General Relativity and Cosmology

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These notes are not endorsed by the lecturers. All errors are mine.

Contents

1. Special Relativity

1.1. The Lorentz transformations

Consider a system of pointlike particles interacting through gravity. We can write the force on the Nth particle as

$$m_N \frac{\mathrm{d}^2 \boldsymbol{r}_N}{\mathrm{d}t^2} = \sum_M \frac{G m_N m_M}{|\boldsymbol{r}_M - \boldsymbol{r}_N|^2} \frac{\boldsymbol{r}_M - \boldsymbol{r}_N}{|\boldsymbol{r}_M - \boldsymbol{r}_N|}$$

which is just Newtons law of gravitation. This equation is easily seen to be invariant under Galilean transformations

$$\boldsymbol{r}' = R\boldsymbol{r} + \boldsymbol{v}t + \boldsymbol{d}; \quad t' = t + t_0$$

where R is a member of O(3). These form a 10 parameter group called the Galilean group. Newtonian physics is only valid in frames related by the Galilean group. We call these *inertial frames*. But why are they special? Newton proposed there exists an *absolute frame* with all inertial frames having some velocity v_0 (including $v_0 = 0$) with respect to the absolute frame.

The above becomes a larger problem when we consider electrodynamics. Maxwell's equations are famously not invariant under the Galilean group! An easy way to see this is the constancy of the speed of light c. However, they are invariant under Lorentz transformations

$$X'^{\alpha} = \Lambda^{\alpha}{}_{\beta} X^{\beta} + a^{\alpha}$$

with $\Lambda^{\alpha}{}_{\beta}$ being defined by the condition

$$\Lambda^{\alpha}{}_{\gamma} \Lambda^{\beta}{}_{\delta} \eta_{\alpha\beta} = \eta_{\gamma\delta}$$

where $\eta_{\alpha\beta}$ is the Minkowski metric. We take this as a symmetry of nature.

Since a^{α} is constant we have

$$\mathrm{d}x'^{\alpha} = \Lambda^{\alpha}{}_{\gamma}\, \mathrm{d}x^{\gamma}$$

We define the proper time τ by

$$\mathrm{d}\tau^2 \equiv -\eta_{\alpha\beta}\, \mathrm{d}x^{\alpha}\, \mathrm{d}x^{\beta} = \mathrm{d}t^2 - \mathrm{d}\boldsymbol{x}^2$$

since $\mathrm{d}s^2 = -\mathrm{d}\tau^2$. We use *natural* units where $c = 1$. This is invariant

$$\begin{aligned} \mathrm{d}\tau'^2 &= -\eta_{\alpha\beta}\, \mathrm{d}x'^{\alpha}\, \mathrm{d}x'^{\beta} \\ &= -\eta_{\alpha\beta} \Lambda^{\alpha}{}_{\gamma} \Lambda^{\beta}{}_{\delta}\, \mathrm{d}x^{\gamma}\, \mathrm{d}x^{\delta} \end{aligned}$$

$$= -\eta_{\gamma\delta}\,\mathrm{d}x^\gamma\,\mathrm{d}x^\delta = \mathrm{d}\tau^2$$

Consider

$$\left|\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t}\right| \overset{\text{for light}}{=} 1$$

implying $\mathrm{d}\tau = 0$ for light. We just showed $\mathrm{d}\tau' = \mathrm{d}\tau$ so this speed is a constant!

The Lorentz transformations form the Lorentz group. We are interested in the *proper* Lorentz group satisfying

$$\Lambda^0{}_0 \geq 1; \quad \det\Lambda = +1$$

this excludes non-physical transformations. Taking $a^\alpha \overset{!}{=} 0$ gives the *homogeneous* proper Lorentz group satisfying

$$\Lambda^i{}_j = R_{ij}; \quad \Lambda^i{}_0 = \Lambda^0{}_i = 0; \quad \Lambda^0{}_0 = 0$$

This is very similar to the Galilean group except we include boosts. We will now show this by determining $\Lambda^\mu{}_\nu$.

Assume an observer O sees a particle at rest while another observer O' sees it having velocity $\boldsymbol{v}$. Then

$$\mathrm{d}x'^\alpha = \Lambda^\alpha{}_\beta\,\mathrm{d}x^\beta$$

but $\mathrm{d}\boldsymbol{x} = 0$ so

$$\mathrm{d}x'^i = \Lambda^i{}_0\,\mathrm{d}t\,; \quad \mathrm{d}t' = \Lambda^0{}_0\,\mathrm{d}t$$

and we obtain

$$\Lambda^i{}_0 = v^i\Lambda^0{}_0$$

where $v^i = \mathrm{d}x'^i\,/\,\mathrm{d}t'$. Then using $\eta_{\gamma\delta} = \Lambda^\alpha{}_\gamma\Lambda^\beta{}_\delta\eta_{\alpha\beta}$ we find

$$\begin{aligned}
-1 &= \Lambda^\alpha{}_0\Lambda^\beta{}_0\eta_{\alpha\beta} \\
&= \sum_i \left(\Lambda^i{}_0\right)^2 - \left(\Lambda^0{}_0\right)^2 = \left[\sum_i \left(v^i\right)^2 - 1\right]\left(\Lambda^0{}_0\right)^2 \\
\Lambda^0{}_0 &= \left[1 - \underbrace{\sum_i \left(v^i\right)^2}_{\boldsymbol{v}^2}\right]^{-1/2} \equiv \gamma
\end{aligned}$$

so $\Lambda^i{}_0 = \gamma v^i$.

