Munkres Topology Solutions

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Contents

1 Set Theory and Logic **1**

1.1 Fundamental Concepts . . . 1
1.2 Functions . . . 4
1.3 Relations . . . 9
1.4 The Integers and the Real Numbers . . . 14
1.5 Cartesian Products . . . 17
1.6 Finite Sets . . . 18
1.7 Countable and Uncountable Sets . . . 20

Forward

These solutions are for the 2[nd] edition Topology textbook by Munkres.

I am slowly going to evolve this guide as I work through the problems, but I'm going to start summarizing each chapter before I do the problems, not only to help me learn each chapter section, but also as a good study guide and reference while doing the problems.

Chapter 1

Set Theory and Logic

1.1 Fundamental Concepts

Exercise 1.1.1
We will check $\cup, \cap$ in DeMorgan's laws.

Let's use

- $A = \{1, 2, 3, 4\}$
- $B = \{-1, 2, 3, 5\}$
- $C = \{3, 9, 11\}$

Check

$$\begin{aligned} A - (B \cup C) &= \{1, 2, 3, 4\} - \{-1, 2, 3, 5, 9, 11\} \\ &= \{1, 4\} \\ &= (A - B) \cap (A - C) \\ &= \{1, 4\} \cap \{1, 2, 4\} = \{1, 4\} \end{aligned}$$

$$\begin{aligned} A - (B \cap C) &= \{1, 2, 3, 4\} - \{3\} \\ &= \{1, 2, 4\} \\ &= (A - B) \cup (A - C) \\ &= \{1, 4\} \cup \{1, 2, 4\} = \{1, 2, 4\} \end{aligned}$$

Exercise 1.1.2

(a) $\implies$ is true. $\impliedby$ is not true, consider $A = \{1, 2, 3\}, B = \{1, 3\}, C = \{2\}$.

(b) $\implies$ is true. $\impliedby$ is not true, consider $A = \{1, 2, 3\}, B = \{1, 3\}, C = \{2\}$.

(c) True.

(d) $\implies$ is not true. Consider $A = \{1\} \subset B = \{1, 2\}, C = \emptyset$. $\impliedby$ is true.

(e) Not true. Consider $A = \{1\}, B = \{2\}$. I think $\subset$ works.

(f) Not true. Consider $A = \{1, 2\}, B = \{2, 3\}$. LHS is equivalent to A, so this should be $\supset$.

(g) True.

(h) $\supset$

(i) True.

(j) True.

(k) Not true, if $A = \emptyset$ for example, we have $(A \times B) \subset (C \times D) = \emptyset \subset (C \times D)$, but we can set B to whatever and this statement is still true, so we can make B have an element that is not in D, and therefore $B \not\subset D$.

(l) True.

(m) $\subset$

(n) $\subset$

(o) True.

(p) I think this is true at first glance...at least $\subset$ looks good.

(q) $\supset$

Exercise 1.1.3

(a)
- **Original**: If $x < 0$ then $x^2 - x > 0$. True.
- **Contrapositive**: If $x^2 - x \leq 0$ then $x \geq 0$. True.
- **Converse**: If $x^2 - x > 0$ then $x < 0$. False.

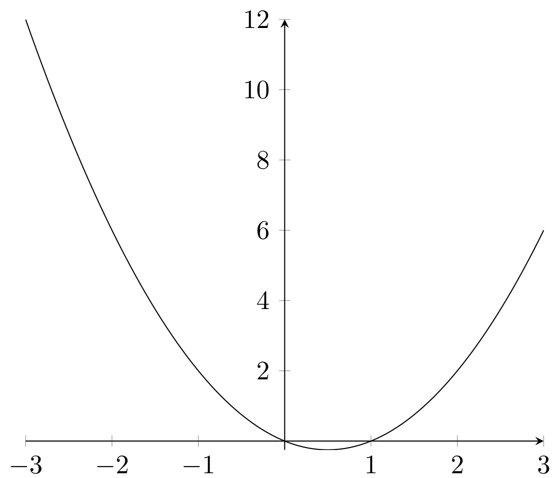

Figure 1.1: Showing how to visualize where $x^2 - x > 0$

(b)
- **Original**: If $x > 0$ then $x^2 - x > 0$. False.
- **Contrapositive**: If $x^2 - x \leq 0$ then $x \leq 0$. False.
- **Converse**: If $x^2 - x > 0$ then $x > 0$. False.

Exercise 1.1.4

(a) $\exists a \in A$ such that $a^2 \notin B$

(b) $\forall a \in A, a^2 \notin B$

(c) $\exists a \in A$ such that $a^2 \in B$.

(d) $\exists a \notin A$ such that $a^2 \notin B$.

Exercise 1.1.5

(a) True. True.

(b) False. True.

(c) True. False.

(d) True. True.

Exercise 1.1.6
TODO too lazy

Exercise 1.1.7

$$
\begin{aligned}
D &= A \cap (B \cup C) \\
E &= (A \cap B) \cup C \\
F &= A
\end{aligned}
$$

For F, I was thinking $x \in B \implies x \in C$ means that either $x \in B$ and $x \in C$, or $x \notin B$ and x can be anything. This sounds like x can be anything in the second case, so we have $A \cap \mathcal{U} = A$.

Exercise 1.1.8
$A = \{0, 1\}$. $\mathcal{P}(A) = \{\emptyset, \{0\}, \{1\}, \{0, 1\}\}$.

If A has one element, $|\mathcal{P}(A)| = 2$. It is called the power set because it contains all the subsets of A, and that $|\mathcal{P}(A)| = 2^{|A|}$.

Exercise 1.1.9
TODO: You can honestly find this everywhere online. Standard proof.

Exercise 1.1.10

(a) $\mathbb{Z} \times \mathbb{R}$

(b) $\mathbb{R} \times (0, 1]$

(c) No. You can do a contradiction proof with cases that the first and second set are disjoint, and then that they are not disjoint.

(d) Yes, $(\mathbb{R} - \mathbb{Z}) \times \mathbb{Z}$

(e) No. The cartesian product will produce a box, while this set is a circle.

