Introductory Functional Analysis with Applications
Solutions

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Part I

Metric Spaces

Chapter 1

Metric Spaces

1.1 Metric Space

We spend this chapter introducing a metric space and building some intuition around it. From the definition below, if we can show they are true, then we have a metric space.

Definition 1 A **metric space** is a pair (X, d) where X is a set, and d is a *metric on* X (or *distance function on* X), that is, a function defined on $X \times X$ such that for all $x, y, z \in X$ we have:

1. d is real-valued, finite and nonnegative
2. $d(x, y) = 0$ if and only if $x = y$
3. $d(x, y) = d(y, x)$ (Symmetry)
4. $d(x, y) \leq d(x, z) + d(z, y)$ (**Triangle inequality**)

Exercise 1.1.1
Let's show that $\mathbb{R}$ is a metric space. Since the author didn't specify, I assume they mean the standard metric $d(x, y) = |x - y|$.

In this case, we can show

1. $d(x, y) = |x - y| \geq 0$, by definition of absolute value. It is finite, since x, y are finite, and you can bound $|x - y|$ by $|x| + |y|$ which is also finite. And it is $\in \mathbb{R}$ by definition of absolute value.
2. If $x = y$ then $d(x, y) = |x - y| = 0$. If $d(x, y) = 0$, then $|x - y| = 0$, which means $x - y = 0$, or $x = y$.
3. $d(x, y) = |x - y| = |y - x| = d(y, x)$.
4. $d(x, y) = |x - y| = |x - z + z - y| \leq |x - z| + |z - y| = d(x, z) + d(z, y)$

Exercise 1.1.2
Same as 1.1.1, but for $d(x, y) = (x - y)^2$.

Counterexample for Triangle Inequality: Let $x = 0, y = 2, z = 1$. Then, we have

$$\begin{gathered} d(x, y) = (2 - 0)^2 = 4 \\ d(x, z) + d(z, y) = (0 - 1)^2 + (1 - 2)^2 = 1 + 1 = 2 \\ 4 \not\leq 2 \end{gathered}$$

So it is **not a metric**.

Exercise 1.1.3
Same as 1.1.1, but for $d(x, y) = \sqrt{|x - y|}$.

We can show

1. $d(x,y) = \sqrt{|x-y|} \geq 0$, by definition of absolute value and square root. It is finite, since x, y are finite, and you can bound $\sqrt{|x-y|}$ by $\sqrt{|x|+|y|}$ which is also finite. And it is $\in \mathbb{R}$ by definition of absolute value and square root.

2. If $x = y$ then $d(x,y) = \sqrt{|x-y|} = 0$. If $d(x,y) = 0$, then $\sqrt{|x-y|} = 0$, which means $|x-y| = 0$, or $x - y = 0$, or $x = y$.

3. $d(x,y) = \sqrt{|x-y|} = \sqrt{|y-x|} = d(y,x)$.

4. $d(x,y) = \sqrt{|x-y|} = \sqrt{|x-z+z-y|} \leq \sqrt{|x-z|} + \sqrt{|z-y|} = d(x,z) + d(z,y)$
   The inequality step comes from the fact that $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$ for nonnegative a, b, because

$$(\sqrt{a} + \sqrt{b})^2 \,?\, (\sqrt{a+b})^2$$
$$a + b + 2\sqrt{ab} \geq a + b$$

So it is a metric.

Exercise 1.1.4
We need to find all metrics on a set X consisting of two points.

Let $X = \{a, b\}$ with $a \neq b$. Suppose we have the following metric:

$$d(x,y) = \begin{cases} 0 & x = y \\ c & x \neq y, c > 0 \end{cases}$$

We can verify 1, 2, 3, from 1 very easily. The main caveat, we needed $c > 0$ for 2, for $d(x,y) = 0$ iff $x = y$. Then, for 4 (Triangle Inequality), we can check all possible cases:

1. If $x = y$, then $d(x,y) = 0 \leq d(x,z) + d(z,y)$

2. if $x \neq y$, then we have two more cases
   (a) if $x = z$, then $z \neq y$ and we have $d(x,z) + d(z,y) = c \geq c = d(x,y)$
   (b) if $x \neq z$, then we have $z = y$ and similarly $d(x,z) + d(z,y) = c \geq c = d(x,y)$

If X consists of one point, then the only metric is the trivial one where $d(x,x) = 0$. This is because, if we have any other metric, then $d(x,x) > 0$ which violates property 2.

Exercise 1.1.5
Suppose d is a metric on X.

