Munkres Topology Solutions

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These solutions are for the 2^{nd} edition Topology textbook by Munkres.

I am slowly going to evolve this guide as I work through the problems, but I'm going to start summarizing each chapter before I do the problems, not only to help me learn each chapter section, but also as a good study guide and reference while doing the problems.

Chapter 1

Set Theory and Logic

1.1 Fundamental Concepts

Exercise 1.1.1
We will check $\cup, \cap$ in DeMorgan's laws.

Let's use

- $A = \{1, 2, 3, 4\}$
- $B = \{-1, 2, 3, 5\}$
- $C = \{3, 9, 11\}$

Check

$$\begin{aligned} A - (B \cup C) &= \{1, 2, 3, 4\} - \{-1, 2, 3, 5, 9, 11\} \\ &= \{1, 4\} \\ &= (A - B) \cap (A - C) \\ &= \{1, 4\} \cap \{1, 2, 4\} = \{1, 4\} \end{aligned}$$

$$\begin{aligned} A - (B \cap C) &= \{1, 2, 3, 4\} - \{3\} \\ &= \{1, 2, 4\} \\ &= (A - B) \cup (A - C) \\ &= \{1, 4\} \cup \{1, 2, 4\} = \{1, 2, 4\} \end{aligned}$$

Exercise 1.1.2

(a) $\implies$ is true. $\impliedby$ is not true, consider $A = \{1, 2, 3\}, B = \{1, 3\}, C = \{2\}$.

(b) $\implies$ is true. $\impliedby$ is not true, consider $A = \{1, 2, 3\}, B = \{1, 3\}, C = \{2\}$.

(c) True.

(d) $\implies$ is not true. Consider $A = \{1\} \subset B = \{1, 2\}, C = \emptyset$. $\impliedby$ is true.

(e) Not true. Consider $A = \{1\}, B = \{2\}$. I think $\subset$ works.

(f) Not true. Consider $A = \{1, 2\}, B = \{2, 3\}$. LHS is equivalent to A, so this should be $\supset$.

(g) True.

(h) $\supset$

(i) True.

(j) True.

(k) Not true, if $A = \emptyset$ for example, we have $(A \times B) \subset (C \times D) = \emptyset \subset (C \times D)$, but we can set B to whatever and this statement is still true, so we can make B have an element that is not in D, and therefore $B \not\subset D$.

(l) True.

(m) $\subset$

(n) $\subset$

(o) True.

(p) I think this is true at first glance...at least $\subset$ looks good.

(q) $\supset$

Exercise 1.1.3

(a)
- **Original**: If $x < 0$ then $x^2 - x > 0$. True.
- **Contrapositive**: If $x^2 - x \leq 0$ then $x \geq 0$. True.
- **Converse**: If $x^2 - x > 0$ then $x < 0$. False.

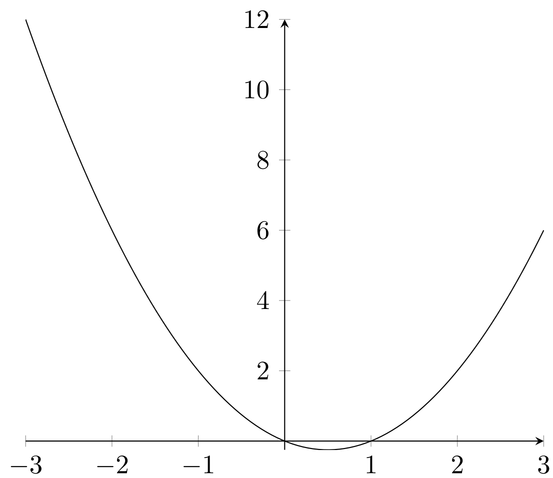

Figure 1.1: Showing how to visualize where $x^2 - x > 0$

(b)
- **Original**: If $x > 0$ then $x^2 - x > 0$. False.
- **Contrapositive**: If $x^2 - x \leq 0$ then $x \leq 0$. False.
- **Converse**: If $x^2 - x > 0$ then $x > 0$. False.