1.2 Functions

Definition 1.2.1
We define the **composite** $g \circ f$ of $f : A \to B$ and $g : B \to C$ as the function $g \circ f : A \to C$ defined by the equation

$$(g \circ f)(a) = g(f(a)) \tag{1.1}$$

and

$$\{(a, c) \mid \text{for some } b \in B, f(a) = b \text{ and } g(b) = c\} \tag{1.2}$$

Definition 1.2.2
A function f is **injective** if

$$[f(a) = f(a')] \implies [a = a'] \tag{1.3}$$

Definition 1.2.3
A function f is **surjective** if

$$[b \in B] \implies [\exists a \in A, b = f(a)] \tag{1.4}$$

Lemma 1 *Let $f : A \to B$. If there are functions $g : B \to A$ and $h : B \to A$ such that $g(f(a) = a, \forall a \in A$ and $f(h(b)) = b, \forall b \in B$, then f is bijective and $g = h = f^{-1}$.*

Definition 1.2.4
$f(A_0)$ is the **image** of A_0, and

$$f(A_0) = \{b \mid \exists a \in A_0, b = f(a)\} \tag{1.5}$$

Definition 1.2.5
$f^{-1}(B_0)$ is the preimage of B_0 under f, formally

$$f^{-1}(B_0) = \{a \mid f(a) \in B_0\} \tag{1.6}$$

Exercises

Exercise 1.2.1
(a) Suppose $x \in A_0$. Then consider

$$f^{-1}(f(A_0)) = \{x' \mid f(x') \in f(A_0)\}.$$

Since $x \in A_0$, we know $f(x) \in f(A_0)$, and so we can conclude $x \in f^{-1}(f(A_0))$.

If f is injective, then we know $f(x) = f(x'), x' \in A_0$ implies that $x = x' \implies x \in A_0$, proving equality.

(b) Suppose $y \in f(f^{-1}(B_0)) = \left\{y' \mid \exists x \in f^{-1}(B_0), y' = f(x)\right\}$.

We have $x \in \{x' \mid f(x') \in B_0\}$, which means $y' = f(x) \in B_0$, so therefore $y \in B_0$.

If f is surjective, then we know $y \in B_0 \implies \exists x \in A$ such that $f(x) = y$.

In particular, $f(x) = y \in B_0$, this set of $x \in f^{-1}(B_0)$, so we can conclude that $y \in f(f^{-1}(B_0))$.

Exercise 1.2.2
(a) We can write some definitions first

- $f^{-1}(B_0) = \{x \mid f(x) \in B_0\}$
- $f^{-1}(B_1) = \{x \mid f(x) \in B_1\}$

If we know $B_0 \subset B_1$, then for some $x \in B_0$, we know $x \in B_1$.

This means for some $f(x) \in B_0, f(x) \in B_1$ as well, so therefore $f^{-1}(B_0) \subset f^{-1}(B_1)$.

(b)

$$\begin{aligned} f^{-1}(B_0 \cup B_1) &= \{x \mid f(x) \in B_0 \cup B_1\} \\ &= \{x \mid f(x) \in B_0 \text{ or } f(x) \in B_1\} \\ &= \{x \mid f(x) \in B_0\} \cup \{x \mid f(x) \in B_1\} \\ &= f^{-1}(B_0) \cup f^{-1}(B_1) \end{aligned}$$

(c) Basically the same proof as (b).

(d) Basically the same proof as (b).

(e) Suppose $x \in A_0$ means $x \in A_1$ as well. Consider $y \in f(A_0) = \{y' \mid y' = f(x) \text{ for some } x \in A_0\}$. Because of our assumptions, it is also the case that $y \in \{y' \mid y' \text{ for some } x \in A_1\} = f(A_1)$.

Notice that $f(A_0) \subset f(A_1)$ does not imply that $A_0 \subset A_1$. E.g. think parabola.

(f)

$$\begin{aligned} f(A_0 \cup A_1) &= \{y \mid y = f(x) \text{ for some } x \in A_0 \cup A_1\} \\ &= \{y \mid y = f(x) \text{ for some } x \in A_0 \text{ or } x \in A_1\} \\ &= \{y \mid y = f(x) \text{ for some } x \in A_0\} \cup \{y \mid y \text{ for some } x \in A_1\} \\ &= f(A_0) \cup f(A_1) \end{aligned}$$

(g)

$$f(A_0 \cap A_1) = \{y \mid y = f(x) \text{ for some } x \in A_0 \cap A_1\}$$

which implies that $f(A_0 \cap A_1) \in f(A_0)$, since $x \in A_0$ and $f(A_0 \cap A_1) \in f(A_1)$, since $x \in A_1$.

This means that $f(A_0 \cap A_1) \subset f(A_0) \cap f(A_1)$.

Now, if f is injective, then if we start with $y \in f(A_0) \cap f(A_1)$, we know $y \in \{y' \mid y' = f(x), x \in A_0\}$ and $y \in \{y' \mid y' = f(x), x \in A_1\}$. Since f is injective, the common y' values in $f(A_0)$ and $f(A_1)$ will map to the same x values in A_0 and A_1, which means $y \in \{y' \mid y' = f(x), x \in A_0 \cap A_1\} = f(A_0 \cap A_1)$.

Exercise 1.2.3
too lazy

Exercise 1.2.4
(a) Consider

$$\begin{aligned} (g \circ f)^{-1}(C_0) &= \{a \mid (g \circ f)(x) \in C_0\} && \text{(by definition)} \\ &= \{a \mid \text{for some } b \in B, f(a) = b, g(b) = c \in C_0\} \\ &= \big\{a \mid \text{for some } b \in B, f(a) = b, b \in g^{-1}(C_0)\big\} && \text{(we know } b \in g^{-1}(C_0) \text{ since } C_0 \subset C) \\ &= \big\{a \mid f(a) \in g^{-1}(C_0)\big\} \\ &= f^{-1}\big(g^{-1}(C_0)\big) \end{aligned}$$

Just a note for this problem, it's easy to get caught up with definitions and forget why we need assumptions. It might seem easy to do this problem without the fact that $C_0 \subset C$, but if you look at the step where we use that property, if $C_0 \not\subset C$, we cannot assume that $b \in g^{-1}(C_0)$, e.g. if C_0 contains elements that are not in C.

(b) Suppose we have

$$\begin{aligned} (g \circ f)(a) &= (g \circ f)(a') \\ g(f(a)) &= g(f(a')) && \text{(by def.)} \\ f(a) &= f(a') && \text{(because } g \text{ is injective)} \\ a &= a' && \text{(because } f \text{ is injective)} \end{aligned}$$

therefore we conclude that $g \circ f$ is also injective.

(c) If we know that $g \circ f$ is injective,

- AFSOC f is not injective. Then $\exists a_1, a_2$ such that $f(a_1) = f(a_2)$ but $a_1 \neq a_2$. If this is the case, then $(g \circ f)(a_1) = g(f(a_1)) = g(f(a_2)) = (g \circ f)(a_2)$ which shows that $g \circ f$ is not injective. Which is a contradiction. Therefore, f must be injective.
- It is possible for g to not be injective. We can have some $b \in B$ that $\nexists a \in A$ such that $f(a) = b$. In this case, we will not be able to find some input $a \neq a' \in$ where we break injectivity for $g \circ f$.

(d) Suppose f and g are surjective. Now consider some $c \in C$. Since g is surjective, we know $\exists b \in B$ such that $g(b) = c$. For this $b \in B$, since f is surjective, we know that $\exists a \in A$ such that $f(a) = b$. This means for any $c \in C$, we know $\exists a$ such that $g(f(a)) = (g \circ f)(a) = c$, which means $g \circ f$ is surjective.