1. Suppose we have kd. Then this is still a metric if $k > 0$. We can verify the four properties:
   (a) $kd(x,y) \geq 0$ since $d(x,y) \geq 0$ and $k > 0$. It is finite since d is finite and k is a finite scalar.
   (b) $kd(x,y) = 0$ iff $d(x,y) = 0$ iff $x = y$.
   (c) $kd(x,y) = kd(y,x)$, we can just scale property 3
   (d) $kd(x,y) \leq kd(x,z) + kd(z,y)$ we can just scale property 4 the triangle inequality.

2. Suppose we have $d + k$. Then this is only a metric for $k = 0$. If $k \neq 0$ then property 2 is immediately dissatisfied, since $d(x,x) + k = k \neq 0$.

Exercise 1.1.6
We want to show the sequnce space l^∞ is a metric space.

1. $d(x,y) = \sup_{n \in \mathbb{N}} |x_n - y_n| \geq 0$ since absolute value is nonnegative. It is finite since $x, y \in l^\infty$ means they are bounded sequences by $c_x \in \mathbb{R}$, so their difference is also bounded. And it is $\in \mathbb{R}$ since absolute value produces real numbers.

2. If $x = y$, then $d(x, y) = \sup |x_n - y_n| = 0$. If $d(x, y) = 0$, then $\sup |x_n - y_n| = 0$, which means for all n, $|x_n - y_n| = 0$, or $x_n = y_n$ for all n, so $x = y$.
3. $d(x, y) = \sup |x_n - y_n| = \sup |y_n - x_n| = d(y, x)$.
4. $d(x, y) = \sup |x_n - y_n| = \sup |x_n - z_n + z_n - y_n| \leq \sup |x_n - z_n| + \sup |z_n - y_n| = d(x, z) + d(z, y)$.

Exercise 1.1.7
Supposed A is the subset of l^∞ where each sequence is composed of 0 or 1.

The induced metric then behaves like this:

$$d(x, y) = \begin{cases} 0 & x = y \\ 1 & x \neq y \end{cases}$$

Exercise 1.1.8
We have X as the set of all real-valued functions, and are defined and continuous on a closed interval $J = [a, b]$. We now choose the metric

$$d(x, y) = \int_a^b |x(t) - y(t)| \,\mathrm{d}t \tag{1.1}$$

Let's check this is a metric

1. $d(x, y) = \int_a^b |x(t) - y(t)| \,\mathrm{d}t \geq 0$ since absolute value is nonnegative. It is finite since x, y are continuous on a closed interval, so they are bounded, and the integral of a bounded function over a finite interval is finite. And it is $\in \mathbb{R}$ since the integral of a real-valued function is real.
2. If $x = y$, then $d(x, y) = \int_a^b |x(t) - y(t)| \,\mathrm{d}t = \int_a^b |0| \,\mathrm{d}t = 0$. If $d(x, y) = 0$, then $\int_a^b |x(t) - y(t)| \,\mathrm{d}t = 0$, which implies $|x(t) - y(t)| = 0$ for all $t \in [a, b]$, so $x(t) = y(t)$ for all $t \in [a, b]$.
3. $d(x, y) = \int_a^b |x(t) - y(t)| \,\mathrm{d}t = \int_a^b |y(t) - x(t)| \,\mathrm{d}t = d(y, x)$.
4. $d(x, y) = \int_a^b |x(t) - y(t)| \,\mathrm{d}t = \int_a^b |x(t) - z(t) + z(t) - y(t)| \,\mathrm{d}t \leq \int_a^b |x(t) - z(t)| \,\mathrm{d}t + \int_a^b |z(t) - y(t)| \,\mathrm{d}t = d(x, z) + d(z, y)$.

Exercise 1.1.9
We want to show that the discrete metric for X is a metric, this is

$$d(x, x) = 0, d(x, y) = 1 \text{ for } x \neq y \tag{1.2}$$

1. $d(x, y)$ is either 0 or 1, so it is nonnegative, finite, and real-valued.
2. By definition, $d(x, y) = 0$ iff $x = y$.
3. By definition, $d(x, y) = d(y, x)$.
4. For the triangle inequality, we have two cases:
   (a) If $x = y$, then $d(x, y) = 0 \leq d(x, z) + d(z, y)$
   (b) If $x \neq y$, then we have two more cases
      i. if $x = z$, then $z \neq y$ and we have $d(x, z) + d(z, y) = 1 \geq 1 = d(x, y)$
      ii. if $x \neq z$, then we have $z = y$ and similarly $d(x, z) + d(z, y) = 1 \geq 1 = d(x, y)$

Exercise 1.1.10
(Hamming distance) Let X be all set of ordered triples of 0s and 1s.

We immediately see that X has $2^3 = 8$ elements.