Exercise 1.1.4

(a) $\exists a \in A$ such that $a^2 \notin B$

(b) $\forall a \in A, a^2 \notin B$

(c) $\exists a \in A$ such that $a^2 \in B$.

(d) $\exists a \notin A$ such that $a^2 \notin B$.

Exercise 1.1.5

(a) True. True.

(b) False. True.

(c) True. False.

(d) True. True.

Exercise 1.1.6
TODO too lazy

Exercise 1.1.7

$$\begin{aligned} D &= A \cap (B \cup C) \\ E &= (A \cap B) \cup C \\ F &= A \end{aligned}$$

For F, I was thinking $x \in B \implies x \in C$ means that either $x \in B$ and $x \in C$, or $x \notin B$ and x can be anything. This sounds like x can be anything in the second case, so we have $A \cap \mathcal{U} = A$.

Exercise 1.1.8
$A = \{0, 1\}$. $\mathcal{P}(A) = \{\emptyset, \{0\}, \{1\}, \{0, 1\}\}$.

If A has one element, $|\mathcal{P}(A)| = 2$. It is called the power set because it contains all the subsets of A, and that $|\mathcal{P}(A)| = 2^{|A|}$.

Exercise 1.1.9
TODO: You can honestly find this everywhere online. Standard proof.

Exercise 1.1.10

(a) $\mathbb{Z} \times \mathbb{R}$

(b) $\mathbb{R} \times (0, 1]$

(c) No. You can do a contradiction proof with cases that the first and second set are disjoint, and then that they are not disjoint.

(d) Yes, $(\mathbb{R} - \mathbb{Z}) \times \mathbb{Z}$

(e) No. The cartesian product will produce a box, while this set is a circle.

1.2 Functions

Definition 1.2.1
We define the **composite** $g \circ f$ of $f : A \to B$ and $g : B \to C$ as the function $g \circ f : A \to C$ defined by the equation

$$(g \circ f)(a) = g(f(a)) \tag{1.1}$$

and

$$\{(a, c) \mid \text{for some } b \in B, f(a) = b \text{ and } g(b) = c\} \tag{1.2}$$

Definition 1.2.2
A function f is **injective** if

$$[f(a) = f(a')] \implies [a = a'] \tag{1.3}$$

Definition 1.2.3
A function f is **surjective** if

$$[b \in B] \implies [\exists a \in A, b = f(a)] \tag{1.4}$$

Lemma 1 *Let $f : A \to B$. If there are functions $g : B \to A$ and $h : B \to A$ such that $g(f(a) = a, \forall a \in A$ and $f(h(b)) = b, \forall b \in B$, then f is bijective and $g = h = f^{-1}$.*

Definition 1.2.4
$f(A_0)$ is the **image** of A_0, and

$$f(A_0) = \{b \mid \exists a \in A_0, b = f(a)\} \tag{1.5}$$

Definition 1.2.5
$f^{-1}(B_0)$ is the preimage of B_0 under f, formally

$$f^{-1}(B_0) = \{a \mid f(a) \in B_0\} \tag{1.6}$$

Exercises

Exercise 1.2.1
(a) Suppose $x \in A_0$. Then consider

$$f^{-1}(f(A_0)) = \{x' \mid f(x') \in f(A_0)\}.$$

Since $x \in A_0$, we know $f(x) \in f(A_0)$, and so we can conclude $x \in f^{-1}(f(A_0))$.

If f is injective, then we know $f(x) = f(x'), x' \in A_0$ implies that $x = x' \implies x \in A_0$, proving equality.

(b) Suppose $y \in f(f^{-1}(B_0)) = \left\{y' \mid \exists x \in f^{-1}(B_0), y' = f(x)\right\}$.

We have $x \in \{x' \mid f(x') \in B_0\}$, which means $y' = f(x) \in B_0$, so therefore $y \in B_0$.

If f is surjective, then we know $y \in B_0 \implies \exists x \in A$ such that $f(x) = y$.

In particular, $f(x) = y \in B_0$, this set of $x \in f^{-1}(B_0)$, so we can conclude that $y \in f(f^{-1}(B_0))$.