The rest of these notes use the Einstein summation convention where repeated indices are summed

$$\sum_i \left(\Lambda^i{}_0\right)^2 \equiv \left(\Lambda^i{}_0\right)^2$$
$$\sum_i \Lambda^i{}_0 \Lambda^i{}_j \equiv \Lambda^i{}_0 \Lambda^i{}_j$$

Consider

$$\begin{aligned} 0 &= \Lambda^i{}_0 \Lambda^i{}_j - \Lambda^0{}_0 \Lambda^0{}_j \\ &= \gamma v^i \Lambda^i{}_j - \gamma \Lambda^0{}_j \\ \Lambda^0{}_j &= v^i \Lambda^i{}_j \end{aligned}$$

so we need to determine $\Lambda^i{}_j$. We impose rotational symmetry about $\hat{\boldsymbol{v}}$ and $\Lambda^i{}_j \to \mathbb{1}$ when $\boldsymbol{v} = 0$. Under these $\Lambda^i{}_j$ takes the form

$$\Lambda^i{}_j = \delta^i{}_j + A v^i v_j$$

with A being a function of $\boldsymbol{v}^2$. Consider

$$\begin{aligned} \delta_{ij} &= \Lambda^k{}_i \Lambda^k{}_j - \Lambda^0{}_i \Lambda^0{}_j \\ &= \Lambda^k{}_i \Lambda^k{}_j - v^m \Lambda^m{}_i v^n \Lambda^n{}_j \end{aligned}$$

We compute

$$\begin{aligned} \Lambda^k{}_i \Lambda^k{}_j &= \delta_{ij} + (2A + A^2 \boldsymbol{v}^2) v_i v_j \\ v^k \Lambda^k{}_i &= v_i (1 + A \boldsymbol{v}^2) \end{aligned}$$

substituting these give

$$0 = A^2 + \frac{2}{\boldsymbol{v}^2} A - \frac{\gamma^2}{\boldsymbol{v}^2}$$

this is a quadratic in A giving

$$A = \frac{\gamma - 1}{\boldsymbol{v}^2}$$

We find

$$\Lambda^i{}_j = \delta^i{}_j + \frac{\gamma - 1}{\boldsymbol{v}^2} v^i v_j; \quad \Lambda^0{}_j = \gamma v_j$$

1.2. Tensors in special relativity

By definition we have

$$\mathrm{d}x'^{\alpha} = \Lambda^{\alpha}{}_{\beta}\,\mathrm{d}x^{\beta}$$

we call anything that transforms like $\mathrm{d}x^{\beta}$ a contravariant four-vector. Then

$$\mathrm{d}x^{\beta} = \left(\Lambda^{\alpha}{}_{\beta}\right)^{-1}\mathrm{d}x'^{\alpha}$$

implying

$$\frac{\partial x^{\beta}}{\partial x'^{\alpha}} = \left(\Lambda^{\alpha}{}_{\beta}\right)^{-1}$$

Then we have

$$\partial_{\alpha'} \equiv \frac{\partial}{\partial x'^{\alpha}} = \left(\Lambda^{\alpha}{}_{\beta}\right)^{-1}\frac{\partial}{\partial x^{\beta}}$$

we call anything that transforms like $\partial_{a'}$ a covariant four-vector. These therefore transform inversely by definition. We can write

$$\left(\Lambda^{\alpha}{}_{\beta}\right)^{-1} = \Lambda_{\alpha}{}^{\beta} = \eta_{\alpha\delta}\eta^{\gamma\beta}\Lambda^{\delta}{}_{\gamma}$$

since

$$\Lambda_{\alpha}{}^{\gamma}\Lambda^{\alpha}{}_{\beta} = \eta_{\alpha\delta}\eta^{\gamma\varepsilon}\Lambda^{\delta}{}_{\varepsilon}\Lambda^{\alpha}{}_{\beta} = \eta_{\varepsilon\beta}\eta^{\gamma\varepsilon} = \delta^{\gamma}{}_{\beta}$$

where we use $\eta^{\beta\delta}\eta_{\alpha\delta} = \delta^{\beta}{}_{\alpha}$. This makes contractions invariant

$$U'_{\alpha}V'^{\alpha} = \Lambda_{\alpha}{}^{\gamma}\Lambda^{\alpha}{}_{\beta}U_{\gamma}V_{\beta} = \delta^{\beta}{}_{\gamma}U_{\gamma}V^{\beta} = U_{\gamma}V^{\gamma}$$

which is very useful! All contravariant V^{β} have a covariant friend given by

$$V_{\alpha} \equiv \eta_{\alpha\beta}V^{\beta}$$

and similarly all covariant U_{β} have a contravariant friend $U^{\alpha} \equiv \eta^{\alpha\beta}U_{\beta}$. Then we can raise and lower indices with $\eta^{\alpha\beta}$ and $\eta_{\alpha\beta}$

$$\eta^{\alpha\beta}V_{\beta} = \eta^{\alpha\beta}\eta_{\beta\gamma}V^{\gamma} = V^{\alpha}$$

To see V_{α} is covariant consider

$$V'_{\alpha} = \eta_{\alpha\beta}V'^{\beta} = \eta_{\alpha\beta}\Lambda^{\beta}{}_{\gamma}V^{\gamma} = \eta_{\alpha\beta}\eta^{\gamma\delta}\Lambda^{\beta}{}_{\gamma}V_{\delta} = \Lambda_{\alpha}{}^{\delta}V_{\delta}$$

similarly one can show U^{α} is contravariant.

Above we see four-vectors have one index. Tensors are more general objects with multiple indices. These transform in the obvious way

$$T^{\gamma}{}_{\alpha\beta} \to T'^{\gamma}{}_{\alpha\beta} = \Lambda^{\gamma}{}_{\delta}\Lambda_{\alpha}{}^{\varepsilon}\Lambda_{\beta}{}^{\rho}T^{\delta}{}_{\varepsilon\rho}$$

We can contract tensors by $T^{\alpha\gamma} \equiv T^{\alpha}{}_{\beta}{}^{\gamma\beta}$

$$\begin{aligned} T'^{\alpha\gamma} = T'^{\alpha}{}_{\beta}{}^{\gamma\beta} &= \Lambda^{\alpha}{}_{\delta}\Lambda_{\beta}{}^{\varepsilon}\Lambda^{\gamma}{}_{\rho}\Lambda^{\beta}{}_{\kappa}T^{\delta}{}_{\varepsilon}{}^{\rho\kappa} \\ &= \Lambda^{\alpha}{}_{\delta}\Lambda^{\gamma}{}_{\rho}\delta_{\kappa}{}^{\varepsilon}T^{\delta}{}_{\varepsilon}{}^{\rho\kappa} \\ &= \Lambda^{\alpha}{}_{\delta}\Lambda^{\gamma}{}_{\rho}T^{\delta}{}_{\varepsilon}{}^{\rho\varepsilon} = \Lambda^{\alpha}{}_{\delta}\Lambda^{\gamma}{}_{\rho}T^{\delta\rho} \end{aligned}$$