Now, $d(x, y) =$ number of positions where x and y differ.

1. $d(x, y) = 0, 1, 2, 3$ so it is nonnegative, finite, and real-valued.

2. If $x = y$, then they differ in 0 positions, so $d(x, y) = 0$. If $d(x, y) = 0$, then they differ in 0 positions, so x and y must be the same in all positions, so $x = y$.

3. The number of positions where x and y differ is the same as the number of positions where y and x differ, so $d(x, y) = d(y, x)$.

4. For the triangle inequality, we can prove this my induction using the discrete metric space, as the base case. Then, suppose we have n-tuples, and we add one more position to each tuple to make it an $(n + 1)$-tuple. There are two cases:

   (a) If the new position is the same for all three tuples, then the number of differing positions is unchanged, so the triangle inequality holds by the inductive hypothesis.

   (b) If the new position is different for at least one of the tuples, then the left-hand side of the triangle inequality increases by at most 1, and the right-hand side also increases by at least 1, because in z, its new position must be different from at least x or y.

Exercise 1.1.11
We want to prove the generalized triangle inequality:

$$d(x_1, x_n) \leq d(x_1, x_2) + d(x_2, x_3) + \cdots + d(x_{n-1}, x_n) \tag{1.3}$$

We can do this by induction.

Base case: $n = 2$ is just the regular triangle inequality. Inductive step: Suppose it is true for $n = k$. Then, for $n = k + 1$, we have

$$\begin{aligned} d(x_1, x_{k+1}) &\leq d(x_1, x_k) + d(x_k, x_{k+1}) && \text{(Triangle Inequality)} \\ &\leq d(x_1, x_2) + d(x_2, x_3) + \cdots + d(x_{k-1}, x_k) + d(x_k, x_{k+1}) && \text{(Inductive Hypothesis)} \end{aligned}$$

Thus, by induction, the generalized triangle inequality holds for all $n \geq 2$.

Exercise 1.1.12
We can see that

$$\begin{aligned} |d(x, y) - d(z, w)| &\leq d(x, z) + d(y, w) \\ d(x, y) - d(z, w) &\leq d(x, z) + d(y, w) && \text{(Case 1, we'll do the other case next)} \\ d(x, y) &\leq d(x, z) + d(y, w) + d(z, w) \\ d(x, y) &\leq d(x, z) + d(w, y) + d(z, w) && \text{(Symmetry)} \\ d(x, y) &\leq d(x, z) + d(z, w) + d(w, y) && \text{(Triangle Inequality)} \end{aligned}$$

We need to do the other case as well:

$$\begin{aligned} d(z, w) - d(x, y) &\leq d(x, z) + d(y, w) \\ d(z, w) &\leq d(x, z) + d(y, w) + d(x, y) \\ d(z, w) &\leq d(z, x) + d(x, y) + d(y, w) && \text{(Symmetry)} \\ d(z, w) &\leq d(z, x) + d(x, y) + d(y, w) && \text{(Triangle Inequality)} \end{aligned}$$

Exercise 1.1.13

$$\begin{aligned} |d(x, z) - d(y, z)| &\leq d(x, y) \\ d(x, z) - d(y, z) &\leq d(x, y) && \text{(Case 1, we'll do the other case next)} \\ d(x, z) &\leq d(x, y) + d(y, z) && \text{(Triangle Inequality)} \end{aligned}$$

Let's do the other case:

$$
\begin{aligned}
d(y,z) - d(x,z) &\leq d(x,y) \\
d(y,z) &\leq d(x,y) + d(x,z) \\
d(y,z) &\leq d(y,x) + d(x,z) && \text{(Symmetry)} \\
d(y,z) &\leq d(y,x) + d(x,z) && \text{(Triangle Inequality)}
\end{aligned}
$$

Exercise 1.1.14

We want to show that properties 2-4 in the 1 definition imply that the metric must be nonnegative.

AFSOC that there exists some $x, y \in X$ such that $d(x,y) < 0$. Then, by property 4, we have

$$d(x,x) \leq d(x,y) + d(y,x)$$

By property 3, we have $d(y,x) = d(x,y)$, so

$$d(x,x) \leq 2d(x,y)$$

But by property 2, we have $d(x,x) = 0$, so

$$0 \leq 2d(x,y)$$

However, we assumed $d(x,y) < 0$, so $2d(x,y) < 0$, which is a contradiction. Therefore, our assumption is false, and we conclude that for all $x, y \in X$, $d(x,y) \geq 0$.

1.2 Further Examples of Metric Spaces

The l^p proof is a bit annoying, I had to read through it carefully to really get it.

