$

We get

$$\begin{aligned}
\lim_{n\to\infty} \sqrt{n^2+n} - n &= \lim_{n\to\infty} n\left(1+\frac{1}{n}\right)^{1/2} - n \\
&= \lim_{n\to\infty} n\left(1 + \frac{1}{2}\frac{1}{n} + \frac{1}{2}\cdot\frac{-1}{2}\cdot\frac{1}{2!}\cdot\frac{1}{\cdot}n^2\right) - n \\
&= \lim_{n\to\infty} \frac{1}{2} + \frac{1}{n^2}\cdot\left(\text{constant or power } n^k \text{ where } k<0\right) \\
&= \frac{1}{2}
\end{aligned}$$

The other solution that I was thinking about was pretty similar to the textbook solution [1], is to divide

[1] which is to "rationalize" by $\sqrt{n^2+n} - n$ btw

by n and take it from there.

So trying that...

$$\begin{aligned}\frac{\left(\sqrt{n^2+n}-n\right)/n}{1/n} &= \frac{\sqrt{1+1/n}-1}{1/n}\\ &= \frac{\frac{1}{2}(1+1/n)^{-1/2}\cdot \lg n}{\lg n} && \text{(L'Hopital's rule}^2\text{)}\end{aligned}$$

from the result here, we can see the limit as $n \to \infty$ makes this expression $1/2$ as well.

Exercise 3.3
TODO

Exercise 3.4
Trying out some values, we get $0, 0, 1/2, 1/4, 3/4, 3/8, 7/8, 7/16, \ldots$. It seems like we are approaching 1 and 1/2 (so this sequence does not converge).

Now taking a look at our equations,

$$\begin{aligned}s_1 &= 0\\ s_{2m} &= \frac{s_{2m-1}}{2}\\ s_{2m+1} &= \frac{1}{2} + s_{2m}\end{aligned}$$

we can deduce that

$$\begin{aligned}s_{2m} &= \frac{1}{4} + \frac{s_{2m-2}}{2}\\ s_{2m+1} &= \frac{1}{2} + \frac{s_{2m-1}}{2}\end{aligned}$$

For s_{2m}, if we expand this, we get $1/4 + 1/8 + \cdots$ until the term is equal to 0. For $s_{2m} + 1$, if we expand this, we get $1/2 + 1/4 + 1/8 + \cdots$ until the term is equal to 0.

To formally prove the bounds, we have to use induction, but we can conclude that our lower limit is 0 and 1 here.[3]

Exercise 3.5
The intuition here is that we take these a_n and b_n that form the $\limsup_{n\to\infty}(a_n + b_n)$, and we notice that by definition, $\lim_{n\to\infty}$ for these two sequences must be $\leq$ both $\limsup_{n\to\infty} a_n$ and b_n respectively, by definition. So we have $a_n \to A$ and $b_n \to B$ and we know that $A \leq \limsup_{n\to\infty} a_n$ and $B \leq \limsup_{n\to\infty} a_n$ and we can conclude that

$$\limsup_{n\to\infty}(a_n + b_n) = A + B \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n \tag{3.3}$$

Here's an attempt at doing it with AFSOC...[4]

AFSOC $\limsup_{n\to\infty}(a_n + b_n) > \limsup_{n\to\infty} a_n + \limsup n \to \infty b_n$.

Then label

$$\begin{aligned}\limsup_{n\to\infty}(a_n + b_n) &= C'\\ \limsup_{n\to\infty} a_n &= A\\ \limsup_{n\to\infty} a_n &= B\\ C &= A + B\end{aligned}$$

[3]Sorry too lazy to just chug through the induction here...I think the problem solving here is more important.
[4]It's way more convoluted so doing it directly is probs preferred.

Now, if we have $\limsup_{n\to\infty}(a_n + b_n) = C'$, then let the $a_n \to A'$ in this sequence $a_n + b_n$. Then if we consider the sequence $(a_n + b_n) - a_n$, we have

$$\limsup_{n\to\infty}[(a_n + b_n) - a_n] = C' - A'$$

we have a few cases, here

- $\limsup_{n\to\infty} a_n > A'$: This is not possible, since we could've chosen this a_n sequence instead and gotten a larger C'.
- $\limsup_{n\to\infty} a_n < A'$: This is not possible, since we could've chosen this a_n sequence a gotten a larger A.
- $\limsup_{n\to\infty} a_n = A'$: In this case, we have

$$\begin{aligned}
\limsup_{n\to\infty}[(a_n + b_n) - a_n] &= \lim_{n\to\infty} b_n' \\
&= C' - A' \\
&= C' - A \\
&> C - A && (C' > C \text{ by assumption}) \\
\implies &> B
\end{aligned}$$

  which is a contradiction, since we know that $\limsup_{n\to\infty} b_n = B$, so there can be no subsequence of b_n' that goes to a larger limit.

