Munkres Topology Solutions

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These solutions are for the 2^{nd} edition Topology textbook by Munkres.

Chapter 1

Set Theory and Logic

1.1 Fundamental Concepts

Exercise 1.1.1
We will check $\cup, \cap$ in DeMorgan's laws.

Let's use

- $A = \{1, 2, 3, 4\}$
- $B = \{-1, 2, 3, 5\}$
- $C = \{3, 9, 11\}$

Check

$$\begin{aligned} A - (B \cup C) &= \{1, 2, 3, 4\} - \{-1, 2, 3, 5, 9, 11\} \\ &= \{1, 4\} \\ &= (A - B) \cap (A - C) \\ &= \{1, 4\} \cap \{1, 2, 4\} = \{1, 4\} \end{aligned}$$

$$\begin{aligned} A - (B \cap C) &= \{1, 2, 3, 4\} - \{3\} \\ &= \{1, 2, 4\} \\ &= (A - B) \cup (A - C) \\ &= \{1, 4\} \cup \{1, 2, 4\} = \{1, 2, 4\} \end{aligned}$$

Exercise 1.1.2

(a) $\Longrightarrow$ is true. $\Longleftarrow$ is not true, consider $A = \{1, 2, 3\}, B = \{1, 3\}, C = \{2\}$.

(b) $\Longrightarrow$ is true. $\Longleftarrow$ is not true, consider $A = \{1, 2, 3\}, B = \{1, 3\}, C = \{2\}$.

(c) True.

(d) $\Longrightarrow$ is not true. Consider $A = \{1\} \subset B = \{1, 2\}, C = \emptyset$. $\Longleftarrow$ is true.

(e) Not true. Consider $A = \{1\}, B = \{2\}$. I think $\subset$ works.

(f) Not true. Consider $A = \{1, 2\}, B = \{2, 3\}$. LHS is equivalent to A, so this should be $\supset$.

(g) True.

(h) $\supset$

(i) True.

(j) True.

(k) Not true, if $A = \emptyset$ for example, we have $(A \times B) \subset (C \times D) = \emptyset \subset (C \times D)$, but we can set B to whatever and this statement is still true, so we can make B have an element that is not in D, and therefore $B \not\subset D$.

(l) True.

(m) $\subset$

(n) $\subset$

(o) True.

(p) I think this is true at first glance...at least $\subset$ looks good.

(q) $\supset$

Exercise 1.1.3

(a)
- **Original**: If $x < 0$ then $x^2 - x > 0$. True.
- **Contrapositive**: If $x^2 - x \leq 0$ then $x \geq 0$. True.
- **Converse**: If $x^2 - x > 0$ then $x < 0$. False.

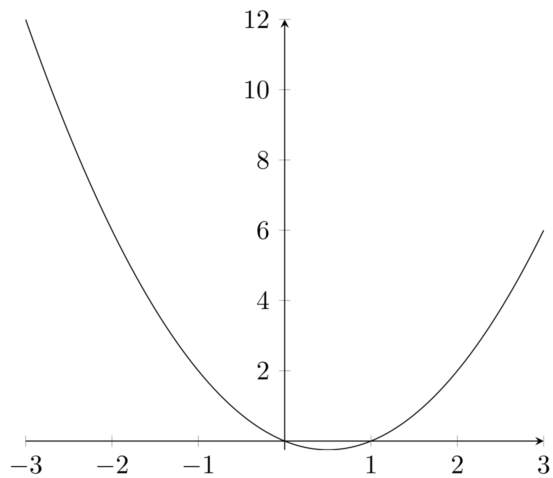

Figure 1.1: Showing how to visualize where $x^2 - x > 0$

(b)
- **Original**: If $x > 0$ then $x^2 - x > 0$. False.
- **Contrapositive**: If $x^2 - x \leq 0$ then $x \leq 0$. False.
- **Converse**: If $x^2 - x > 0$ then $x > 0$. False.

Exercise 1.1.4

(a) $\exists a \in A$ such that $a^2 \notin B$

(b) $\forall a \in A, a^2 \notin B$

(c) $\exists a \in A$ such that $a^2 \in B$.

(d) $\exists a \notin A$ such that $a^2 \notin B$.

Exercise 1.1.5

(a) True. True.

(b) False. True.

(c) True. False.

(d) True. True.

Exercise 1.1.6
TODO too lazy

Exercise 1.1.7

$$\begin{aligned}
D &= A \cap (B \cup C) \\
E &= (A \cap B) \cup C \\
F &= A
\end{aligned}$$

For F, I was thinking $x \in B \implies x \in C$ means that either $x \in B$ and $x \in C$, or $x \notin B$ and x can be anything. This sounds like x can be anything in the second case, so we have $A \cap \mathcal{U} = A$.