Exercise 1.2.2
(a) We can write some definitions first

- $f^{-1}(B_0) = \{x \mid f(x) \in B_0\}$
- $f^{-1}(B_1) = \{x \mid f(x) \in B_1\}$

If we know $B_0 \subset B_1$, then for some $x \in B_0$, we know $x \in B_1$.

This means for some $f(x) \in B_0, f(x) \in B_1$ as well, so therefore $f^{-1}(B_0) \subset f^{-1}(B_1)$.

(b)

$$\begin{aligned} f^{-1}(B_0 \cup B_1) &= \{x \mid f(x) \in B_0 \cup B_1\} \\ &= \{x \mid f(x) \in B_0 \text{ or } f(x) \in B_1\} \\ &= \{x \mid f(x) \in B_0\} \cup \{x \mid f(x) \in B_1\} \\ &= f^{-1}(B_0) \cup f^{-1}(B_1) \end{aligned}$$

(c) Basically the same proof as (b).

(d) Basically the same proof as (b).

(e) Suppose $x \in A_0$ means $x \in A_1$ as well. Consider $y \in f(A_0) = \{y' \mid y' = f(x) \text{ for some } x \in A_0\}$. Because of our assumptions, it is also the case that $y \in \{y' \mid y' \text{ for some } x \in A_1\} = f(A_1)$.

Notice that $f(A_0) \subset f(A_1)$ does not imply that $A_0 \subset A_1$. E.g. think parabola.

(f)

$$\begin{aligned} f(A_0 \cup A_1) &= \{y \mid y = f(x) \text{ for some } x \in A_0 \cup A_1\} \\ &= \{y \mid y = f(x) \text{ for some } x \in A_0 \text{ or } x \in A_1\} \\ &= \{y \mid y = f(x) \text{ for some } x \in A_0\} \cup \{y \mid y \text{ for some } x \in A_1\} \\ &= f(A_0) \cup f(A_1) \end{aligned}$$

(g)

$$f(A_0 \cap A_1) = \{y \mid y = f(x) \text{ for some } x \in A_0 \cap A_1\}$$

which implies that $f(A_0 \cap A_1) \in f(A_0)$, since $x \in A_0$ and $f(A_0 \cap A_1) \in f(A_1)$, since $x \in A_1$.

This means that $f(A_0 \cap A_1) \subset f(A_0) \cap f(A_1)$.

Now, if f is injective, then if we start with $y \in f(A_0) \cap f(A_1)$, we know $y \in \{y' \mid y' = f(x), x \in A_0\}$ and $y \in \{y' \mid y' = f(x), x \in A_1\}$. Since f is injective, the common y' values in $f(A_0)$ and $f(A_1)$ will map to the same x values in A_0 and A_1, which means $y \in \{y' \mid y' = f(x), x \in A_0 \cap A_1\} = f(A_0 \cap A_1)$.

Exercise 1.2.3
too lazy

Exercise 1.2.4
(a) Consider

$$\begin{aligned} (g \circ f)^{-1}(C_0) &= \{a \mid (g \circ f)(x) \in C_0\} && \text{(by definition)} \\ &= \{a \mid \text{for some } b \in B, f(a) = b, g(b) = c \in C_0\} \\ &= \big\{a \mid \text{for some } b \in B, f(a) = b, b \in g^{-1}(C_0)\big\} && \text{(we know } b \in g^{-1}(C_0) \text{ since } C_0 \subset C) \\ &= \big\{a \mid f(a) \in g^{-1}(C_0)\big\} \\ &= f^{-1}\big(g^{-1}(C_0)\big) \end{aligned}$$

Just a note for this problem, it's easy to get caught up with definitions and forget why we need assumptions. It might seem easy to do this problem without the fact that $C_0 \subset C$, but if you look at the step where we use that property, if $C_0 \not\subset C$, we cannot assume that $b \in g^{-1}(C_0)$, e.g. if C_0 contains elements that are not in C.