We can also take the direct product of two tensors giving a new tensor

$$T^{\alpha}{}_{\beta}{}^{\gamma} \equiv A^{\alpha}{}_{\beta}B^{\gamma}$$

this is how we take derivatives

$$T_{\alpha}{}^{\beta\gamma} \equiv \partial_{\alpha}T^{\beta\gamma}$$

The linear combination of tensors is also a tensor

$$T^{\alpha}{}_{\beta} \equiv aR^{\alpha}{}_{\beta} + bS^{\alpha}{}_{\beta}$$

These imply that $\delta^{\alpha}{}_{\beta}$ is a tensor since $\eta_{\alpha\beta}$ is a tensor. Similarly raising or lowering an index preserves *tensorness*

2. The equivalence principle

2.1. Newtonian field theory

Newtonian gravity is summarized by

$$\boldsymbol{F} = -G_N\frac{mM}{r^2}\hat{\boldsymbol{r}} = m\boldsymbol{g}$$

where we define the gravitational field

$$\boldsymbol{g} = -G_N\frac{M}{r^2}\hat{\boldsymbol{r}}$$

We can integrate this to find

$$\underbrace{\oint_S \boldsymbol{g}\cdot \mathrm{d}\boldsymbol{A}}_{\substack{\text{gravitational flux}\\ \text{through } S}} = -4\pi G_N M_{\text{enclosed}}$$

using Gauss' law we can rewrite this as

$$\int \boldsymbol{\nabla} \cdot \boldsymbol{g}\, \mathrm{d}V = -4\pi G_N \int \rho\, \mathrm{d}V$$

this is true for any volume V meaning

$$\boldsymbol{\nabla} \cdot \boldsymbol{g} = -4\pi G_N \rho$$

We define $\boldsymbol{g} \equiv -\boldsymbol{\nabla}\Phi$ with Φ being the gravitational potential giving

$$\nabla^2 \Phi = 4\pi G_N \rho$$

this is the Newtonian field equation. This equation describes how some matter distribution ρ shapes Φ. Newton's second law can also be written in terms of Φ giving

$$\frac{\mathrm{d}^2 \boldsymbol{r}}{\mathrm{d}t^2} = -\boldsymbol{\nabla}\Phi$$

this equation describes how a particle moves given Φ.

These two equations make up the Newtonian field theory of gravity. We immediately see these are incompatible with special relativity since time and space are not treated equally. Actually it is a static theory due to the *action at a distance* description which underlies Newtonian gravity.

2.2. The principle

We have two kinds of mass in Newtonian physics. The inertial mass m_i and the gravitational mass m_g. These are defined by

$$\boldsymbol{F} = m_i \boldsymbol{a}; \quad \boldsymbol{F} = m_g \boldsymbol{g}$$

The basic statement of the equivalence principle is $m_i \stackrel{!}{=} m_g$ meaning we can identify

$$\ddot{\boldsymbol{x}} = \boldsymbol{g}$$

Assuming $\ddot{\boldsymbol{x}} = \boldsymbol{g}$ we can always do a coordinate transformation to a local frame with no acceleration $\ddot{\boldsymbol{y}} = 0$ by

$$\boldsymbol{y} = \boldsymbol{x} - \frac{1}{2}\boldsymbol{g}t^2$$

as we will see this is very important! We call these locally inertial frames for freely falling elevators.

Einstein generalized this to all of physics by the strong equivalence principle.

In any gravitational field it is possible to select a locally inertial system such that the laws of physics are the same as in special relativity.

So we can always do a coordinate transformation to a freely falling elevator with no gravity!

3. The geodesic equation

3.1. The metric

Above we had

$$\mathrm{d}\tau^2 = -\eta_{\alpha\beta}\,\mathrm{d}y^\alpha\,\mathrm{d}y^\beta$$

under the assumption of no gravity. With gravity this is only true locally $y^\alpha \to y^\alpha(x)$

$$\begin{aligned}
\mathrm{d}\tau^2 &= -\eta_{\alpha\beta}\,\mathrm{d}y^\alpha(x)\,\mathrm{d}y^\beta(x) \\
&\overset{\text{chain rule}}{=} -\eta_{\alpha\beta}\frac{\partial y^\alpha(x)}{\partial x^\mu}\frac{\partial y^\beta(x)}{\partial x^\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu \\
&= -g_{\mu\nu}(x)\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu
\end{aligned}$$

this is a global relation. We have defined the metric g

$$g_{\mu\nu}(x) = \eta_{\alpha\beta}\frac{\partial y^\alpha(x)}{\partial x^\mu}\frac{\partial y^\beta(x)}{\partial x^\nu}$$

which is now space-dependent meaning space becomes non-Euclidean. Therefore g contains the effect gravity!

3.2. The equation

Locally we have

$$\frac{\mathrm{d}^2 y^\alpha(x)}{\mathrm{d}\tau^2} = 0$$

We can rewrite this as

$$\begin{aligned}
0 &= \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\mathrm{d}y^\alpha(x)}{\mathrm{d}\tau}\right) \\
0 &= \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\partial y^\alpha}{\partial x^\mu}\frac{\partial x^\mu(\tau)}{\partial \tau}\right) \\
0 &= \frac{\partial y^\alpha}{\partial x^\mu}\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\tau^2} + \frac{\partial^2 y^\alpha}{\partial x^\mu \partial x^\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} \\
0 &= \frac{\partial x^\lambda}{\partial y^\alpha}\left(\frac{\partial y^\alpha}{\partial x^\mu}\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\tau^2} + \frac{\partial^2 y^\alpha}{\partial x^\mu \partial x^\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\right)
\end{aligned}$$

$$
\begin{aligned}
0 &= \delta^\lambda{}_\mu \frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\tau^2} + \frac{\partial x^\lambda}{\partial y^\alpha}\frac{\partial^2 y^\alpha}{\partial x^\mu \partial x^\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} \\
0 &= \frac{\mathrm{d}^2 x^\lambda}{\mathrm{d}\tau^2} + \Gamma^\lambda{}_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} \equiv \ddot{x}^\lambda + \Gamma^\lambda{}_{\mu\nu}\dot{x}^\mu \dot{x}^\nu
\end{aligned}
$$

This is called the *geodesic equation*. We have defined

$$
\Gamma^\lambda{}_{\mu\nu} \equiv \frac{\partial x^\lambda}{\partial y^\alpha}\frac{\partial^2 y^\alpha}{\partial x^\mu \partial x^\nu} \overset{\text{flat space}}{=} 0
$$

which is the *Christoffel symbol*. With our definition it is *torsion-free* meaning symmetric in the lower indices $\Gamma^\lambda{}_{\mu\nu} = \Gamma^\lambda{}_{\nu\mu}$.

