Baby Rudin Solutions

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These solutions are for the 3rd edition *Principles of Mathematical Analysis* textbook by the legendary Walter Rudin.

I will do my best to provide some guidance along the way in this book.

Chapter 1

The Real and Complex Number Systems

The most important definition in this chapter is **1.10** about the *least-upper-bound property*, or the sup, and the corresponding inf. Without these concepts it's hard to prove anything else, so please pay attention to this definition!

Although you don't have to understand the R construction from Q perfectly, it is a good exercise. It is quite set-theory heavy, so if you haven't read *Naive Set Theory* by Paul Halmos, I would highly recommend give that book a skim.

I thought it was helpful to think the cuts in the construction as subsets of Q that kinda look like $(-\infty, r)$. This isn't super precise, but it should aid with the understanding quite a bit (the book should honestly include a diagram for this idea). I believe these cuts are known as Dedekind cuts in the real world.

Exercises

Exercise 1

Suppose they are trivial, then we have

(a) $r + x = q \in Q$ so $x = q - r \in Q$ which is a contradiction

(b) $rx = q \in Q$ so $x = \frac{q}{r} \in Q$ which is a contradiction

so we conclude both are irrational.

Exercise 2

This is the classic $\sqrt{2}$ is irrational proof, but you start with AFSOC $q \in Q$ such that

$$\begin{aligned} q^2 &= 12 \\ \left(\frac{m}{n}\right)^2 &= 12 && (m, n \in Z,\ m, n \text{ relatively prime}) \\ m^2 &= 12n^2 \end{aligned}$$

Now, m must be a multiple of 3, so let $m = 3k, k \in Z$.

$$m^2 = 9k^2 = 12n^2 3k^2 \qquad = 4n^2$$

and we conclude the same for n, which is a contradiction since we assume $\frac{m}{n}$ was represented as a fraction in simplest forms.

Exercise 3

These are pretty trivial.

Exercise 4
Pick an arbitrary $x \in E$, then we have $\alpha \leq x$ by lower bound definition and $x \leq \beta$ by upper bound definition. Putting these together, we get

$$\alpha \leq x \leq \beta \implies \alpha \leq \beta$$

Exercise 5
Let $\alpha = -\sup(-A) \implies -\alpha = \sup(-A)$. Then we know

$$\begin{aligned} -\alpha &> y && (y \in -A) \\ \alpha &< -y && (y \in -A) \\ \alpha &< y' && (y' = -y \in A) \end{aligned}$$

Therefore α is a lower bound for A. Now, because of the sup property of α on $-A$, we know $\nexists \beta$ such that

$$\beta > \alpha \text{ and } \beta < y \text{ for } y \in -A$$

which means α is the largest lower bound for A as well, so therefore we conclude $\alpha = \inf(A)$

Exercise 6

(a) I think the idea here is to show that $m/n = p/q$ AFSOC means $p = km, q = kn, k \in Q$, and then you say

$$\begin{aligned} (b^m)^{\frac{1}{n}} = A \implies b^m &= A^n && \text{(Theorem 1.21)} \\ b^{km} &= A^{kn} && \text{(Repeatedly multiply both sides } k \text{ times)} \\ (b^p)^{\frac{1}{q}} = \left(b^{km}\right)^{\frac{1}{kn}} &= A \end{aligned}$$

(b) Substitute fractions in for r, s and you can derive the rest with help from (a).

(c) Any $r' \leq r$ will have $b^{r'} \in B(r)$ by definition of $B(r)$. If $r' > r$, then $r' \notin B(r)$, hence we see that $b^r = \sup B(r)$.

(d) Add $B(x)$ and $B(y)$ and use (c).

Exercise 7
TODO

Exercise 8
$i^2 = -1 < 0$ so it is not ordered.

Exercise 9
TODO but you just run through the definition and verify.

Exercise 10
TODO

Exercise 11
We just have to set $r = |z|$ and divide z's coefficients by r to get w. These are unique because $|z|$ is unique.

Exercise 12
We can use induction to prove this, starting with the $n = 2$ case which is given by axioms.

Exercise 13
Start with triangle inequality and manipulate. I think you sould WLOG $|x| < |y|$ at some point when you try to get the $|x| - |y|$ term, so you can get your inequality.

Exercise 14

$$\begin{aligned}|1+z|^2+|1-z|^2 &= (1+z)(1+\bar{z})+(1-z)(1-\bar{z}) \\ &= 1+z+\bar{z}+z\bar{z}+1-z-\bar{z}+z\bar{z} \\ &= 4 && (z\bar{z}=1)\end{aligned}$$

You can also do this problem geometrically, realizing that $|z|=1$ means z is on the unit circle in the complex plane, and that

$$|1+z|^2+|1-z|^2 = |1-(-z)|^2+|1-z|^2 \tag{1.1}$$

which means we are finding the sum of square distances of $z,-z$ from 1 on the complex plane, which, since $z,-z$ are endpoints of a diameter, their sum distances are the sums of squares of two legs of a right triangle with the diameter as the hypotenuse, which has diameter length 2. We then conclude their square sums $= 2^2 = 4$.

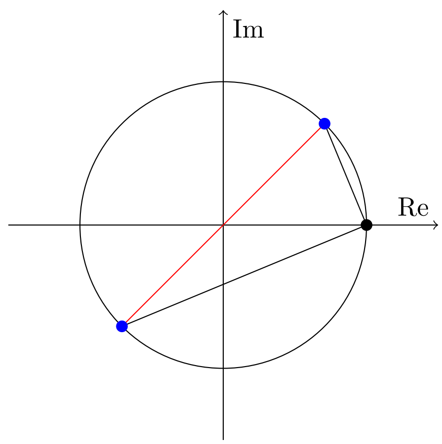

Figure 1.1: $z,\bar{z}$ and their square sum distance from 1 is the diameter

Exercise 15
Taking a look at our derivation, equality would happen when every $|Ba_j - Cb_j| = 0$.

Exercise 16

(a) For some reason I'm only getting k solutions here, since you have the locus of points that are r away from $\boldsymbol{x},\boldsymbol{y}$ respectively and you find their intersection.

(b) There is only one point that is equidistant from $\boldsymbol{x}$ and $\boldsymbol{y}$ and sums up to d, which is the midpoint of the two points.

(c) This is impossible because there is no point that can be $< r$ to both points but also cover the d distance.

Exercise 17
Proving equality is just an algebra exercise, use the conjugate definition.

Geometrically, we see that $|\boldsymbol{x}+\boldsymbol{y}|^2+|\boldsymbol{x}-\boldsymbol{y}|^2$ are the sum of squares of the diagonals, and the rest are the sums of squares of the side lengths. And we've proved they are equal.

Exercise 18
In $k=1$ this is not possible because $xy=0$ implies one of them is 0.

To some up with a general $\boldsymbol{y}$ for any k, observe that

$$\begin{aligned}
\boldsymbol{x} \cdot \boldsymbol{y} &= \sum_{i=1}^{k} x_i y_i \\
&= x_1 \cdot \frac{1}{x_1} + x_2 \cdot \frac{1}{x_2} + \cdots + x_k \cdot \frac{-(k-1)}{x_k} \\
&= k - 1 - (k-1) = 0
\end{aligned}$$

Geometrically, any perpendicular vector would have a dot product of 0.

Exercise 19
TODO

Exercise 20
TODO