(e) If we know that $g \circ f$ is surjective,

- It is possible for f to not be surjective. Intuitively, a counterexample would show that there is some $b \in B$ such that $\nexists a \in A$ such that $f(a) = b$. But all we have to make sure in our example is that whatever $g(b) = c$ maps to, $\exists a' \neq a \in A$ such that $(g \circ f)(a') = c$.
- AFSOC g is not surjective. Then $\exists c \in C$ such that $\nexists b \in B$ such that $g(b) = c$. If this is the case, then $\nexists a \in A$ such that $(g \circ f)(a) = c$, which means $g \circ f$ is not surjective. This is a contradiction, therefore g must be surjective.

(f) Summary should be pretty clear :) from the results above.

Exercise 1.2.5

Alright let's consider the two cases in this problem

- f has a left inverse, i.e. $\exists g, g \circ f = i_A$.

  AFSOC f is not injective. This means that $f(a) = f(a') = b \in B$ but $a \neq a' \in A$.

  If this is the case, then $g(b) = a$ or $g(b) = a'$, but cannot be both, by the definition of a function, which means that $g \circ f$ is not i_A by counterexample of either a or a'.

- f has a right inverse, i.e. $\exists h, f \circ h = i_B$.

  AFSOC f is not surjective. Then $\exists b \in B$ such that $\nexists a \in A$ such that $f(a) = b$. If this is the case, then for this b, $(f \circ h)(b) \neq b$, in which case $f \circ h \neq i_B$.

The proofs above in 1.2.5 are by contradiction, and illustrate how to construct such a counterexample.

The proofs above in 1.2.5 are by contradiction, and illustrate how to construct such a counterexample.

No, these left/right inverses are unique. Not rigorous, but if you have an identity mapping and you change any of the mappings, it will no longer be an identity mapping.

So we have f that has both a left and right inverse, g, h respectively.In that case, we know that f is injective and surjective, by the results of 1.2.5, so we can conclude that f is bijective. We have $g = h = f^{-1}$ by Lemma 2.1 in the text.

(We could've just used Lemma 2.1 directly, but I think it's important to remind ourselves that showing a function is injective and surjective is a problem solving technique for showing a function is bijective.)

Exercise 1.2.6
Let us draw out our function first,

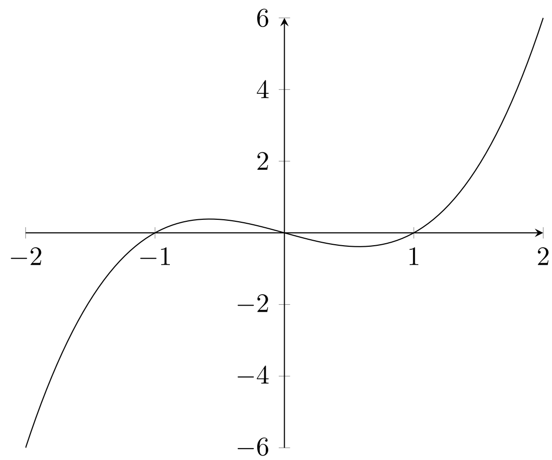

Figure 1.2: Plotting $f(x) = x^3 - x$

A less formal way to imagine injective functions is to use the horizontal line test. If you sweep a horizontal line and it intersects the plot in more than 1 spot, then you know the function is not injective. We see our function $f(x)$ here has issues between $[-1, 1]$.

For surjectivity, we just need to map all of $\mathbb{R}$ on the y-axis. Our function does this nicely already, so we just have to make our function injective, and then make sure to keep the surjectivity.

We have two choices here, we can either restrict our domain to be

- $D = (-\infty, -1) \cup [-1, \infty)$
- $D = (-\infty, -1] \cup (-1, \infty)$

If we use the first choice, we will get the following plot as g:

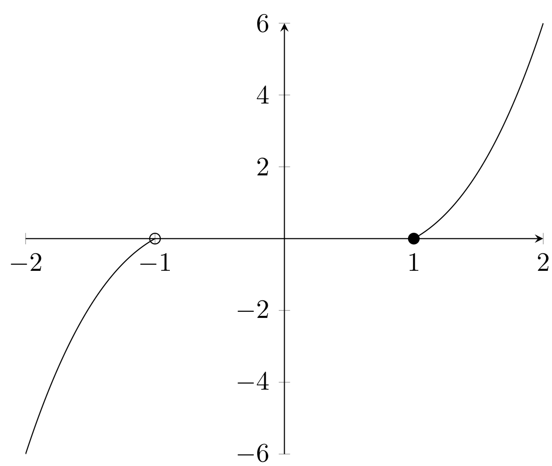

Figure 1.3: Plotting $g(x)$, a bijective function. Open circles are exclusive, closed circles are inclusive.

To find g^{-1}, a classic algebra way to do this is to solve the function in terms of x. Because of the horizontal line issues from before, we will encounter, some issues, but because of our domain restriction, things should be ok.

Instead of solving for x, since we already have a plot, we can just do a reflection across $y = x$, and we will get the inverse function g^{-1}. The intuitive way to think about this is that we are essentially swapping all (x, y) coordinates to become (y, x).

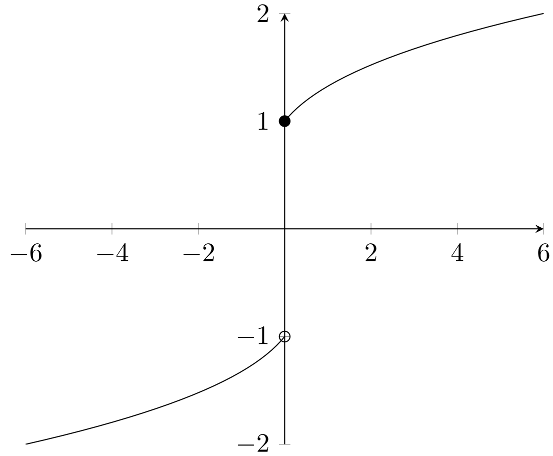

Figure 1.4: Plotting $g(x)$, a bijective function. Open circles are exclusive, closed circles are inclusive.

1.3 Relations

Definition 1.3.1
A **relation** on a set A is a subset C of the cartesian product $A \times A$.

For a relation C on A, we use the notation xCy to mean $(x, y) \in C$, or "x is in the relation C to y."