Exercise 1.1.8
$A = \{0, 1\}$. $\mathcal{P}(A) = \{\emptyset, \{0\}, \{1\}, \{0, 1\}\}$.

If A has one element, $|\mathcal{P}(A)| = 2$. It is called the power set because it contains all the subsets of A, and that $|\mathcal{P}(A)| = 2^{|A|}$.

Exercise 1.1.9
TODO: You can honestly find this everywhere online. Standard proof.

Exercise 1.1.10

(a) $\mathbb{Z} \times \mathbb{R}$

(b) $\mathbb{R} \times (0, 1]$

(c) No. You can do a contradiction proof with cases that the first and second set are disjoint, and then that they are not disjoint.

(d) Yes, $(\mathbb{R} - \mathbb{Z}) \times \mathbb{Z}$

(e) No. The cartesian product will produce a box, while this set is a circle.

1.2 Functions

Exercise 1.2.1

(a) Suppose $x \in A_0$. Then consider

$$f^{-1}(f(A_0)) = \{x' \mid f(x') \in f(A_0)\}.$$

Since $x \in A_0$, we know $f(x) \in f(A_0)$, and so we can conclude $x \in f^{-1}(f(A_0))$.

If f is injective, then we know $f(x) = f(x'), x' \in A_0$ implies that $x = x' \implies x \in A_0$, proving equality.

(b) Suppose $y \in f(f^{-1}(B_0)) = \left\{y' \mid \exists x \in f^{-1}(B_0), y' = f(x)\right\}$.

We have $x \in \{x' \mid f(x') \in B_0\}$, which means $y' = f(x) \in B_0$, so therefore $y \in B_0$.

If f is surjective, then we know $y \in B_0 \implies \exists x \in A$ such that $f(x) = y$.

In particular, $f(x) = y \in B_0$, this set of $x \in f^{-1}(B_0)$, so we can conclude that $y \in f(f^{-1}(B_0))$.

Exercise 1.2.2

(a) We can write some definitions first

- $f^{-1}(B_0) = \{x \mid f(x) \in B_0\}$
- $f^{-1}(B_1) = \{x \mid f(x) \in B_1\}$

If we know $B_0 \subset B_1$, then for some $x \in B_0$, we know $x \in B_1$.

This means for some $f(x) \in B_0, f(x) \in B_1$ as well, so therefore $f^{-1}(B_0) \subset f^{-1}(B_1)$.

(b)

$$\begin{aligned}
f^{-1}(B_0 \cup B_1) &= \{x \mid f(x) \in B_0 \cup B_1\} \\
&= \{x \mid f(x) \in B_0 \text{ or } f(x) \in B_1\} \\
&= \{x \mid f(x) \in B_0\} \cup \{x \mid f(x) \in B_1\} \\
&= f^{-1}(B_0) \cup f^{-1}(B_1)
\end{aligned}$$

(c) Basically the same proof as (b).

(d) Basically the same proof as (b).

(e) Suppose $x \in A_0$ means $x \in A_1$ as well. Consider $y \in f(A_0) = \{y' \mid y' = f(x) \text{ for some } x \in A_0\}$. Because of our assumptions, it is also the case that $y \in \{y' \mid y' \text{ for some } x \in A_1\} = f(A_1)$.

Notice that $f(A_0) \subset f(A_1)$ does not imply that $A_0 \subset A_1$. E.g. think parabola.

(f)

$$\begin{aligned}
f(A_0 \cup A_1) &= \{y \mid y = f(x) \text{ for some } x \in A_0 \cup A_1\} \\
&= \{y \mid y = f(x) \text{ for some } x \in A_0 \text{ or } x \in A_1\} \\
&= \{y \mid y = f(x) \text{ for some } x \in A_0\} \cup \{y \mid y \text{ for some } x \in A_1\} \\
&= f(A_0) \cup f(A_1)
\end{aligned}$$

(g)

$$f(A_0 \cap A_1) = \{y \mid y = f(x) \text{ for some } x \in A_0 \cap A_1\}$$

which implies that $f(A_0 \cap A_1) \in f(A_0)$, since $x \in A_0$ and $f(A_0 \cap A_1) \in f(A_1)$, since $x \in A_1$.

This means that $f(A_0 \cap A_1) \subset f(A_0) \cap f(A_1)$.

Now, if f is injective, then if we start with $y \in f(A_0) \cap f(A_1)$, we know $y \in \{y' \mid y' = f(x), x \in A_0\}$ and $y \in \{y' \mid y' = f(x), x \in A_1\}$. Since f is injective, the common y' values in $f(A_0)$ and $f(A_1)$ will map to the same x values in A_0 and A_1, which means $y \in \{y' \mid y' = f(x), x \in A_0 \cap A_1\} = f(A_0 \cap A_1)$.

Exercise 1.2.3
too lazy

Exercise 1.2.4
(a)