Schrodinger equation

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1 Schrodinger Equation

A derivation

1.1 In 1 dimension

From Einstein, we know that for photons:

$$E = \hbar\omega, \;\; p = \hbar k$$

From de Broglie, we know that these are true for other particles as well. QM is linear $\implies$ describing plane waves is sufficient Let's consider a plane wave like:

$$\psi(x,t) = \psi_0 e^{i(kx-\omega t)}$$

Now let's find the energy and momentum operators. We know that:

$$\frac{\partial\psi}{\partial t} = -i\omega\psi, \;\; \frac{\partial\psi}{\partial x} = ik\psi$$

So:

$$\frac{\hbar}{-i}\frac{\partial\psi}{\partial t} = E\psi = i\hbar\frac{\partial\psi}{\partial t},$$

$$\frac{\hbar}{i}\frac{\partial\psi}{\partial x} = p\psi = -i\hbar\frac{\partial\psi}{\partial x}$$

From Newton's laws we get that:

$$p = mv \implies K = \frac{1}{2}mv^2 = \frac{p^2}{2m}$$

And since energy is conserved, let's introduce a potential energy operator V, so that the potential energy of our particle is $V\psi$. Then, we get:

$$E = K + V$$

$$\boxed{i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V\psi}$$

1.2 In 3 dimensions

Now x, p and k are vectors $\mathbf{x}$, $\mathbf{p}$ and $\mathbf{k}$. So the plane wave is:

$$\psi(\mathbf{x},t) = \psi_0 e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}$$

The energy doesn't change:

$$\frac{\partial\psi}{\partial t} = -i\omega\psi \implies E\psi = i\hbar\frac{\partial\psi}{\partial t}$$

But the spacial derivative is now the gradient:

$$\nabla\psi = \begin{pmatrix}\partial_x\\ \partial_y\\ \partial_z\end{pmatrix}\psi_0 e^{i(k_x x_x + k_y x_y + k_z x_z - \omega t)} = \begin{pmatrix}ik_x\\ ik_y\\ ik_z\end{pmatrix}\psi = i\mathbf{k}\psi$$

Which leaves us with the momentum:

$$\frac{\hbar}{i}\nabla\psi = \mathbf{p}\psi = -i\hbar\nabla\psi$$

But for the energy we need p^2, so let's see what happens if we take the next spacial derivative, the divergence (since $\mathbf{p}\psi$ is a vector).

$$\nabla\cdot(\mathbf{p}\psi) = \partial_x(p_x\psi) + \partial_y(p_y\psi) + \partial_z(p_z\psi)$$

Since $\mathbf{p}$ is independent of $\mathbf{x}$ ($\mathbf{p}$ is a constant), we can pull its components from the derivatives:

$$= p_x\partial_x\psi + p_y\partial_y\psi + p_z\partial_z\psi$$

$$= (p_x ik_x + p_y ik_y + p_z ik_z)\psi$$

$$= i(\mathbf{p} \cdot \mathbf{k})\psi = i\hbar(\mathbf{k} \cdot \mathbf{k})\psi = i\hbar k^2 \psi$$

$$\implies \frac{\hbar}{i} \nabla \cdot (\mathbf{p}\psi) = p^2 \psi = -\hbar^2 \nabla^2 \psi$$

Now, just like we did in 1D, we can use $K = p^2/2m$ for the conservation of energy and we get the Schrodinger equation in 3d:

$$\boxed{i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi + V\psi}$$