Definition 1.3.2
An **equivalence relation** on a set A is a relation C on A having the following 3 properties: (we use $\sim$ to denote the equivalence relation)

1. Reflexivity: $xCx \quad \forall x \in A$ $(x \sim x)$
2. Symmetry: If xCy then yCx $(x \sim y \implies y \sim x)$
3. Transitivity: If xCy and yCz then xCz $(x \sim y \wedge y \sim z \implies x \sim z)$

Definition 1.3.3
We call a subset of E of A the **equivalence class** determined by x as the equation

$$E = \{y \mid y \sim x\} \tag{1.7}$$

Definition 1.3.4
A **partition** of a set A is a collection of disjoint nonempty subsets of A whose union is all of A.

Definition 1.3.5
A relation C on a set A is called an **order relation** if it has the following properties:

1. Comparability: For every $x, y \in A$ for which $x \neq y$, either xCy or yCx
2. Nonreflexitivity: For no $x \in A$ does xCx hold
3. Transtivity: If xCy and yCz then xCz

Definition 1.3.6
if X is a set and $<$ is an order relation on X, and if $a < b$, we use the notation (a, b) to denote the set

$$\{x \mid a < x < b\}; \tag{1.8}$$

it is called an **open interval** in X. If this set if empty, we call

- a the **immediate predecessor** of b
- b the **immediate successor** of a

Definition 1.3.7
Suppose A, B are two sets with order relations $<_A$ and $<_B$ respectively. We say that A, B have the same **order type** if there is a bijective correspondence between them that preserves order; that is, if there exists a bijective function $f : A \to B$ such that

$$a_1 <_A a_2 \implies f(a_1) <_B f(a_2) \tag{1.9}$$

Definition 1.3.8
An ordered set A has the **least upper bound property** if every nonempty subset A_0 of A that is bounded above has a least upper bound. Analogously, the set A is said to have the **greatest lower bound property** if every nonempty subset A_0 of A that is bounded below has a greatest lower bound.

Exercises

Exercise 1.3.1
Check equivalence relation:

- Reflexivity is obvious
- Equality is symmetric, so the relation is too
- Equality is transitive, so the relation is too

This looks like a bunch of parabolas of the form $y = x^2 + C$ on the plane, see figure 1.5

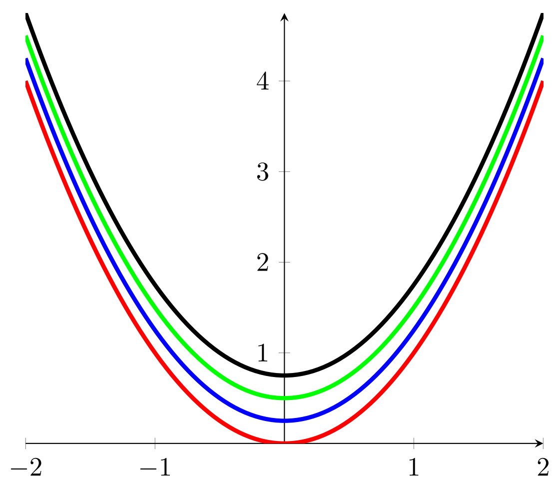

Figure 1.5: Plotting the partition defined by the equivalence relation

Exercise 1.3.2
Reflexivity will still hold in A_0, since $A_0 \in A$, and C applies to any element $x \in A$. Symmetry still holds, since $x, y \in A$, and transitivity also holds since $x, y, z \in A$. The idea is that A_0's elements are contained in A, so all the equivalence relation properies still hold.

Exercise 1.3.3
We are assuming $\exists b$ such that aCb. If there is no such b, then we do not have aCa.

Exercise 1.3.4

(a) Let us check the properties

- Reflexive: $f(a) = f(a)$ is trivial
- Symmetric: If we have $f(a) = f(b)$, then $f(b) = f(a)$
- Transitive: Equality is transitive, so this also holds

(b) A^* is a partition of B, so a bijective correspondence exists.

Exercise 1.3.5
(a) S' is an equivalence relation because we can imagine partitions of $y - x = z$ for $z \in \mathbb{Z}$. In S, we notice that $y - x = 1$, and $1 \in \mathbb{Z}$, so we know every relation in S is also in S', therefore $S \subset S'$. See 1.6 for how to visualize these equivalence classes.

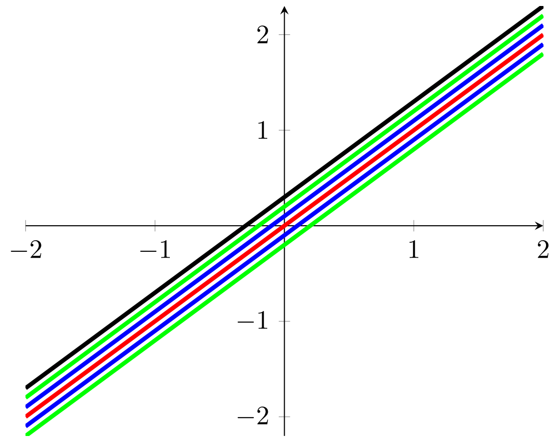

Figure 1.6: Plotting the partition defined by the equivalence relation

(b) If their intersection is empty, then this is trivially true.

Otherwise, if we have some nonempty intersection, since we know any elements in this intersection are also part of some equivalence relation, all the equivalence relation properties apply, so this intersection is also an equivalence relation on A.

(c) **TODO**: I'm pretty confused about this question. Wouldn't the intersection of all equivalence relations that contain S just end up with S?

Exercise 1.3.6
Showing it is an order relation

- Comparability: Per the rule, we will always have xCy or yCx.
- Nonreflexitivity: We have a tiebreaker rule that prevents xCx.
- Transitivity: Intuitively, there is an ordering at the highest level with $y - x^2$ value. If there are ties there, we check with x values for ordering.

Yeah sorry I didn't feel like doing the order relation formally, but hopefully the geometric solution can help explain more. There's just a lot of casework and mundane proof so I didn't feel like writing it out.

Geometrically, this is like the partition in 1.5, except parabolas that are higher up are ordered "higher". Within a parabola, the values on the left are less than the values on the right.

Exercise 1.3.7
A restriction is a subset of the larger set, so all the order relation properties will still hold.

Exercise 1.3.8
I assume the author wants us to check the $x^2 < y^2$, if $x^2 = y^2$, then $x < y$.

1. Comparability: If $x^2 \neq y^2$, either $x^2 < y^2$ or $x^2 > y^2$. If $x^2 = y^2$, then it must be the case that $x, y \neq 0$, and one of x, y is negative and the other positive, so we either have $x > y$ or $x < y$.
2. Nonreflexitivity: xCx means $x^2 = x^2$, so we would check $x <$ or $> x$, but this is not possible since $x = x$.
3. Transitivity: If xCy then $x^2 < y^2$ or $x < y$, and if we have yCz then $y^2 < z^2$ or $y < z$. In all cases, we can conclude $x^2 < z^2$ or $x < z$. I'm being lazy with casework here.

Exercise 1.3.9
We want to check that a dictionary order relation is an order relation.