Since the Christoffel symbol vanishes in flat space there must be some dependence on g. We have

$$
\Gamma^\lambda{}_{\mu\nu}\frac{\partial y^\beta}{\partial x^\lambda} = \frac{\partial^2 y^\beta}{\partial x^\mu \partial x^\nu}
$$

Then

$$
\begin{aligned}
\frac{\partial g_{\mu\nu}}{\partial x^\lambda} &= \frac{\partial}{\partial x^\lambda}\left(\eta_{\alpha\beta}\frac{\partial y^\alpha}{\partial x^\mu}\frac{\partial y^\beta}{\partial x^\nu}\right) \\
&= \eta_{\alpha\beta}\frac{\partial^2 y^\alpha}{\partial x^\lambda \partial x^\mu}\frac{\partial y^\beta}{\partial x^\nu} + \eta_{\alpha\beta}\frac{\partial^2 y^\beta}{\partial x^\lambda \partial x^\nu}\frac{\partial y^\alpha}{\partial x^\mu} \\
&\overset{\text{by above}}{=} \eta_{\alpha\beta}\frac{\partial y^\beta}{\partial x^\nu}\Gamma^\rho{}_{\lambda\mu}\frac{\partial y^\alpha}{\partial x^\rho} + \eta_{\alpha\beta}\frac{\partial y^\alpha}{\partial x^\mu}\Gamma^\rho{}_{\lambda\nu}\frac{\partial y^\beta}{\partial x^\rho} \\
&= g_{\rho\nu}\Gamma^\rho{}_{\mu\lambda} + g_{\rho\mu}\Gamma^\rho{}_{\nu\lambda}
\end{aligned}
$$

Using this and symmetry we have

$$
\frac{\partial g_{\mu\nu}}{\partial x^\lambda} + \frac{\partial g_{\lambda\nu}}{\partial x^\mu} - \frac{\partial g_{\mu\lambda}}{\partial x^\nu} = 2 g_{\sigma\nu}\Gamma^\rho{}_{\lambda\mu}
$$

To isolate Γ we need the inverse metric $g^{\mu\nu}$ satisfying

$$
g_{\mu\nu}(x) g^{\nu\sigma}(x) = \delta^\sigma{}_\mu
$$

We claim

$$
g^{\nu\sigma} = \eta^{\alpha\beta}\frac{\partial x^\nu}{\partial y^\alpha}\frac{\partial x^\sigma}{\partial y^\beta}
$$

This is simple to show

$$g_{\mu\nu}g^{\nu\sigma} = \eta_{\gamma\delta}\eta^{\alpha\beta}\frac{\partial y^\gamma}{\partial x^\mu}\underbrace{\frac{\partial y^\delta}{\partial x^\nu}\frac{\partial x^\nu}{\partial y^\alpha}}_{\delta^\delta{}_\alpha}\frac{\partial x^\sigma}{\partial y^\beta} = \underbrace{\eta_{\gamma\delta}\eta^{\delta\beta}}_{\delta^\beta{}_\gamma}\frac{\partial y^\gamma}{\partial x^\mu}\frac{\partial x^\sigma}{\partial y^\beta} = \delta^\sigma{}_\mu$$

Then we find

$$\Gamma^\lambda{}_{\mu\nu} = \frac{1}{2}g^{\lambda\sigma}\left[\frac{\partial g_{\nu\sigma}}{\partial x^\mu} + \frac{\partial g_{\mu\sigma}}{\partial x^\nu} - \frac{\partial g_{\mu\nu}}{\partial x^\sigma}\right] = \frac{1}{2}g^{\lambda\sigma}\left[\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} \underbrace{-\partial_\sigma g_{\mu\nu}}_{\text{symmetric}}\right]$$

where the dependence on g is manifest.

3.3. The Newtonian limit

For the above to be consistent we should be able to recover Newtonian gravity in what we call *the Newtonian limit*. In this limit all velocities are small

$$\left|\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}\tau}\right| \ll 1$$

and the metric g is static. We also assume gravity is weak meaning we can write

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x)$$

with $|h| \ll 1$.

Assuming small velocities the geodesic equation reduces to

$$\begin{aligned} 0 &= \ddot{x}^\mu + \Gamma^\mu{}_{\nu\lambda}\dot{x}^\nu\dot{x}^\lambda \\ &\simeq \ddot{x}^\mu + \Gamma^\mu{}_{00}\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 \end{aligned}$$

Assuming a static g the Christoffel symbol becomes

$$\begin{aligned} \Gamma^\mu{}_{00} &= \frac{1}{2}g^{\mu\sigma}[\partial_0 g_{0\sigma} + \partial_0 g_{0\sigma} - \partial_\sigma g_{00}] \\ &\overset{\text{static } g}{\simeq} -\frac{1}{2}g^{\mu\sigma}\partial_\sigma g_{00} \end{aligned}$$

Assuming gravity is weak we have to order $\mathcal{O}(h)$

$$\Gamma^\mu{}_{00} \simeq -\frac{1}{2}\eta^{\mu\sigma}\partial_\sigma h_{00}$$

Since h_{00} does not depend on time we find $\Gamma^0{}_{00} = 0$ meaning

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = \text{constant}$$

by the geodesic equation. Then for $\mu = i$

$$\begin{aligned}
0 &\simeq \frac{\mathrm{d}^2\boldsymbol{x}}{\mathrm{d}\tau^2} - \frac{1}{2}\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 \boldsymbol{\nabla} h_{00}(\boldsymbol{x}) \\
&= \frac{\mathrm{d}^2\boldsymbol{x}}{\mathrm{d}t^2} - \frac{1}{2}\boldsymbol{\nabla} h_{00}(\boldsymbol{x})
\end{aligned}$$

By comparison we identify

$$h_{00} = -2\Phi + \underbrace{\text{constant}}_{\text{can absorb}}$$

meaning

$$g_{00} = -(1 + 2\Phi)$$

which is quite nice!