We have $p \in \mathbb{R}$, and $x = (\xi_j) = (\xi_1, \xi_2, \dots)$ such that

$$\sum_{j=1}^{\infty} |\xi_j|^p < \infty. \qquad (p \geq 1)$$

and the metric is defined by

$$d(x, y) = \left(\sum_{j=1}^{\infty} |\xi_j - \eta_j|^p \right)^{1/p}. \tag{1.4}$$

For $p = 2$, we have the Hilbert space l^2.

We now want to prove that l^p is a metric space.

PROOF Properties 1-3 are pretty straightforward, so we really just need to show convergence and the triangle inequality.

We are starting with a auxiliary inequality

$$\frac{1}{p} + \frac{1}{q} = 1 \tag{1.5}$$

From this inequality, where p, q are **conjugate exponents**, we can derive $1/(p-1) = q - 1$. Therefore, we can say

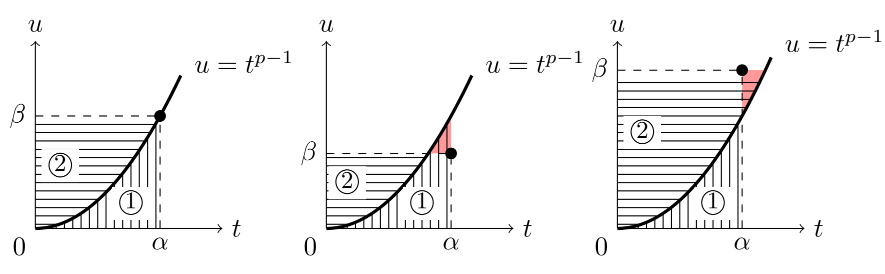
$$u = t^{p-1} \implies t = u^{1/(p-1)} = u^{q-1}. \tag{1.6}$$

Now, suppose $\alpha, \beta \geq 0$. Since $\alpha\beta$ is the area of a rectangle in 1.1, we can bound it above by the integral curves. The intuition here is, we want to consider the area under the curve, vertically and horizontally. We have 3 cases:

1. If the corner (α, β) is exactly at the corner of the rectangle, then the rectangle area is exactly the sum of the two areas.

2. If the end of the curve at α is above β, then the rectangle is smaller than the sum of the two areas, because the curve has more area above the β line.

3. If the end of the curve at α is below β, then the rectangle is still smaller than the sum of the two areas, since it has more horizontal area past α.



Case 1: $\beta = \alpha^{p-1}$ (equality)

Case 2: $\beta < \alpha^{p-1}$ (curve above β)

Case 3: $\beta > \alpha^{p-1}$ (curve below β)

Figure 1.1: Integral inequality cases. The red areas are the excess

The diagrams in 1.1 illustrate these cases. Now, we can derive

$$\alpha\beta \leq \int_0^\alpha t^{p-1}\,\mathrm{d}t + \int_0^\beta u^{q-1}\,\mathrm{d}u = \frac{\alpha^p}{p} + \frac{\beta^q}{q}. \tag{1.7}$$

Now, suppose we have sequences $(\xi_j), (\eta_j)$, such that

$$\sum_{j=1}^\infty |\xi_j|^p = 1, \quad \sum_{j=1}^\infty |\eta_j|^q = 1. \tag{1.8}$$

From (1.7), and summing over j, we have

$$\sum_{j=1}^\infty |\xi_j \eta_j| \leq \sum_{j=1}^\infty \left(\frac{|\xi_j|^p}{p} + \frac{|\eta_j|^q}{q} \right) = \frac{1}{p} + \frac{1}{q} = 1. \tag{1.9}$$

Now, if we want to choose any arbitrary sequences $(\xi_j), (\eta_j)$, we can normalize them by defining

$$\tilde{\xi_j} = \frac{\xi_j}{\left(\sum_{k=1}^\infty |\xi_k|^p\right)^{1/p}}, \quad \tilde{\eta_j} = \frac{\eta_j}{\left(\sum_{k=1}^\infty |\eta_k|^q\right)^{1/q}}, \tag{1.10}$$

and we can now substitute into (1.9) to get

$$\begin{aligned}
\sum_{j=1}^\infty |\xi_j \eta_j| &\leq 1 \\
\sum_{j=1}^\infty \left| \frac{\xi_j}{\left(\sum_{k=1}^\infty |\xi_k|^p\right)^{1/p}} \cdot \frac{\eta_j}{\left(\sum_{k=1}^\infty |\eta_k|^q\right)^{1/q}} \right| &\leq 1 && \text{(Substitute normalizations)} \\
\sum_{j=1}^\infty |\xi_j \eta_j| &\leq \left(\sum_{k=1}^\infty |\xi_k|^p \right)^{1/p} \left(\sum_{k=1}^\infty |\eta_k|^q \right)^{1/q}.
\end{aligned}$$

This now gives us **Hölder's inequality**.