Exercise 3.6

(a) Let us investigate with ratio test

$$\begin{aligned}
\frac{\sqrt{n+1} - \sqrt{n}}{\sqrt{n} - \sqrt{n-1}} &= \frac{\sqrt{n} + \sqrt{n-1}}{\sqrt{n} + \sqrt{n-1}} \cdot \frac{\sqrt{n+1} - \sqrt{n}}{\sqrt{n} - \sqrt{n-1}} \\
&= \frac{\sqrt{n+1} + \sqrt{n}}{\sqrt{n+1} + \sqrt{n}} \cdot \frac{\left(\sqrt{n} + \sqrt{n-1}\right)\left(\sqrt{n+1} - \sqrt{n}\right)}{1} \\
&= \frac{\sqrt{n} + \sqrt{n-1}}{\sqrt{n+1} + \sqrt{n}} \\
&= \frac{1 + \sqrt{1 - 1/n}}{\sqrt{1 + 1/n} + 1}
\end{aligned}$$

this limit is $= 1$, so we conclude this series diverges.

(b) From (a), we can use the ratio test and get to

$$\frac{n}{n+1} \cdot \frac{\sqrt{n} + \sqrt{n-1}}{\sqrt{n+1} + \sqrt{n}} = \frac{1 + \sqrt{1 - 1/n}}{\sqrt{1 + 1/n} + \sqrt{1/n^2 + 1/n^3} + 1 + 1/n} = \frac{1}{2} < 1$$

so the ratio test concludes that this series converges.

(c) **TODO**

(d) **TODO**

Exercise 3.7

I was going to do comparison for this, so take a few cases,

1. $a_i = 0$, then it is trivial that

$$\frac{\sqrt{a_n}}{n} = 0 \leq 0 = a_n$$

2. $a_i \geq 1$. Then we have $\sqrt{a_i} < a_i$, so we have

$$\frac{\sqrt{a_n}}{n} < a_n$$

3. $a_i < 1$. Then we have $\sqrt{a_i} < a_i$, but

$$\frac{\sqrt{a_n}}{n} \geq a_i \implies a_n < \frac{1}{n^2}$$

   In this case, we have

$$\sum \sqrt{a_n}n < \sum \frac{1}{n^3}$$

   which we know converges.

Therefore, by our 3 parts, we have 2 cases which converge because they are bounded above and the series is increasing, and the last case we showed that subsequence converges as it is also bounded by above.

Exercise 3.8
If we just use Theorem 3.42, but at the very last step, instead of setting b_N to be arbitrarily small, we can set it to some bound M_b since we are given that, and we instead have that

$$\left|\sum_{n=1}^{N} a_n b_n\right| = \left|\sum_{n=1}^{N-1} A_n(b_n - b_{n+1}) + A_N b_N - A_0 b_1\right| \leq M_A|2b_1| \leq 2M_A M_B$$

Actually this doesn't feel right... :(

Let me try again with showing that the partial sums form a Cauchy sequence.

We want to find some N given any $\epsilon > 0$ such that $n > m \geq N$ implies

$$\left|\sum_{i=m+1}^{n} a_i b_i\right| < \epsilon.$$

Since we know b_i is bounded, let's say by M_b, we can rewrite our expression as

$$\left|\sum_{i=m+1}^{n} a_i b_i\right| \leq M_b \left|\sum_{i=m+1}^{n} a_i\right|$$

We know that $\sum a_i$ converges, so just choose N such that $\sum_{i=m+1}^{n} < \frac{\epsilon}{M_b}$, so we have

$$\left|\sum_{i=m+1}^{n} a_i b_i\right| \leq M_b \left|\sum_{i=m+1}^{n} a_i\right| < M_b \cdot \frac{\epsilon}{M_b} = \epsilon.$$

I honestly don't feel like I learned this chapter that well, but I wanna start the next chapter, so here we go...