This already leads to non-trivial results. Consider two clocks at rest. Both satisfy

$$\mathrm{d}\tau^2 = -g_{00}\,\mathrm{d}t^2$$

Then since $\omega \propto \mathrm{d}\tau^{-1}$ we have to order $\mathcal{O}(\Phi)$

$$\begin{aligned}
\frac{\omega_2 - \omega_1}{\omega_1} &= \sqrt{\frac{-g_{00}(x_1)}{-g_{00}(x_2)}} - 1 \\
&= \sqrt{\frac{1 + 2\Phi(x_1)}{1 + 2\Phi(x_2)}} - 1 \\
&\overset{\text{Taylor}}{\simeq} \Phi(x_1) - \Phi(x_2)
\end{aligned}$$

so

$$\frac{\Delta\omega}{\omega_1} = -\,\Delta\Phi$$

This is gravitational redshift!

4. The principle of general covariance

4.1. The principle

We want to express the laws of physics such that they hold in all frames. This is called *general covariance*. Specifically we want to write the laws of special relativity in covariant form. Then by definition they would be valid in all frames including those with gravity. This is the principle of general covariance.

Before we see how this is done consider the geodesic equation

$$\ddot{x}^\lambda + \underbrace{\Gamma^\lambda{}_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}_{\text{gravity}} = 0$$

At all points $x = \tilde{x}$ we can by the equivalence principle define a freely falling elevator where

$$g_{\mu\nu}(\tilde{x}) = \eta_{\mu\nu}$$

This requires the Christoffel symbol vanishes at $x = \tilde{x}$ meaning

$$\partial_\sigma g_{\mu\nu}(x)\big|_{x=\tilde{x}} = 0$$

But we do not require

$$\partial_\sigma \partial_\rho g_{\mu\nu}(x)\big|_{x=\tilde{x}} = 0$$

so if we move away from $x = \tilde{x}$ the Christoffel symbol is not guaranteed to vanish. And in fact if this were true then $g_{\mu\nu} = \eta_{\mu\nu}$ everywhere! When we quantify curvature later this becomes important.

4.2. Tensors in general relativity

We saw before how tensors in special relativity are objects that transform in a specific way under Lorentz transformations. This notion generalizes to general tensors being objects that transform in a specific way under general coordinate transformations.

Scalars are objects like $\mathrm{d}\tau$ and $\varphi(x)$. These are invariant under $x \to x'$ meaning they transform trivially.

$\mathrm{d}x^\mu$ transforms as

$$\mathrm{d}x'^\mu \overset{\text{chain rule}}{=} \frac{\partial x'^\mu}{\partial x^\nu}\,\mathrm{d}x^\nu$$

anything transforming like $\mathrm{d}x^\mu$ is a contravariant vector. Anything transforming inversely of this is a covariant vector

$$A'_\mu = \frac{\partial x^\nu}{\partial x'^\mu} A_\nu$$

This is the inverse since contracting gives a scalar

$$A'_\mu U'^\mu = \underbrace{\frac{\partial x^\nu}{\partial x'^\mu}\frac{\partial x'^\mu}{\partial x^\sigma}}_{\delta^\nu{}_\sigma} A_\nu U^\sigma = A_\nu U^\nu$$

We can also form a covariant vector by differentiating a scalar

$$\partial_{\mu'}\varphi'(x') \overset{\text{chain rule}}{=} \frac{\partial x^\nu}{\partial x'^\mu}\underbrace{\partial_\nu \varphi(x)}_{A_\mu}$$

But we will find that $\partial_\mu U^\nu$ is not generally a tensor.

General tensors transform in the obvious way. As an example take the metric $g_{\mu\nu}$

$$g'_{\mu\nu}(x') = \eta_{\alpha\beta}\frac{\partial y^\alpha}{\partial x'^\mu}\frac{\partial y^\beta}{\partial x'^\nu} = \eta_{\alpha\beta}\frac{\partial y^\alpha}{\partial x^\rho}\frac{\partial y^\beta}{\partial x^\sigma}\frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu} = \frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu} g_{\rho\sigma}(x)$$

so it is covariant. Similarly the inverse metric $g^{\mu\nu}$ is contravariant. Using the metric we define

$$\tilde{T}_{\mu\nu} \equiv \overbrace{\underbrace{g_{\mu\sigma}g_{\nu\rho}}_{2\times\text{ covariant}} \quad \underbrace{T^{\sigma\rho}}_{\text{contravariant}}}^{\text{covariant}} \equiv T_{\mu\nu}$$

By comparison $\tilde{T}_{\mu\nu}$ is covariant since the indices $\sigma\rho$ *cancel* leaving the covariant $\mu\nu$. We write $\tilde{T}_{\mu\nu} \equiv T_{\mu\nu}$ with $T_{\mu\nu}$ and $T^{\mu\nu}$ representing the same object. Similarly $\delta^\mu{}_\nu$ is a mixed tensor $\delta^\mu{}_\nu = g^{\mu\sigma}g_{\sigma\nu}$ since the σ *cancels.*

We define $g \equiv \det g_{\mu\nu}$. This is used to define the invariant measure

$$\underbrace{\sqrt{-g}}_{\text{inverse Jacobian}} \quad \mathrm{d}^4x$$

4.3. The covariant derivative

As mentioned $\partial_\mu A^\nu$ is not a tensor. This is a problem since we would like to take derivatives. Consider

$$\begin{aligned}\text{invariant} &= \frac{\mathrm{d}\varphi(x)}{\mathrm{d}\tau} \\ &= \frac{\partial\varphi}{\partial x^\mu}\left(\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\right)\end{aligned}$$