We now need to prove **Minkoswki's inequality**, which is the triangle inequality for l^p. We start by using $\omega_j = \xi_j + \eta_j$. Then, what we can do is use the triangle inequality to get

$$\begin{aligned}
|\omega_j| &\leq |\xi_j + \eta_j| \\
|\omega_j|^p &\leq |\xi_j + \eta_j||\omega_j|^{p-1} \\
|\omega_j|^p &\leq (|\xi_j| + |\eta_j|)|\omega_j|^{p-1} && \text{(Triangle inequality)} \\
\sum |\omega_j|^p &\leq \sum |\xi_j||\omega_j|^{p-1} + \sum |\eta_j||\omega_j|^{p-1} && \text{(Applying sum)}
\end{aligned}$$

Now we can apply Hölder's inequality to each of the sums on the right-hand side, where we have conjugate exponents p and q, which gives us:

$$\begin{aligned}
\sum |\omega_j|^p &\leq \left(\sum |\xi_j|^p\right)^{1/p} \left(\sum |\omega_j|^{(p-1)q}\right)^{1/q} + \left(\sum |\eta_j|^p\right)^{1/p} \left(\sum |\omega_j|^{(p-1)q}\right)^{1/q} \\
&\leq \left(\sum |\xi_j|^p\right)^{1/p} \left(\sum |\omega_j|^p\right)^{1/q} + \left(\sum |\eta_j|^p\right)^{1/p} \left(\sum |\omega_j|^p\right)^{1/q} && \text{(Since } (p-1)q = p\text{)} \\
&\leq \left[\left(\sum |\xi_j|^p\right)^{1/p} + \left(\sum |\eta_j|^p\right)^{1/p} \right] \left(\sum |\omega_j|^p\right)^{1/q}
\end{aligned}$$

Finally, dividing both sides by $\left(\sum |\omega_j|^p\right)^{1/q}$:

$$\left(\sum |\omega_j|^p\right)^{1-1/q} \leq \left(\sum |\xi_j|^p\right)^{1/p} + \left(\sum |\eta_j|^p\right)^{1/p}$$

Since $1 - \frac{1}{q} = \frac{1}{p}$ (from $\frac{1}{p} + \frac{1}{q} = 1$), we have

$$\left(\sum |\xi_j + \eta_j|^p\right)^{1/p} \leq \left(\sum |\xi_j|^p\right)^{1/p} + \left(\sum |\eta_j|^p\right)^{1/p} \tag{1.11}$$

which is **Minkowski's inequality**.

Finally, we need to show the triangle inequality for the metric.

We take $x = (\xi_j), y = (\eta_j), z = (\zeta_j)$, and we have

$$\begin{aligned} d(x,z) &= \left(\sum |\xi_j - \zeta_j|^p\right)^{1/p} \\ &= \left(\sum |\xi_j - \eta_j + \eta_j - \zeta_j|^p\right)^{1/p} \\ &\leq \left(\sum |\xi_j - \eta_j|^p\right)^{1/p} + \left(\sum |\eta_j - \zeta_j|^p\right)^{1/p} && \text{(Minkowski's inequality)} \\ &= d(x,y) + d(y,z). \end{aligned}$$

This completes the proof that l^p is a metric space.

Exercise 1.2.1
if we have μ_j such that $\sum \mu_j$ converges instead of $1/2^j$, then we just need to show

$$d(x,y) = \sum_{j=1}^{\infty} \mu_j \frac{|\xi_j - \eta_j|}{1 + |\xi_j - \eta_j|}$$

is a metric. The properties 1-3 are easy, we just need to show convergence. But this is easy since $\frac{|\xi_j - \eta_j|}{1+|\xi_j - \eta_j|} \leq 1$, so we have

$$d(x,y) \leq \sum_{j=1}^{\infty} \mu_j,$$

which converges by assumption.