1. Comparability: For any two $(a_1, b_1), (a_2, b_2)$, we have 2 cases:

(a) $a_1 <_A a_2$ or $a_2 <_A a_1$ since $<_A$ is an order relation on A. Then we know $(a_1, b_1) < (a_2, b_2)$ or $(a_1, b_1) > (a_2, b_2)$.

(b) $a_1 = a_2$. Then we use the same argument with b_1, b_2, that either $b_1 <_B b_2$ or $b_2 <_B b_1$, which then shows the corresponding $<$ and $>$ on the tuple.

2. Nonreflexitivity: if we have some (a, b), we know by $<_A$ that $a <_A a$ does not hold, so the overall order relation is not possible.

3. Transitivity: If we have $(a_1, b_1) < (a_2, b_2) < (a_3, b_3)$, then we have 2 cases for the first tuple, and 2 cases for the second tuple.

   (a) $a_1 <_A a_2 <_A a_3$: then we can use the transitive property of $<_A$

   (b) $a_1 <_A a_2 = a_3$: we can see that $a_1 <_A a_3$

   (c) $a_1 = a_2 <_A a_3$: we can see that $a_1 <_A a_3$

   (d) $a_1 = a_2 = a_3$: then we must have $b_1 <_B b_2 <_B b_3$, so we can use the transitive property of $<_B$

Exercise 1.3.10

- One way to see that this is an order preserving function is that the derivative is always positive between $(-1, 1)$,

$$f'(x) = \frac{x^2 + 1}{(1 - x^2)^2},$$

  which means the function is monotonically increasing, and thus will preserve the order, since monotonically increasing functions have the property that

$$a < b \implies f(a) < f(b).$$

- This is just an algebra exercise...pretty easy to verify

Exercise 1.3.11

AFSOC there is more than one immediate successor to some a, call them b and c. Then by order set properties, we know that either $b < c$ or $c < b$. In either case, we end up finding that b or c cannot be immediate successors, since for example, if $b < c$, we have that (a, c) is not empty.

The argument for immediate predecessor is symmetric to this argument.

To show there can only be one smallest element, we can AFSOC there is more than one. If we call these a, a', we know from ordering properties that WLOG $a < a'$, then a' is not the smallest element, so this is a contradiction.

The argument for the largest element is symmetric.

Exercise 1.3.12

(i*) Every element has an immediate predecessor. For some (x, y), the immediate predecessor is $(x, y + 1)$. There is no smallest element, since you can always find a smaller element, i.e. for any (x, y), $(x - 1, y) < (x, y)$.

(ii*) The immediate predecessor for some (x, y) is $(x + 1, y + 1)$. There is no smallest element, since you can always find $(x - 1, y) < (x, y)$ for any (x, y).

(iii*) The immediate predecessor for some (x, y) is $(x - 1, y + 1)$. There is no smallest element, since for any (x, y), you have $(x - 1, y) < (x, y)$.

Not rigorous, but geometrically, the first ordering is like a zigzag on the plane, the second one is like $y = x + C$, and the third is $y = -x + C$, so these orderings are all different.

Exercise 1.3.13

Suppose A has the least upper bound property, meaning every nonempty subset A_0 of A is bounded above by some least upper bound. AFSOC A does not have the greatest lower bound property, that is $\exists A_1 \subset A$

such that A_1 does not have a greatest lower bound. If this is the case, then consider the lower bound a for this set A_1, and consider the set A_2, which we define as

$$A_2 = \{a' \mid a' \geq a\}$$

Notice that A_2 cannot be empty, or else a is the only, and therefore greatest lower bound for A_1. Now, from the definition of A_2, we can see that a is an upper bound for A_2. However, we now claim that there is no least upper bound for A_2. Because if there were, call it some a_2, then a_2 would be the greatest lower bound for A_1, since a_2 is larger than all lower bounds of A_1. This is a contradiction, since we assumed that A_1 does not have a greatest lower bond. Therefore, we must conclude that every subset of A has a greatest lower bound.

Exercise 1.3.14

(a) If C is symmetric, then $(a, b) \in C \implies (b, a) \in C$, which means $D \subset C$.

If $C = D$, then $(a, b) \in C \implies (b, a) \in D = C$, so therefore C is symmetric.

(b) We will check the order relation properties for D

  (a) Comparability: For any $(b, a) \in D$, we know $(a, b) \in C$, so we know either $b < a$ or $a < b$.

  (b) Nonreflexitivity: $(b, b) \in D$ would imply $(b, b) \in C$, but C is an order relation so this is not possible.

  (c) Transitivity: $(c, b) \in D, (b, a) \in D$. We know that $(a, b), (b, c) \in C$, so we know $(a, c) \in C$, so therefore $(c, a) \in D$, which proves transitivity.

(c) The other direction of the argument is symmetric.

Exercise 1.3.15

(a) Let us show that

- 1 is the least upper bound for $[0, 1]$. We can see this because if you pick any smaller of an upper bound u, $u < 1$ and thus is not an upper bound for the set.
- 1 is the least upper bound for $[0, 1)$. Suppose we have some other least upper bound u such that $x \in [0, 1), x \leq u$, but $u < 1$. Then consider $u' = u + \epsilon/2$, where $\epsilon = \frac{1-u}{2}$. Then u' is < 1 but $u' > u$ and is also an upper bound, which means u was not the least upper bound. Therefore we have reached a contradiction and conclude that 1 is the least upper bound.

(b) $[0, 1] \times [0, 1]$ with dictionary ordering has least upper bound property. This is because for any subset, if we look at the first coordinate, it is in $[0, 1]$, which we showed has the least upper bound property, so call this upper bound u_1. Similarly, for the second coordinate it is also in $[0, 1]$, so we have a least upper bound u_2 for this coordinate. Then we have (u_1, u_2) is a least upper bound for any subset in $[0, 1] \times [0, 1]$.

This argument holds for $[0, 1] \times [0, 1)$ and $[0, 1) \times [0, 1]$.