This implies

$$\text{invariant} = \frac{\mathrm{d}^2\varphi}{\mathrm{d}\tau^2}$$

$$= \frac{\partial^2\varphi}{\partial x^\mu \partial x^\nu}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} + \frac{\partial\varphi}{\partial x^\mu}\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\tau^2}$$

Assuming $x^\mu(\tau)$ is described by the geodesic equation we find

$$\underbrace{\frac{\mathrm{d}^2\varphi}{\mathrm{d}\tau^2}}_{\text{scalar}} = \overbrace{\left[\frac{\partial^2\varphi}{\partial x^\mu \partial x^\nu} - \Gamma^\sigma{}_{\mu\nu}\frac{\partial\varphi}{\partial x^\sigma}\right]}^{\text{must be covariant}} \times \underbrace{\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}}_{\text{contravariant}}$$

We call the object in [...] the covariant derivative. Let $V_\mu = \partial_\mu\varphi$ then [...] becomes

$$D_\mu V_\nu = \partial_\mu V_\nu - \Gamma^\sigma{}_{\mu\nu} V_\sigma$$

and similarly

$$D_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu{}_{\mu\sigma} V^\sigma$$

which generalizes in the obvious way.

We see $D_\mu \to \partial_\mu$ in the freely falling elevator since Γ vanishes. This also implies that both $\Gamma^\lambda{}_{\mu\nu}$ and ∂_μ are not tensors.

Consider $D_\sigma g_{\mu\nu}$. For a freely falling elevator we have

$$D_\sigma g_{\mu\nu} \underset{\substack{\text{possible by}\\ \text{equivalence principle}}}{\overset{\text{free fall}}{=}} \partial_\sigma g_{\mu\nu} \overset{\text{flat locally}}{=} 0$$

Since $D_\sigma g_{\mu\nu}$ is written in covariant form then the above is true in all frames when it is true in one frame! This relies on us being able to pick a freely falling elevator, this is possible by the equivalence principle. Generally any equation that is true in flat space becomes true in all frames by the replacement $\partial_\mu \to D_\mu$.

Consider

$$\begin{aligned} \frac{DV^\nu}{D\tau} &\equiv \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} D_\mu V^\nu \\ &= \frac{\mathrm{d}V^\nu}{\mathrm{d}\tau} + \Gamma^\nu{}_{\mu\sigma} V^\sigma \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} \end{aligned}$$

We have $D_\tau \to d_\tau$ in the freely falling elevator. We can now immediately derive the geodesic equation by

$$\frac{DV^\mu}{D\tau} = 0$$

which reduces to

$$\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\tau^2} = 0$$

in the freely falling elevator.

4.4. Electrodynamics

As an example of the procedure mentioned above we consider classical electrodynamics.

We define the antisymmetric Faraday tensor

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$$

and the four-current $J^\mu \equiv (\rho, \boldsymbol{J})$. Then Maxwell's equations can be written as

$$\begin{aligned} \partial_\mu F^{\mu\nu} &= -J^\nu \\ 0 &= \partial_\mu F_{\nu\gamma} + \partial_\gamma F_{\mu\nu} + \partial_\nu F_{\gamma\mu} \end{aligned}$$

To include gravity we replace $\partial_\mu \to D_\mu$

$$\begin{aligned} D_\mu F^{\mu\nu} &= -J^\nu \\ 0 &= D_\mu F_{\nu\lambda} + D_\lambda F_{\mu\nu} + D_\nu F_{\lambda\mu} \overset{\Gamma \text{ cancel}}{=} \partial_\mu F_{\nu\gamma} + \partial_\gamma F_{\mu\nu} + \partial_\nu F_{\gamma\mu} \end{aligned}$$

We can show

$$D_\mu F^{\mu\nu} = \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} F^{\mu\nu})$$

which holds for any antisymmetric $F^{\mu\nu}$. Then by $D_\mu F^{\mu\nu} = -J^\nu$ we have

$$\partial_\mu (\sqrt{-g} F^{\mu\nu}) = -\sqrt{-g} J^\nu$$

Then

$$\underbrace{\partial_\nu \partial_\mu}_{\text{symmetric}} \overbrace{(\sqrt{-g} F^{\mu\nu})}^{\text{antisymmetric}} = 0$$

implying

$$\partial_\nu (\sqrt{-g} J^\nu) = 0$$

This is the covariant form of $\partial_\mu J^\mu = 0$.

Consider the Lorentz force

$$f^\mu = e F^\mu{}_\nu U^\nu$$

this is already covariant!

5. The Einstein field equations

5.1. The energy-momentum tensor

The Newtonian limit gave

$$\Phi = -\frac{1 + g_{00}}{2}$$

Then

$$\begin{aligned}\nabla^2\Phi &= -\frac{1}{2}\nabla^2 g_{00} \\ &\stackrel{!}{=} 4\pi G_N \rho\end{aligned}$$

or

$$\nabla^2 g_{00}(\boldsymbol{x}) = -8\pi G_N \rho(\boldsymbol{x})$$

with $\rho(\boldsymbol{x})$ being the matter density. We want to generalize this equation. We see by the LHS that $\rho(\boldsymbol{x})$ should generalize to some tensor. The natural choice is the energy-momentum tensor of some relativistic flow

$$T^{\mu\nu} = \rho(x) U^\mu(x) U^\nu(x)$$

where $\rho(x)$ is the energy density seen by an observer moving with the flow.

To see that $T^{\mu\nu}$ does describe flow consider T^{00}

$$\begin{aligned}T^{00} &= \rho(x)\left(\frac{\mathrm{d}x^0}{\mathrm{d}\tau}\right)^2 \\ &= \frac{\rho}{1 - \boldsymbol{v}^2}\end{aligned}$$

where we use for $g_{\mu\nu} = \eta_{\mu\nu}$

$$\mathrm{d}\tau^2 = \left(\mathrm{d}x^0\right)^2\left(1 - \boldsymbol{v}^2\right)$$

with $\boldsymbol{v} = \mathrm{d}\boldsymbol{x} \,/\, \mathrm{d}x^0$. Taking the Newtonian limit we have

$$T^{00} \simeq \rho$$

Then

$$\nabla^2 g_{00} = -8 G_N T_{00}$$

The other components of $T^{\mu\nu}$ are

$$T^{ij} = \underbrace{\frac{\rho v_i v_j}{1-\boldsymbol{v}^2}}_{\substack{\text{momentum}\\\text{current}}}$$

$$T^{i0} = \underbrace{\frac{\rho v_i}{1-\boldsymbol{v}^2}}_{\substack{\text{momentum}\\\text{density}}} = T^{0i}$$

$T^{\mu\nu}$ is the energy-momentum tensor of a system (with $g_{\mu\nu} = \eta_{\mu\nu}$) with no pressure in the rest-frame. With the above identifications.