Exercise 1.2.2
Suppose we have $\alpha, \beta > 0$. Then from (1.7), we have

$$\alpha\beta \leq \frac{\alpha^p}{p} + \frac{\beta^q}{q}.$$

If we plug in $p = q = 2$, we have

$$\begin{aligned} \alpha\beta &\leq \frac{\alpha^2}{2} + \frac{\beta^2}{2} \\ \frac{\alpha\beta}{2} &\leq \frac{\alpha^2 + \beta^2}{4} \\ \alpha\beta &\leq \frac{\alpha^2 + 2\alpha\beta + \beta^2}{4} \\ \sqrt{\alpha\beta} &\leq \frac{\alpha + \beta}{2}. \end{aligned}$$

Exercise 1.2.3
Let's start with the Cauchy-Schwarz inequality for sums:

$$\sum_{j=1}^{n} |\xi_j \eta_j| \leq \sqrt{\sum_{k=1}^{n} |\xi_k|^2} \sqrt{\sum_{m=1}^{n} |\eta_m|^2}$$

Now, choose

$$\eta_j = \begin{cases} 1 & j = 1, 2, \dots, n \\ 0 & \text{otherwise} \end{cases} \tag{1.12}$$

Then, if we square both sides of the inequality, we get

$$\left(\sum_{j=1}^{n} |\xi_j| \right)^2 \leq n \sum_{k=1}^{n} |\xi_k|^2.$$

Exercise 1.2.4
Consider the sequence $\xi_n = \frac{1}{\log(n+1)}$.

Converges to 0: Since $\log(n+1) \to \infty$ as $n \to \infty$, we have $\xi_n \to 0$.

Not in any l^p: For any $p \geq 1$, we show that

$$\sum_{n=1}^{\infty} |\xi_n|^p = \sum_{n=1}^{\infty} \frac{1}{(\log(n+1))^p} = \infty.$$

Since $\log(n+1)$ grows slower than any positive power of n, for large n we have

$$\frac{1}{(\log(n+1))^p} > \frac{1}{n}$$

and the harmonic series $\sum 1/n$ diverges. By comparison, $\sum \frac{1}{(\log(n+1))^p}$ diverges for all $p \geq 1$.

Therefore $(\xi_n) = \left(\frac{1}{\log(n+1)} \right)$ converges to 0 but is not in any lp space.

Exercise 1.2.5
Consider the sequence $\xi_n = \frac{1}{n}$.

In l^p for $p > 1$:

$$\sum_{n=1}^{\infty} |\xi_n|^p = \sum_{n=1}^{\infty} \frac{1}{n^p} < \infty$$

since the p-series converges for $p > 1$.

Not in l^1:

$$\sum_{n=1}^{\infty} |\xi_n| = \sum_{n=1}^{\infty} \frac{1}{n} = \infty$$

since the harmonic series diverges.

Exercise 1.2.6
AFSOC $A \subset B$ and $\delta(A) > \delta(B)$.

Then, $\exists x, y \in A$ such that $d(x, y) > d(x', y')$ for all $x', y' \in B$. But this is a contradition, since $A \subset B$, so all $x, y \in A$ are also in B. Therefore, we must have $\delta(A) \leq \delta(B)$.

Exercise 1.2.7
If $\delta(A) = 0$, then $\forall x, y \in A$, we have $d(x, y) = 0$. By property 2 of metric spaces, this implies $x = y$, so A contains a single point.

Suppose A contains a single point, then $\forall x, y \in A$, we have $x = y$, so $d(x, y) = 0$. Thus, $\delta(A) = 0$.

Exercise 1.2.8
To explain D in plain English, it is the minimum distance of two points coming from two sets A, B. Now, the question is, does D define a metric on the power set of X?

We see that, property 2 is violated, because as long as A and B share a common point, we have $D(A, B) = 0$, but A and B could be different sets.

Exercise 1.2.9
The $\emptyset$ in the text looks like ϕ (phi). Anyways, if $A \cap B \neq \emptyset$, then $\exists x \in A \cap B$. This means that $\exists x \in A, \in B$, so $d(x, x) = 0$, so $D(A, B) = 0$.

Now the converse, if $D(A, B) = 0$, then $\inf\{d(x, y) : x \in A, y \in B\} = 0$. Suppose $A \cap B = \emptyset$, then $d(x, y) > 0$. But we can choose $\epsilon = d(x, y)/2$, which leads to a contradiction. Thus, we must have $A \cap B \neq \emptyset$

Exercise 1.2.10
For any $x, y \in X$, $b \in B$, we have from the triangle inequality that

$$d(x, b) \leq d(x, y) + d(y, b)$$

Taking the infimum over $b \in B$ on both sides:

$$\inf_{b \in B} d(x, b) \leq \inf_{b \in B} [d(x, y) + d(y, b)]$$

Since $d(x, y)$ does not depend on b, we can pull it out of the infimum:

$$D(x, B) \leq d(x, y) + \inf_{b \in B} d(y, b) = d(x, y) + D(y, B)$$

Therefore $D(x, B) - D(y, B) \leq d(x, y)$.

Now, applying the same argument but starting with $d(y, b) \leq d(y, x) + d(x, b)$:

$$D(y, B) \leq d(y, x) + D(x, B) = d(x, y) + D(x, B)$$

Therefore $D(y, B) - D(x, B) \leq d(x, y)$.