Chapter 4

Continuity

Short chapter, and I think pretty intuitive, except it does rely quite a bit on Chapter 2, so if you didn't learn that well...now is the time to review those definitions and theorems.

Stressing the convenience and importance of compactness for sets for certain continuity theorems was really cool, as without compactness, we saw that functions aren't able to behave as nicely as we wanted to. I think this is also the first time we see why it's useful to have the finite open cover for any open cover property of compact sets – we are able to use the finite-ness to use things like taking a minimum over a finite set for example.

There was a brief mention of adding ∞ to neighborhood definitions at the end, but don't sweat, all it's allowing us to do is now define the limit of some $f(t) \to \infty$, which was previously not defined.

4.1 Exercises

Exercise 4.1
For any $\epsilon > 0$, we have to find some δ-radius neighorhood around x such that

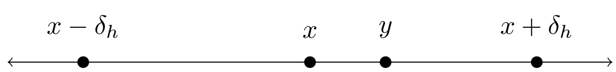
$$|f(x) - f(y)| < \epsilon,$$

for $y \in N_\delta(x)$.

By the definition of the limit, we can choose h such that $|h| < \delta_h$ and

$$|f(x+h) - f(x-h)| < \epsilon_h,$$

So it turns out from here, it's really hard to prove this, and this might hint that maybe this isn't true. To give some more insight into this, the situation we have now can be illustrated as



From the figure, we see the issue is that despite knowing the limit of the function of some endpoints goes to zero, we don't know anything about y. Something natural to try here is to choose h such that one of $x \pm \delta_h = y$, but the issue is that we can't do that, because we have to choose δ_h first.

Therefore, this leads us to think that trying to prove this is not possible, aided by the help that if $f(y)$ is just vastly different from $f(x \pm h)$, then we can make f discontinuous. One such function that works is a counterexample is $f(x) = 1$ if rational 0 otherwise.

Exercise 4.2
Suppose $x \in f(\overline{E})$. Then we can consider cases

- $x \in f(E)$: Then it is trivial that $x \in \overline{f(E)}$ because $f(E) \subset \overline{f(E)}$.

- $x \in \overline{E} \setminus E$: This means x is a limit point of E.

  If we consider $f(x)$, by continuity of f, we know that given any $\epsilon > 0$, we can find some $p \in N_\delta(x)$ such that $d_Y(f(x), f(p)) < \epsilon$. Since x is a limit point, there are infinite such p, which means there are infinite $f(p)$ that satisfy this ϵ as well. Since $p \in E, f(p) \in f(E)$, which means $f(x)$ is a limit point of $f(E)$.

  We can now conclude that $f(x) \in \overline{f(E)}$, so $f(E) \subset \overline{f(E)}$ in this case as well.

For a proper subset, we can consider $f(x) = 1/x, x \in (1, \infty)$. We notice that 0 is a limit point of $f(x)$ in this set, but even the closure $[1, \infty)$ won't contain this limit point.

Exercise 4.3
$Z(f)$ is the inverse image of 0, which is closed, so $Z(f)$ is closed as well.

Exercise 4.4
Since E is dense in X, we know that $\overline{E} = E \cup E' = X$, meaning that $f(X) = f(\overline{E}) \subset \overline{f(E)}$, from 4.2. We just need to prove $\overline{f(E)} \subset f(X)$ now. The case if we consider a non-limit point of $\overline{f(E)}$ is trivial since we know that $f(E) \subset f(X)$. If we take a limit point of $\overline{f(E)}$, AFSOC it is $\notin f(X)$, then this implies a contradiction since the limit point property shows that we have infinitely many points near this limit point, in $f(E)$, which we know is $\subset f(X)$, which means...**TODO**[1]

I think the direct proof might be easier, where you try to prove every point of $f(X)$ is a limit point of $f(E)$. We can conclude this because we know E is dense, so by continuity and limit point properties, we can conclude that for any $x \in X$, we can find close points $p \in E$ such that $|f(p) - f(x)| < \epsilon$, and this will show that $f(x)$ is a limit point for $f(E)$.