Assuming the system is closed then $T^{\mu\nu}$ is conserved

$$\partial_\nu T^{\mu\nu} = 0$$

Taking $\mu = 0$ we find

$$\begin{aligned}\partial_\nu T^{0\nu} &= \frac{\partial}{\partial t}\Big(\frac{\rho}{1-\boldsymbol{v}^2}\Big) + \nabla\Big(\frac{\rho\boldsymbol{v}}{1-\boldsymbol{v}^2}\Big)\\ &\overset{!}{=} 0\end{aligned}$$

which is the relativistic Euler equation. Taking $\mu = i$ we find

$$\begin{aligned}\partial_\nu T^{i\nu} &= \frac{\partial}{\partial t}\Big(\frac{\rho v_i}{1-\boldsymbol{v}^2}\Big) + \frac{\partial}{\partial x}\Big(\frac{\rho v_i v_x}{1-\boldsymbol{v}^2}\Big) + \frac{\partial}{\partial y}\Big(\frac{\rho v_i v_y}{1-\boldsymbol{v}^2}\Big) + \frac{\partial}{\partial z}\Big(\frac{\rho v_i v_z}{1-\boldsymbol{v}^2}\Big)\\ &= \frac{\rho}{1-\boldsymbol{v}^2}\left(\frac{\partial v^i}{\partial t} + \boldsymbol{v}\cdot\nabla v^i\right)\\ &\underset{\text{four-velocity constant}}{\overset{!}{=}} 0\end{aligned}$$

which is the relativistic Navier-Stokes equation. Then $\partial_\nu T^{\mu\nu} = 0$ is a direct consequence of relativistic fluid mechanics.

5.2. The perfect fluid

A perfect fluid includes pressure. We define a perfect fluid by a field $\boldsymbol{v}(x)$ with the property that any observer moving with the fluid sees it as isotropic everywhere. The rest frame requires

$$\begin{aligned}T^{00} &= \rho\\ T^{i0} &= 0 = T^{0i}\\ T^{ij} &= p\delta_{ij} = T^{ji}\end{aligned}$$

Assuming the fluid has some velocity $\boldsymbol{v}(x)$ then

$$T^{\mu\nu} = p\eta^{\mu\nu} + (p+\rho)U^\mu U^\nu$$

This reduces to the rest-frame $T^{\mu\nu}$. The generally covariant form is obtained by $\eta^{\mu\nu} \to g^{\mu\nu}$

$$T^{\mu\nu} = pg^{\mu\nu} + (p+\rho)U^\mu U^\nu$$

which is very important!

The fluid equations in general relativity are

$$D_\nu T^{\mu\nu} = 0$$

but the total energy-momentum defined by

$$P^\mu = \int \mathrm{d}^3x \sqrt{-g} T^{\mu 0}$$

is not conserved!

5.3. The curvature tensor

We want some covariant object that vanishes in flat space but not in curved space. This would let us determine if space is flat independent of coordinates. We already have a candidate since

$$\left[\partial_\mu, \partial_\nu\right] \overset{\text{flat space}}{=} 0$$

meaning if

$$\left[D_\mu, D_\nu\right] \overset{\text{curved space}}{\neq} 0$$

we would be done since $D_\mu \to \partial_\mu$ in flat space.

We now show $\left[D_\mu, D_\nu\right] \neq 0$. By definition

$$D_\kappa T_{\mu\nu} = \partial_\kappa T_{\mu\nu} - \Gamma^\lambda{}_{\nu\kappa} T_{\mu\lambda} - \Gamma^\lambda{}_{\mu\kappa} T_{\lambda\nu}$$

Let $T_{\mu\nu} \equiv D_\mu V_\nu$. Then

$$D_\kappa D_\mu V_\nu = \partial_\kappa D_\mu V_\nu - \Gamma^\lambda{}_{\nu\kappa} D_\mu V_\lambda - \Gamma^\lambda{}_{\mu\kappa} D_\lambda V_\nu$$

and similarly

$$D_\mu D_\kappa V_\nu = \partial_\mu D_\kappa V_\nu - \Gamma^\lambda{}_{\nu\mu} D_\kappa V_\lambda - \Gamma^\lambda{}_{\kappa\mu} D_\lambda V_\nu$$

Then

$$\begin{aligned}
\left[D_\kappa, D_\mu\right] V_\nu &= D_\kappa D_\mu V_\nu - D_\mu D_\kappa V_\nu \\
&= \partial_\kappa D_\mu V_\nu - \partial_\mu D_\kappa V_\nu - \Gamma^\lambda{}_{\nu\kappa} D_\mu V_\lambda + \Gamma^\lambda{}_{\nu\mu} D_\kappa V_\lambda
\end{aligned}$$

$$
\begin{aligned}
&= \partial_\kappa\big(\partial_\mu V_\nu - \Gamma^\sigma{}_{\mu\nu}V_\sigma\big) - \partial_\mu(\partial_\kappa V_\nu - \Gamma^\sigma{}_{\kappa\nu}V_\sigma) \\
&\quad -\Gamma^\lambda{}_{\nu\kappa}\big(\partial_\mu V_\lambda - \Gamma^\sigma{}_{\mu\lambda}V_\sigma\big) + \Gamma^\lambda{}_{\nu\mu}(\partial_\kappa V_\lambda - \Gamma^\sigma{}_{\kappa\lambda}V_\sigma) \\
&= -\partial_\kappa\Gamma^\sigma{}_{\mu\nu}V_\sigma - \Gamma^\sigma{}_{\mu\nu}\partial_\kappa V_\sigma + \partial_\mu\Gamma^\sigma{}_{\kappa\nu}V_\sigma + \Gamma^\sigma{}_{\kappa\nu}\partial_\mu V_\sigma \\
&\quad -\Gamma^\lambda{}_{\nu\kappa}\partial_\mu V_\lambda + \Gamma^\lambda{}_{\nu\kappa}\Gamma^\sigma{}_{\mu\lambda}V_\sigma + \Gamma^\lambda{}_{\nu\mu}\partial_\kappa V_\lambda - \Gamma^\lambda{}_{\nu\mu}\Gamma^\sigma{}_{\kappa\lambda}V_\sigma \\
&= \big[-\partial_\kappa\Gamma^\sigma{}_{\mu\nu} + \partial_\mu\Gamma^\sigma{}_{\kappa\nu} - \Gamma^\lambda{}_{\nu\mu}\Gamma^\sigma{}_{\kappa\lambda} + \Gamma^\lambda{}_{\nu\kappa}\Gamma^\sigma{}_{\mu\lambda}\big]V_\sigma \\
&= -R^\sigma{}_{\nu\mu\kappa}V_\sigma
\end{aligned}
$$