Combining both inequalities:

$$|D(x, B) - D(y, B)| \leq d(x, y)$$

Exercise 1.2.11
We want to show that

$$\tilde{d}(x, y) = \frac{d(x, y)}{1 + d(x, y)} \tag{1.13}$$

is still a metric.

Let's walk through the four properties:

1. Since $d(x, y) \geq 0$, we have $\tilde{d}(x, y) \geq 0$. It is real valued because d is, and it's finite because d is finite, and $\tilde{d}(x, y)$ is bounded by $(0, 1)$.

2. If $\tilde{d}(x, y) = 0$, then $d(x, y) = 0$, which implies $x = y$. Conversely, if $x = y$, then $d(x, y) = 0$, so $\tilde{d}(x, y) = 0$.

3. $\tilde{d}(x, y) = \frac{d(x,y)}{1+d(x,y)} = \frac{d(y,x)}{1+d(y,x)} = \tilde{d}(y, x)$.

4. For the triangle inequality, we can show

$$\begin{aligned}
\tilde{d}(x, z) &= \frac{d(x, z)}{1 + d(x, z)} \\
&\leq \frac{d(x, y) + d(y, z)}{1 + d(x, y) + d(y, z)} && \text{(By triangle inequality)} \\
&\leq \frac{d(x, y)}{1 + d(x, y) + d(y, z)} + \frac{d(y, z)}{1 + d(x, y) + d(y, z)} \\
&\leq \frac{d(x, y)}{1 + d(x, y)} + \frac{d(y, z)}{1 + d(y, z)} \\
&\text{(Since } d(x, y), d(y, z) \geq 0\text{, this makes the denom smaller, thus the fraction bigger)} \\
&\leq \tilde{d}(x, y) + \tilde{d}(y, z)
\end{aligned}$$

Exercise 1.2.12
Since A, B are bounded sets, we know $\exists p_A, p_B$ such that $\forall x \in A, y \in B$, $\exists M_A, M_B \in \mathbb{R}$ such that

$$d(x, p_A) \leq M_A, \quad d(y, p_B) \leq M_B.$$

Let's choose p_A to be our anchor point. Then, for any $p \in A \cup B$, we have two cases

1. $p \in A$, then $d(p, p_A) \leq M_A$ by definition.
2. $p \in B$, then we can do:

$$\begin{aligned} d(p, p_A) &\leq d(p, p_B) + d(p_B, p_A) && \text{(Triangle inequality)} \\ &\leq M_B + d(p_B, p_A) && \text{(Since } p \in B\text{)} \\ &\leq M_B + K \end{aligned}$$

where $K = d(p_B, p_A)$ is a constant, and this is true because d is a metric and is finite and real-valued, and $p_B, p_A \in X$ (the overall metric space).

Therefore, we can bound $A \cup B$ by $\max(M_A, M_B + K)$, so $A \cup B$ is bounded.

Exercise 1.2.13
We have the metric for the $X = X_1 \times X_2$ space defined as

$$d(x, y) = d_1(x_1, y_1) + d_2(x_2, y_2).$$

We want to show that this is a metric.

1. Real, ≥ 0, finite, follows from d_1, d_2 are both real-valued and non-negative and finite.
2. $d(x, y) = 0$ implies $d_1(x_1, y_1) + d_2(x_2, y_2) = 0$. Since both $d_1, d_2 \geq 0$, we must have $d_1(x_1, y_1) = 0$ and $d_2(x_2, y_2) = 0$, which implies $x_1 = y_1$ and $x_2 = y_2$, so $x = y$. Conversely, if $x = y$, then $d(x, y) = d_1(x_1, y_1) + d_2(x_2, y_2) = 0$.
3. Symmetry follows from the symmetry of d_1, d_2:

$$\begin{aligned} d(x, y) &= d_1(x_1, y_1) + d_2(x_2, y_2) \\ &= d_1(y_1, x_1) + d_2(y_2, x_2) \\ &= d(y, x). \end{aligned}$$

4. Triangle inequality:

$$\begin{aligned} d(x, z) &= d_1(x_1, z_1) + d_2(x_2, z_2) \\ &\leq d_1(x_1, y_1) + d_1(y_1, z_1) + d_2(x_2, y_2) + d_2(y_2, z_2) && \text{(Triangle inequality for } d_1, d_2\text{)} \\ &= d(x, y) + d(y, z). \end{aligned}$$