If we have $g(p) = f(p), p \in E$, then AFSOC $\exists p' \in X, g(p') \neq f(p')$. We know that p' is a limit point of E by density, so by continuity of f, g, we can find a $p \in N_\delta(p')$, such that $p \in E$ and $f(p)$ and $g(p)$ get arbitrarily close to $f(p')$ and $g(p')$ respectively. But for all of these p, we have that $f(p) = g(p)$, so we run into a contradiction that

$$|f(p') - g(p')| \leq |f(p') - f(p) - (g(p') - f(p))| \leq |f(p') - f(p)| + |g(p') - g(p)| < \epsilon$$

for appropriately chosen δ to make $|f(p') - f(p)| < \epsilon/2$ and same for g. This means we must conclude that $\forall p \in X, g(p) = f(p)$.

[1] idk why this is so difficult to prove the rest. I think I need to use continuity and density of E here.

Chapter 5

Differentiation

Short chapter here. The proofs are fairly straightforward and if you have some calculus background, this chapter should've have been too hard. The hardest proof for me was L'Hospital's Theorem, namely the step where we have

$$\frac{f(x)}{g(x)} < r - r\frac{g(y)}{g(x)} + \frac{f(y)}{g(x)}$$

and we conclude by letting $x \to a$, that we have

$$\frac{f(x)}{g(x)} < q.$$

This is confusing because when $x \to a$, and we assume $g(x) \to \infty$ as $x \to a$, I don't see how the equation reduces nicely. Meaning, it seems like $f(x)/g(x) \to 0$ as well, but maybe it's because we have that y is fixed, so we know *for sure* that

$$r\frac{g(y)}{g(x)} + \frac{f(y)}{g(x)}$$

all go to zero, so we are left with

$$\frac{f(x)}{g(x)} < r < q.$$

I've noticed the problems in this chapter tend to be pretty MVT (Mean Value Theorem) heavy, so make sure to pull that out when you can.

5.1 Exercises

Exercise 5.1
I think we can change this into

$$\frac{|f(x) - f(y)|}{x - y} \leq x - y$$

and if we let $y \to x$, then we have

$$|f'(x)| \leq 0$$

which implies that $f'(x) = 0$, for all x, and thus f is constant.

Exercise 5.2
If we have $f'(x) > 0, x \in (a, b)$, then it must be strictly increasing, because AFSOC $\exists x_1 < x_2 \in (a, b), f(x_1) \geq f(x_2)$, then we can show that $\exists x_3 \in (x_1, x_2)$ such that

$$f'(x_3) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq 0$$

by the mean value theorem, which contradicts the fact that $f'(x) > 0 \forall x \in (a, b)$.

Since $g(x) = f^{-1}(x)$, we have

$$\begin{aligned} g(f(x)) &= f^{-1}(f(x)) = x \\ g'(f(x))f'(x) &= 1 && \text{(product rule)} \\ g'(f(x)) &= \frac{1}{f'(x)} \end{aligned}$$

We only really needed $f'(x) \neq 0$ I believe, not the strictly increasing part.

Exercise 5.3

For injective (one-to-one) proofs we usually start with $f(x_1) = f(x_2)$ and try to prove that $x_1 = x_2$. If we do that here, we get

$$\begin{aligned} f(x_1) &= f(x_2) \\ x_1 + \epsilon g(x_1) &= x_2 + \epsilon g(x_2) \\ g'(x_1) &= g'(x_2) \end{aligned}$$

I'm honestly not sure what to do here, especially with the bounded derivative property of g.

TODO

Exercise 5.4

The solution for this is to choose

$$P(x) = \sum_{i=0}^{n} \frac{C_i}{i+1} x^{i+1}$$

and see that $P(0), P(1) = 0$ from the problem statement, and that $P'(0) = 0$ follows by the mean value theorem.

What I can add to this problem is that I was confused about the

$$P'(x) = C_0 + C_1 x + \cdots + C_n x_n = 0$$

It seems like by taking $x = 0$, we can show that $C_0 = 0$. Now, if we differentiate to get $P''(x)$, we can take $x = 0$ and show $C_1 = 0$ and so on...unsure why this is flawed[1].

Exercise 5.5

Applying MVT to $f(x+1) - f(x)$ we get

$$f(x+1) - f(x) = (x + 1 - x)f'(y)$$

for $y \in (x, x+1)$. Now, since we know that $\lim_{x\to\infty} f'(x) = 0$, we can choose some x so large so that we have this choice of y imply that $|f'(y)| < \epsilon$, meaning that $g(x) = f(x+1) - f(x) < \epsilon$, implying that $\lim_{x\to\infty} g(x) = 0$.