(b) Suppose we have

$$\begin{aligned} (g \circ f)(a) &= (g \circ f)(a') \\ g(f(a)) &= g(f(a')) && \text{(by def.)} \\ f(a) &= f(a') && \text{(because } g \text{ is injective)} \\ a &= a' && \text{(because } f \text{ is injective)} \end{aligned}$$

therefore we conclude that $g \circ f$ is also injective.

(c) If we know that $g \circ f$ is injective,

- AFSOC f is not injective. Then $\exists a_1, a_2$ such that $f(a_1) = f(a_2)$ but $a_1 \neq a_2$. If this is the case, then $(g \circ f)(a_1) = g(f(a_1)) = g(f(a_2)) = (g \circ f)(a_2)$ which shows that $g \circ f$ is not injective. Which is a contradiction. Therefore, f must be injective.
- It is possible for g to not be injective. We can have some $b \in B$ that $\nexists a \in A$ such that $f(a) = b$. In this case, we will not be able to find some input $a \neq a' \in$ where we break injectivity for $g \circ f$.

(d) Suppose f and g are surjective. Now consider some $c \in C$. Since g is surjective, we know $\exists b \in B$ such that $g(b) = c$. For this $b \in B$, since f is surjective, we know that $\exists a \in A$ such that $f(a) = b$. This means for any $c \in C$, we know $\exists a$ such that $g(f(a)) = (g \circ f)(a) = c$, which means $g \circ f$ is surjective.

(e) If we know that $g \circ f$ is surjective,

- It is possible for f to not be surjective. Intuitively, a counterexample would show that there is some $b \in B$ such that $\nexists a \in A$ such that $f(a) = b$. But all we have to make sure in our example is that whatever $g(b) = c$ maps to, $\exists a' \neq a \in A$ such that $(g \circ f)(a') = c$.
- AFSOC g is not surjective. Then $\exists c \in C$ such that $\nexists b \in B$ such that $g(b) = c$. If this is the case, then $\nexists a \in A$ such that $(g \circ f)(a) = c$, which means $g \circ f$ is not surjective. This is a contradiction, therefore g must be surjective.

(f) Summary should be pretty clear :) from the results above.

Exercise 1.2.5

Alright let's consider the two cases in this problem

- f has a left inverse, i.e. $\exists g, g \circ f = i_A$.

  AFSOC f is not injective. This means that $f(a) = f(a') = b \in B$ but $a \neq a' \in A$.

  If this is the case, then $g(b) = a$ or $g(b) = a'$, but cannot be both, by the definition of a function, which means that $g \circ f$ is not i_A by counterexample of either a or a'.

- f has a right inverse, i.e. $\exists h, f \circ h = i_B$.

  AFSOC f is not surjective. Then $\exists b \in B$ such that $\nexists a \in A$ such that $f(a) = b$. If this is the case, then for this b, $(f \circ h)(b) \neq b$, in which case $f \circ h \neq i_B$.

The proofs above in 1.2.5 are by contradiction, and illustrate how to construct such a counterexample.

The proofs above in 1.2.5 are by contradiction, and illustrate how to construct such a counterexample.

No, these left/right inverses are unique. Not rigorous, but if you have an identity mapping and you change any of the mappings, it will no longer be an identity mapping.

So we have f that has both a left and right inverse, g, h respectively.In that case, we know that f is injective and surjective, by the results of 1.2.5, so we can conclude that f is bijective. We have $g = h = f^{-1}$ by Lemma 2.1 in the text.

(We could've just used Lemma 2.1 directly, but I think it's important to remind ourselves that showing a function is injective and surjective is a problem solving technique for showing a function is bijective.)