1.4 The Integers and the Real Numbers

1.
$$
\begin{aligned}
(x+y)+z &= x+(y+z) \\
(x\cdot y)\cdot z &= x\cdot(y\cdot z)
\end{aligned}
$$

2.
$$
\begin{aligned}
x+y &= y+x \\
x\cdot y &= y\cdot x
\end{aligned}
$$

3. $\exists$ element of $\mathbb{R}$ called **zero**, denoted by 0, such that
$$x+0=x, \forall x\in\mathbb{R}$$

4. $\exists$ element of $\mathbb{R}$ called **one**, denoted by 1, such that
$$x\cdot 1=x, \forall x\in\mathbb{R}$$

5.
$$x\cdot(y+z)=(x\cdot y)+(x\cdot z)$$

6.
$$
\begin{aligned}
&\text{If } x>y, \text{ then } x+z>y+z \\
&\text{If } x>y \text{ and } z>0, \text{ then } x\cdot z>y\cdot z
\end{aligned}
$$

- Properties 1-5 define a field
- Properties 1-6 define an ordered field
- Properties 7-8 by themselves define a linear continuum

Definition 1.4.1
A **section** S_n is defined as
$$S_{n+1}=\{1,2,\dots,n\} \tag{1.10}$$

Exercises

Exercise 1.4.1
TODO: Too lazy

Exercise 1.4.2
TODO: too lazy

Exercise 1.4.3

(a) If we take the intersection of any two inductive sets, call this intersection A, we know that $1\in A$, and we still can inductively show that for any $x\in A$, that $x+1\in A$ as well, since x is part of two inductive sets. We can then use induction to prove that the intersection of an arbitrary collection of inductive sets is also inductive.

(b) Proving the basic properties

(1) $1>0$ and $1\in\mathbb{Z}$, so $1\in\mathbb{Z}_+$. Now for any $x\in\mathbb{Z}_+$, the number $x+1$ is still an integer, and since $x>0$, we know $x+1>0$, which means $x+1\in\mathbb{Z}_+$ as well. Therefore, we conclude $\mathbb{Z}_+$ is inductive.

(2) AFSOC $A \neq \mathbb{Z}_+$. Then we know that $\exists x \in \mathbb{Z}_+, x \notin A$. However, if we use induction from 1, we can show that $x \in A$, which is a contradiction. Therefore we conclude $A = \mathbb{Z}_+$.

Exercise 1.4.4

(a) We have $a \in \mathbb{Z}_+$. We just want to inductively show that $a + x \in \mathbb{Z}_+, \forall x \in \mathbb{Z}_+$. We can do this by induction.

(b) Induction on b

(c) We can either casework here, with $a = 0, a \geq 1$, or we can show that $a - 1$ is inductive.

(d) We can use induction on d, via (c)

(e) We can use induction on d, both ways (positive and negative directions)

Exercise 1.4.5

1. $a^n a^m = a^{n+m}$, shown by (a)

2. $(a^n)^m = a^{nm}$, shown by (b)

3. $a^m b^m = (ab)^m$, induction on m, and you have to use the property in 1 for $(ab)^m \cdot ab = (ab)^{m+1}$.

Exercise 1.4.6

TODO: probably just doing induction in another direction

Exercise 1.4.7

(a) This is a corollary of the Axiom of Completness, which states that every nonempty set of real numbers that is bounded above has a least upper bound.

(b) Informally, suppose we have some $\epsilon > 0$ that is the inf of this set, then we can always find $1/n < \epsilon$, so therefore this ϵ is not the inf, and we conclude that 0 is the inf.

(c) **TODO**: induction proof, idea is that a^n gets smaller as n increases, and can be smaller than any ϵ. We have to show this property by induction.

Exercise 1.4.8

(a) Follows from that $\mathbb{Z} \subset \mathbb{R}$.

(b) AFSOC there are two such numbers, $n_1, n_2 \in \mathbb{Z}$. WLOG $n_1 > n_2$, Then we can find that

$$\begin{aligned} & n_1 < x < n_1 + 1 \\ n_2 < x < n_2 + 1 \implies & -n_2 - 1 < -x < -n_2 \\ \implies & n_1 - n_2 - 1 < 0 < n_1 - n_2 + 1 \end{aligned}$$

which is impossible, since $n_1 - n_2 - 1 \geq 0$, which is $\nless 0$, so therefore we cannot have more than one n.

Now to show the existence of this n, we can use a contradiction argument along with an induction argument to show no such n exists would imply $x \notin \mathbb{R}$.

I'm sure there's a more straightforward proof with lower bounds...using the fact that $\mathbb{Z}$ has a greatest lower bound.

(c) We have $n \geq x > y + 1 > y \geq m$ for $m, n \in \mathbb{Z}$.

Either $y+1$ is an integer, in which case $n = y+1$. If $y+1$ is not an integer, then we know it is bounded below by some $b \in \mathbb{Z}$ such that $b < y + 1 < b + 1$, which we know must be $< x$, and $> y$, since if it were $\leq y$, then we could choose $b' = b + 1$, and $b' > y$ and $b' \leq y + 1$, and thus b would not satisfy $b + 1 > y + 1$. In this case, $n = b$, and we have $x > n > y, n \in \mathbb{Z}$.

(d) We know that $x - y > 0$, so we can find some $n \in \mathbb{Z}$ such that $n(x-y) > 1$. Using (c), we can find some $m \in \mathbb{Z}$ such that

$$\begin{aligned} nx > m > ny \\ x > \frac{m}{n} > y \end{aligned}$$

let $z = m/n$.

Exercise 1.4.9

(a) First, let's show

$$\begin{aligned} (x+h)^2 = x^2 + 2xh + h^2 &\leq x^2 + h(2x+1) \\ \implies h^2 &\leq h \\ \implies h &\leq 1 && \text{(Given, since } h < 1) \end{aligned}$$

Notice in the last step, if $h = 0$, then we don't divide and $0^2 \leq 0$ holds.

$$\begin{aligned} (x-h)^2 = x^2 - 2xh + h^2 &\geq x^2 - 2xh \\ \implies h^2 &\geq 0 \\ \implies h &\geq 0 && \text{(Given, since } h \geq 0) \end{aligned}$$

Notice in the last step, if $h = 0$, then we don't divide and $0^2 \geq 0$ holds.

(b) If we have $x^2 < a$, then we know $(x+h)^2 \leq x^2 + h(2x+1)$, so we can find an h by considering the RHS and making it $< a$. We can do this by solving for h, which gives us

$$h < \frac{a - x^2}{2x+1}$$

The other part of the problem is more of the same algebra.

(c) B is bounded above by $1 + a$, and $a/2 \in B$. If we let $b = \sup B$, then b^2 cannot be $< a$, or else we can find some $(b+h)^2 < a, h > 0$, and show that this b is not the sup.

(d) **TODO**: I'm stuck

Exercise 1.4.10

(a) We know from exercises before that since $m/2 \notin \mathbb{Z}$ since m is odd, we know $\exists n \in \mathbb{Z}$ such that $n < m/2 < n+1$. Now consider,

$$2n < m < 2n + 2$$

Since $m \in \mathbb{Z}$, we conclude $m = 2n + 1$.

(b) $p \cdot q = (2m+1) \cdot (2n+1) = 4mn + 2m + 2n + 1 = 2(2mn + m + n) + 1$, which is in the form $2N + 1$, so therefore $p \cdot q$ is odd. For p^n to be odd, we show by induction.

(c) Honestly I think this is by definition of $\mathbb{Q}$. The hint in the textbook implies the proof should be something about finding the denominator in reduced form.