Exercise 5.6

Here is a solution sketch to find motivation.

If we want to show that g is monotonically increasing, we need to show that $g'(x) \geq 0$. Given the definition of g, what we end up getting is that

$$g'(x) = \frac{f'(x)x - f(x)}{x^2} ? \geq 0 x \cdot f'(x) ? \geq f(x)$$

Seeing we need a relationship between $f(x)$ and $f'(x)$, we can apply MVT to $f(x), f(0)$ and go from there. I had to look up the solution for this one, but I wanted to add in the motivation for the solution while I was trying to solve it.

[1] Maybe it's because I'm only considering $x = 0$? At other $x \neq 0$ values, maybe these constants don't have these properties? I guess we're also not given that equation is true for all x, we just want to know for which x we *definitely* know it is true for.

Exercise 5.7
This is pretty straightforward, we have

$$\frac{f'(x)}{g'(x)} = \lim_{t \to x} \frac{f(t) - f(x)}{t - x} \frac{t - x}{g(t) - g(x)} = \lim_{t \to x} \frac{f(t) - f(x)}{g(t) - g(x)} = \lim_{t \to x} \frac{f(t)}{g(t)}$$

The last step comes from the fact that $f(x), g(x) = 0$.

This holds for complex functions since we made no assumptions about R^1 operations here.

Exercise 5.8
This is probably not formal, but since we know that f' is continuous on $[a, b]$, we have that $f'(x)$ exists for $x \in [a, b]$, and that for small enough δ, we can get

$$\frac{f(t) - f(x)}{t - x}$$

arbitrarily close to $f'(x)$.

The question then comes, why can't we do this when f' *isn't* continuous, and I think it's because without continuity, we can't show that the expression in can be arbitrarily close to $f'(x)$.

Intuition says this holds for vector-valued functions since we have a finite number of coordinates and we just need to show the ϵ inequality holds for n dimensions.

Appendix A

Important Theorems and Definitions

There are a lot of important theorems in the book, but some stand out more than others.

A.1 Basic Topology

All of 2.18 are **essential** for this chapter and the book. Reproduced below for convenience

Definition A.1.1
Let X be a metric space. All points and sets mentioned below are understood to be elements and subsets of X.

(a) A *neighborhood* of p is a set $N_r(p)$ consisting of all q such that $d(p, q) < r$, for some $r > 0$. The number r is called the *radius* of $N_r(p)$.

(b) A point p is a *limit point* of the set E if *every* neighborhood of p contains a point $q \neq p$ such that $q \in E$.

(c) If $p \in E$ and p is not a limit point of E, then p is called an *isolated point* of E.

(d) E is *closed* if every limit point of E is a point of E.

(e) A point p is an *interior point* of E if there is a neighborhood N of p such that $N \subset E$.

(f) E is *open* if every point of E is an interior point of E.

(g) The *complement* of E (denoted by E^{c}) is the set of all points $p \in X$ such that $p \notin E$.

(h) E is *perfect* if E is closed and if every point of E is a limit point of E.

(i) E is *bounded* if there is a real number M and a point $q \in X$ such that $d(p, q) < M$ for all $p \in E$.

(j) E is *dense in* X if every point of X is a limit point of E, or a point of E (or both)[1].

A.2 Continuity

Definition A.2.1
Suppose X and Y are metric spaces, $E \subset X, p \in E$ and f maps E into Y. Then f is said to be *continuous* at p if for every $\epsilon > 0$ there exists a $\delta > 0$ such that

$$d_Y(f(x), f(p)) < \epsilon \tag{A.1}$$

for all points $x \in E$ for which $d_X(x, p) < \delta$.

[1] I often find it easier to read this as $X = \overline{E} = E \cup E'$

A.3 Differentiation

Definition A.3.1
Let f be defined (and real-valued) on $[a,b]$. For any $x \in [a,b]$, form the quotient

$$\Phi(t) = \frac{f(t) - f(x)}{t - x} \quad (t \in (a,b), t \neq x), \tag{A.2}$$

and define

$$f'(x) = \lim_{t \to x} \Phi(t). \tag{A.3}$$