Exercise 1.2.6
Let us draw out our function first,

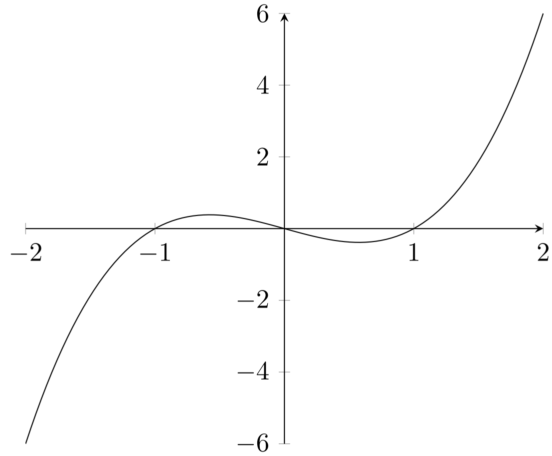

Figure 1.2: Plotting $f(x) = x^3 - x$

A less formal way to imagine injective functions is to use the horizontal line test. If you sweep a horizontal line and it intersects the plot in more than 1 spot, then you know the function is not injective. We see our function $f(x)$ here has issues between $[-1, 1]$.

For surjectivity, we just need to map all of $\mathbb{R}$ on the y-axis. Our function does this nicely already, so we just have to make our function injective, and then make sure to keep the surjectivity.

We have two choices here, we can either restrict our domain to be

- $D = (-\infty, -1) \cup [-1, \infty)$
- $D = (-\infty, -1] \cup (-1, \infty)$

If we use the first choice, we will get the following plot as g:

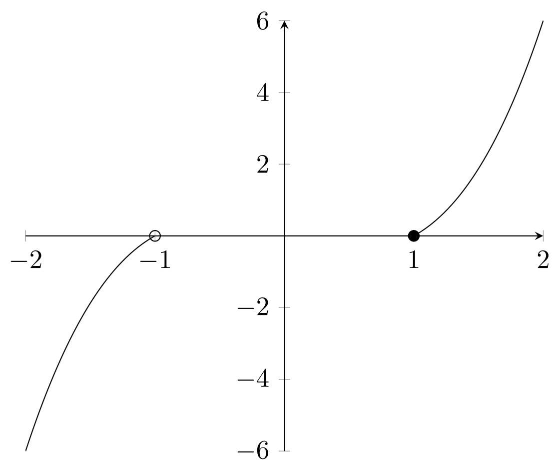

Figure 1.3: Plotting $g(x)$, a bijective function. Open circles are exclusive, closed circles are inclusive.

To find g^{-1}, a classic algebra way to do this is to solve the function in terms of x. Because of the horizontal line issues from before, we will encounter, some issues, but because of our domain restriction, things should be ok.

Instead of solving for x, since we already have a plot, we can just do a reflection across $y = x$, and we will get the inverse function g^{-1}. The intuitive way to think about this is that we are essentially swapping all (x, y) coordinates to become (y, x).

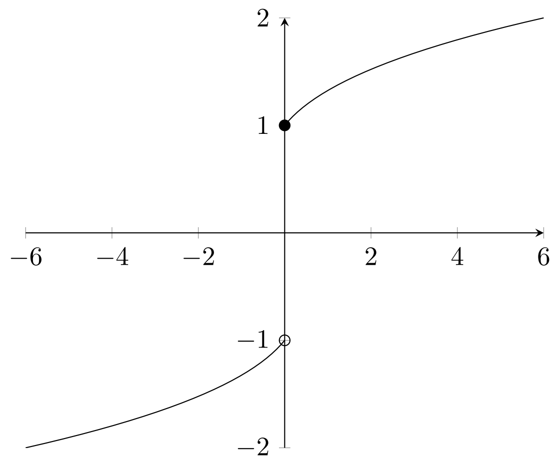

Figure 1.4: Plotting $g(x)$, a bijective function. Open circles are exclusive, closed circles are inclusive.

1.3 Relations

Definition 1.3.1
A **relation** on a set A is a subset C of the cartesian product $A \times A$.

For a relation C on A, we use the notation xCy to mean $(x, y) \in C$, or "x is in the relation C to y."

Definition 1.3.2
An **Equivalence Relation** on a set A is a relation C on A having the following 3 properties: (we use $\sim$ to denote the equivalence relation)

1. Reflexivity: $xCx \quad \forall x \in A$ $(x \sim x)$
2. Symmetry: If xCy then yCX $(x \sim y \implies y \sim x)$
3. Transitivity: If xCy and yCz then xCz $(x \sim y \wedge y \sim z \implies x \sim z)$

Exercises

Exercise 1.3.1