(d) I really hope you can show $\sqrt{2}$ is irrational by the time you take topology...

1.5 Cartesian Products

Definition 1.5.1
Let $\mathcal{A}$ be a nonempty collection of sets. An **indexing function** for $\mathcal{A}$ is a surjective function f from some set J, called the **index set**, to $\mathcal{A}$. The collection $\mathcal{A}$, together with the indexing function f is called an **indexed family of sets**. Given $\alpha \in J$, we shall denote the set $f(\alpha)$ by the symbol A_α. And we shall denote the indexed family itself by the symbol

$$\{A_\alpha\}_{\alpha \in J}, \tag{1.11}$$

which is read as "the family of all A_α, as α ranges over J."

Exercises

Exercise 1.5.1
Consider $f : A \to B$

$$\begin{aligned} f(a, b) &= (b, a) \\ f^{-1}(b, a) &= (a, b) \end{aligned}$$

Exercise 1.5.2
I'm just going to write out functions that you can check are bijective.

(a) $f(a_1, a_2, \dots, a_n) = ((a_1, \dots, a_{n-1}), a_n)$

(b)

$$\begin{aligned} f_{-1}(a_1, a_2, \dots) &= ((a_1, a_2), (a_3, a_4), \dots) = (b_1, b_2, \dots) \\ f^{-1}(b_1, b_2, \dots) &= f^{-1}((a_1, a_2), (a_3, a_4), \dots) \\ &= (a_1, a_2, \dots) \end{aligned}$$

Exercise 1.5.3

(a) For any $b \in B$, we have that every index element $\in A_i$, so therefore $b \in A$. Therefore $B \subset A$.

(b) The converse is if $B \subset A$, then $B_i \subset A_i$. This is true because AFSOC $\exists b \in B$ such that $\exists i$ such that $B_i \not\subset A_i$. Then $b \notin A$, and therefore $B \not\subset A$. But this is a contradiction.

(c) AFSOC $\exists i$ such that A_i is empty. Then, A must be empty, because there is no element that can be in the i^{th} tuple. If every A_i is nonempty, then A must also be nonempty.

(d)
- $A \cup B \subset \prod_i A_i \cup B_i$
- $A \cap B \supset \prod_i A_i \cap B_i$

Exercise 1.5.4

(a) Just map the first m tuples to the first m, and the remainder just map to any element of X

(b) $f((x_1, \dots, x_m), (x'_1, \dots, x'_n)) = (x_1, \dots, x_m, x'_1, \dots, x'_n)$

(c) Just map the first n elements, and for the rest just use any element of X

(d) Map first n, then map the rest. Other way, Map first n, then map the rest

(e) $f((x_1, x_2, \dots), (x'_1, x'_2, \dots)) = (x_1, x'_1, x_2, x'_2, \dots)$

(f) Just cycle through the n coordinates for mapping, similar to (e).

Exercise 1.5.5

(a) Yes, just do $\prod_i \mathbb{Z}$

(b) Easy, just $\prod_i [i, \infty)$

(c) $\prod_{i=1}^{100} \mathbb{R} \times \prod_{i=101}^{\infty} \mathbb{Z}$

(d) Cannot be expressed.

1.6 Finite Sets

Definition 1.6.1
A set is said to be **finite** if there is a bijective correspondence of A with some section of the positive integers. That is, A is finite if it is empty or if there is a bijection

$$f : A \to \{1, \ldots, n\} \tag{1.12}$$

for some positive integer n. In the former case, we say that A as **cardinality 0**, and in the latter case, we say that A has **cardinality** n.

Exercises

Exercise 1.6.1
(a) There are $4 \times 3 \times 2 = 24$ injective mappings,

$$\begin{gathered}
(1,1),(2,2),(3,3)\\
(1,1),(2,2),(3,4)\\
(1,1),(2,3),(3,2)\\
(1,1),(2,3),(3,4)\\
(1,1),(2,4),(3,2)\\
(1,1),(2,4),(3,3)\\
(1,2),(2,1),(3,3)\\
(1,2),(2,1),(3,4)\\
(1,2),(2,3),(3,1)\\
(1,2),(2,3),(3,4)\\
(1,2),(2,4),(3,1)\\
(1,2),(2,4),(3,3)\\
(1,3),(2,1),(3,2)\\
(1,3),(2,1),(3,4)\\
(1,3),(2,2),(3,1)\\
(1,3),(2,2),(3,4)\\
(1,3),(2,4),(3,1)\\
(1,3),(2,4),(3,2)\\
(1,4),(2,1),(3,2)\\
(1,4),(2,1),(3,3)\\
(1,4),(2,2),(3,1)\\
(1,4),(2,2),(3,3)\\
(1,4),(2,3),(3,1)\\
(1,4),(2,3),(3,2)
\end{gathered}$$

(b) $10 \cdot 9 \cdot \cdots \cdot 3 = 1814400 \ldots$, you can tell this is not a fun time.

Exercise 1.6.2
AFSOC A is finite. Then B must be finite since it is a subset of A, but this is a contradiction since b is not finite by assumption.

Exercise 1.6.3
Let the propr subset be $X^\omega - 0, 0, \ldots$. Let the bijection be

$$\begin{aligned} f(000000\ldots) &= 1000000\ldots \\ f(100000\ldots) &= 0100000\ldots \\ f(110000\ldots) &= 0010000\ldots \\ &\ldots \end{aligned}$$

It's just binary written in R to L significant digits, + 1 to shift for the missing $000\ldots$.

Exercise 1.6.4

(a) Induction from empty set as base case. Adding an element gives you two cases, either the new element is the largest, or the existing largest element remains the largest.

(b) **TODO**: What is an order type? I think the idea here is map A into $\mathbb{Z}$ in the order that the elements are in A, and then they will have the order type of the $\mathbb{Z}_+$.

Exercise 1.6.5
No. $A = \emptyset$ and B could be infinite.

Exercise 1.6.6

(a) X^n is basically an n-tuple showing whether or not an element of A is included in the set, so you can make that bijection to $\mathcal{P}(A)$.

(b) $\mathcal{P}(A)$ has a bijection with X^n, which is finite, so $\mathcal{P}(A)$ is finite as well.

Exercise 1.6.7
Consider $C = A \times B$. Any function f is some subset of C. Then the set of all functions is a subset of $\mathcal{P}(C)$, which from 1.6.6 we know is finite since C is finite.

1.7 Countable and Uncountable Sets

Definition 1.7.1
A set A is said to be **infinite** if it is not finite. It is said to be **countably infinite** if there is a bijective correspondence

$$f : A \to \mathbb{Z}_+ \tag{1.13}$$

Definition 1.7.2
A set is said to be **countable** if it is either finite or countably infinite. A set that is not countable is said to be **uncountable**.