Exercise 1.2.14
We want to show the same as the previous exercise, except for the metric defined as

$$d(x, y) = \sqrt{d_1(x_1, y_1)^2 + d_2(x_2, y_2)^2}.$$

The first three properties are straightforward from the properties of d_1, d_2. For the triangle inequality, we can use Minkowski's inequality:

$$\begin{aligned} d(x, z) &= \sqrt{d_1(x_1, z_1)^2 + d_2(x_2, z_2)^2} \\ &\leq \sqrt{\left(d_1(x_1, y_1) + d_1(y_1, z_1)\right)^2 + \left(d_2(x_2, y_2) + d_2(y_2, z_2)\right)^2} && \text{(Triangle inequality for } d_1, d_2\text{)} \\ &\leq \sqrt{d_1(x_1, y_1)^2 + d_2(x_2, y_2)^2} + \sqrt{d_1(y_1, z_1)^2 + d_2(y_2, z_2)^2} && \text{(Minkowski's inequality)} \\ &= d(x, y) + d(y, z). \end{aligned}$$

Exercise 1.2.15
We want to show the metric space defined by

$$d(x, y) = \max(d_1(x_1, y_1), d_2(x_2, y_2))$$

is a metric. The first three properties are straightforward from the properties of d_1, d_2. For the triangle inequality, we have

$$\begin{aligned}
d(x, z) &= \max(d_1(x_1, z_1), d_2(x_2, z_2)) \\
&\leq \max(d_1(x_1, y_1) + d_1(y_1, z_1), d_2(x_2, y_2) + d_2(y_2, z_2)) && \text{(Triangle inequality for } d_1, d_2\text{)} \\
&\leq \max(d_1(x_1, y_1), d_2(x_2, y_2)) + \max(d_1(y_1, z_1), d_2(y_2, z_2)) && \text{(Properties of max)} \\
&= d(x, y) + d(y, z).
\end{aligned}$$

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Chapter 2

Normed Spaces. Banach Spaces

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Part II

Banach Spaces

Chapter 3

Inner Product Spaces. Hilbert Spaces

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Chapter 4

Fundamental Theorems for Normed and Banach Spaces

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Chapter 5

Further Applications: Banach Fixed Point Theorem

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Part III

Hilbert Spaces

Chapter 6

Further Applications: Approximation Theory

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Chapter 7

Spectral Theory of Linear Operators in Normed Spaces

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Chapter 8

Compact Linear Operators on Normed Spaces and Their Spectrum

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Part IV

Operators on Hilbert Spaces

Chapter 9

Spectral Theory of Bounded Self-Adjoint Linear Operators

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Chapter 10

Unbounded Linear Operators in Hilbert Space

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Chapter 11

Unbounded Linear Operators in Quantum Mechanics

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Appendix A

Extras

A.1 Important Definitions

Definition 2 (Metric Space) A **metric space** is a pair (X, d) where X is a set, and d is a *metric on* X (or *distance function on* X), that is, a function defined on $X \times X$ such that for all $x, y, z \in X$ we have:

1. d is real-valued, finite and nonnegative
2. $d(x, y) = 0$ if and only if $x = y$
3. $d(x, y) = d(y, x)$ (Symmetry)
4. $d(x, y) \leq d(x, z) + d(z, y)$ (Triangle inequality)

A.2 Important Inequalities

Theorem 1 (Triangle Inequality) *For any metric space* (X, d) *and points* $x, y, z \in X$*:*

$$d(x, z) \leq d(x, y) + d(y, z)$$

For real or complex numbers:

$$|a + b| \leq |a| + |b|$$

Theorem 2 (Hölder's Inequality) *Let* $p, q > 1$ *be conjugate exponents. For sequences* $(\xi_j), (\eta_j) \in \ell^p$ *and* ℓ^q *respectively:*

$$\sum_{j=1}^{\infty} |\xi_j \eta_j| \leq \left(\sum_{j=1}^{\infty} |\xi_j|^p \right)^{1/p} \left(\sum_{j=1}^{\infty} |\eta_j|^q \right)^{1/q}$$

For functions $f \in L^p$ *and* $g \in L^q$*:*

$$\int |fg| \leq \|f\|_p \|g\|_q$$

Theorem 3 (Minkowski's Inequality) *Let* $p \geq 1$*. For sequences* $(\xi_j), (\eta_j) \in \ell^p$*:*

$$\left(\sum_{j=1}^{\infty} |\xi_j + \eta_j|^p \right)^{1/p} \leq \left(\sum_{j=1}^{\infty} |\xi_j|^p \right)^{1/p} + \left(\sum_{j=1}^{\infty} |\eta_j|^p \right)^{1/p}$$

For functions $f, g \in L^p$*:*

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p$$

This is the triangle inequality for ℓ^p *and* L^p *norms.*