Theorem 1 *Let B be a nonempty set, then the following are equivalent:*

(1) B is countable

(2) There is a surjective function $f : \mathbb{Z}_+ \to B$

(3) There is an injective function $g : B \to \mathbb{Z}_+$

Theorem 2 *A countable union of countable sets is countable.*

Theorem 3 *A finite product of countable sets is countable.*

Exercises

Exercise 1.7.1
Lazy way to prove this, but you can make an injective function $f : \mathbb{Q} \to \mathbb{Z}_+ \times \mathbb{Z}_+ \times \mathbb{Z}_+$ where given any

$$f(\pm\frac{m}{n}) = (m, n, \{1 \text{ if negative}, 2 \text{ if positive}\})$$

And then we know that $\mathbb{Z}_+ \times \mathbb{Z}_+ \times \mathbb{Z}_+$ is a finite product of countable sets, so it is also countable. Since f is an injective function into a countably infinite set, we conclude that $\mathbb{Q}$ is also countably infinite.

Exercise 1.7.2
Checking for bijections. I'm not going to explicitly show this, but you can find inverse functions to show bijectivity. **TODO**.

Exercise 1.7.3
The bijection is whether or not some $n \in \mathbb{Z}_+$ is included in a set or not, which corresponds to 0 (not included) and 1 (included) in the tuple of X^ω.

Exercise 1.7.4

(a) For a given n, the number of algebraic numbers is $\prod_{i=0}^{n-1} \mathbb{Q}$. Then taking the union over $\mathbb{Z}_+$, this would be a countable union of countable sets, which is also countable. Therefore there are a countable number of algebraic numbers.

(b) AFSOC transcendental numbers are countable. Then the real numbers are a union of two countable sets, which could make $\mathbb{R}$ countable, which is a contradiction.

Exercise 1.7.5

(a) This set of functions can be bijected to $\mathbb{Z}_+ \times \mathbb{Z}_+$, which is countable.

(b) This would be $\prod_{i=1}^{n} \mathbb{Z}_+$, which is a finite product of countable sets, which is countable.

(c) This is a countable union of countable sets, which is countable.

(d) Not countable, you can use diagonalization to construct a function that is not included in a countable map.

(e) Not countable, can also use a diagonalization argument to construct a function that differs in the i^{th} spot for f_i.

(f) Countable, since this is essentially a subset of the countable union of $\mathbb{Z}_+$.

(g) Countable, same reasoning as (f)

(h) This is a countable union of a countable union of countable sets, which is still countable.

(i) Countable, this is a subset of $\mathbb{Z}_+ \times \mathbb{Z}_+$.

(j) Countable, this is just $\cup_{i=0}^{\infty} \prod_{j=1}^{i} \mathbb{Z}_+$, which is a countable union of a finite product of countable sets.

Exercise 1.7.6

(a) Originally I was going to define the identity map $I : B \to A$, since $B \subset A$, and say that injections both ways implies bijection, but that's the theorem we are trying to prove in the next step, so let's just do what the problem tells us here.

To understand the hint a bit more, the reason we can plug in B values for f is because $B \subset A$. And the reason that we get an A_i value from the result of f is also because $B \subset A$.

Now, if we define this new function h,

$$h(x) = \begin{cases} f(x) & \text{if } x \in A_n - B_n \\ x & \text{otherwise} \end{cases} \tag{1.14}$$

We have to check this function is bijective.

- **Injective**: We have 2 cases for x. Either it is in B, in which case we just spit out x, or it is in A_n but not in B_n, in which case we apply $f(x)$, and make $x \in B$. This is injective, because cases, for some $x_1, x_2 \in A$,
  (a) $x_1, x_2 \in A_n - B_n$, then we are just looking at f, which we know is injective
  (b) $x_1 \in A_n - B_n, x_2 \notin A_n - B_n$, then $x_1 \neq x_2$, and since $x_1 \in A_n - B_n$, it is never mapped to itself, because it if were, then it would imply it is $\notin A_n - B_n$, which is a contradiction.
  (c) $x_1, x_2 \notin A_n - B_n$, then we are using the identity map, which is injective
- **Surjective**: This is the more difficult part of the proof, but also the more fun.

  The idea of the recursive definition is that we want to partition B into parts, A_i, B_i. The only part that we have trouble with is $A_1 - B_1$, since this part is not in B. However, by applying f to this portion, we can have it map into B, which helps us map B. However, since $f(A_1)$ "takes the spot" of A_2, we then apply f on A_2, so that $f(A_2) \subset B$. $f(A_2)$ takes the spot of A_3, so we keep applying this recursive definition. Eventually, we can see that all of B is mapped, so h is also surjective on B. I know this is an informal proof, but Figure 1.7 explains the hardest part of the proof.

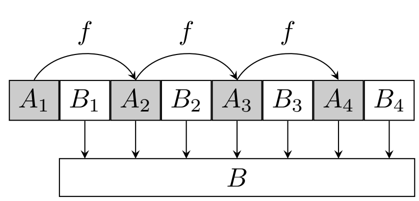

Figure 1.7: Showing the subset partitions for A_i, B_i

Note: To be honest, I'm not sure why we had to go through the trouble of defining the recursive definition. To me, it seems like with f, the scenario is that A maps to B injectively, except there are some some elements in B that are not mapped.

Update: It seems like the reason is that with a naive definition of just $h(x)$ mapping $f(x)$ for $x \in A - B$ and identity for $x \in A \cap B$, we cannot claim that this combo of $f(x), x$ maps B surjectively. See the surjective part of the proof to see why this proof is so fun.

(b) We have two cases, either $A \subset C$ or $C \subset A$. In either case, we can apply part (a).

Exercise 1.7.7
We can see that $E \subset D$. Need $f : D \to E$. We can construct such an f by representing each $x \in \mathbb{Z}_+$ in binary, and mapping it to the corresponding 0-1 tuple in $\{0, 1\}$.

Exercise 1.7.8
TODO. I'm stuck. I assume we want to do the double bijective proof to show equal cardinalities.

Exercise 1.7.9

(a) First rearrange, the formula to be easier to apply:

$$h(n+1) = \sqrt{h(n) + h(n-1)^2}$$

Trying out some values,

$$\begin{aligned} h(3) &= \sqrt{2+1} = \sqrt{3} \\ h(4) &= \sqrt{4 + \sqrt{3}} \\ h(5) &= \sqrt{3 + \sqrt{4 + \sqrt{3}}} \end{aligned}$$

so we know that there exists such a function

(b) Notice that when we took the square root, we could have had $\pm$, so there it is not well defined which one we should take.

(c) If we try to solve for

$$\begin{aligned} h(3) &= \sqrt{2-1} = \pm 1 \\ h(4) &= \sqrt{\pm 1 - 2} \implies \text{square root of negative number...} \end{aligned}$$

imaginary numbers? Never heard of them.