where we have defined the *Riemann-Christoffel curvature tensor*

$$
R^\sigma{}_{\nu\mu\kappa} = \partial_\kappa\Gamma^\sigma{}_{\mu\nu} - \partial_\mu\Gamma^\sigma{}_{\kappa\nu} + \Gamma^\lambda{}_{\nu\mu}\Gamma^\sigma{}_{\kappa\lambda} - \Gamma^\lambda{}_{\nu\kappa}\Gamma^\sigma{}_{\mu\lambda} \overset{\text{flat space}}{=} 0
$$

A more useful form is

$$
\begin{aligned}
R_{\lambda\mu\nu\kappa} &= g_{\lambda\sigma}R^\sigma{}_{\nu\mu\kappa} \\
&= \frac{1}{2}\left[\frac{\partial^2 g_{\lambda\nu}}{\partial x^\kappa\partial x^\mu} + \frac{\partial^2 g_{\mu\kappa}}{\partial x^\nu\partial x^\lambda} - \frac{\partial^2 g_{\mu\nu}}{\partial x^\kappa\partial x^\lambda} - \frac{\partial^2 g_{\lambda\kappa}}{\partial x^\nu\partial x^\mu}\right] \\
&\qquad\qquad + g_{\eta\sigma}\big[\Gamma^\eta{}_{\nu\lambda}\Gamma^\sigma{}_{\mu\kappa} - \Gamma^\eta{}_{\kappa\lambda}\Gamma^\sigma{}_{\mu\nu}\big]
\end{aligned}
$$

so $R \sim \partial^2 g$ as we would expect. This form also shows the symmetries

$$
\begin{aligned}
R_{\lambda\mu\nu\kappa} &= R_{\nu\kappa\lambda\mu} && (\lambda\mu)(\nu\kappa) \leftrightarrow (\nu\kappa)(\lambda\mu) \\
R_{\lambda\mu\nu\kappa} &= -R_{\mu\lambda\nu\kappa} && \lambda \leftrightarrow \mu \\
R_{\mu\lambda\nu\kappa} &= R_{\lambda\mu\kappa\nu} && \lambda \leftrightarrow \mu, \kappa \leftrightarrow \nu \\
R_{\lambda\mu\nu\kappa} &= -R_{\lambda\mu\kappa\nu} && \kappa \leftrightarrow \nu \\
0 &= \underbrace{R_{\lambda\mu\nu\kappa} + R_{\lambda\kappa\mu\nu} + R_{\lambda\nu\kappa\mu}}_{\text{cyclic permutation of } (\mu\nu\kappa)}
\end{aligned}
$$

with the last being the *first Bianchi identity*. To see these it is simplest to consider a freely falling elevator.

We define the *Ricci tensor*

$$
R_{\mu\kappa} = g^{\lambda\nu}R_{\lambda\mu\nu\kappa} = R^\lambda{}_{\mu\lambda\kappa}
$$

which is symmetric. We also define the *Ricci scalar*

$$
R = g^{\mu\kappa}R_{\mu\kappa} = R^\mu{}_\mu
$$

We need the *second Bianchi identity*

$$D_\eta R_{\lambda\mu\nu\kappa} + D_\kappa R_{\lambda\mu\eta\nu} + D_\nu R_{\lambda\mu\kappa\eta} = 0$$

and recall

$$D_\sigma g_{\mu\nu} = 0$$

Then we can rewrite the second Bianchi identity

$$\begin{aligned}
0 &= g^{\lambda\nu}\big(D_\eta R_{\lambda\mu\nu\kappa} + D_\kappa R_{\lambda\mu\eta\nu} + D_\nu R_{\lambda\mu\kappa\eta}\big) \\
0 &= D_\eta R_{\mu\kappa} - D_\kappa R_{\mu\eta} + D_\nu R^\nu{}_{\mu\kappa\eta} \\
0 &= g^{\mu\kappa}\big(D_\eta R_{\mu\kappa} - D_\kappa R_{\mu\eta} + D_\nu R^\nu{}_{\mu\kappa\eta}\big) \\
0 &= D_\eta R - D_\mu R^\mu{}_\eta - D_\nu R^\nu{}_\eta \\
0 &= D_\mu\left(R^\mu{}_\eta - \frac{1}{2}\delta^\mu{}_\eta R\right) \\
0 &= D_\mu \underbrace{\left(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R\right)}_{\text{trace-reversed Ricci tensor}}
\end{aligned}$$

So the *trace-reversed Ricci tensor* is conserved!

5.4. The field equations

We had

$$\nabla^2 g_{00}(x) = -8\pi G_N T_{00}(x)$$

We write this as a tensor equation

$$G_{\mu\nu}(x) = -8\pi G_N T_{\mu\nu}(x)$$

where $G_{\mu\nu}$ depends on g and ∂g. By dimensional analysis the LHS should have units $\left[L^{-2}\right]$ which matches $\partial^2 g$. Then an obvious guess would be

$$G_{\mu\nu} \stackrel{?}{=} R_{\mu\nu}$$

However we require

$$D_\nu G^{\mu\nu} \stackrel{!}{=} 0 \quad \text{since} \quad D_\nu T^{\mu\nu} = 0$$

implying

$$G_{\mu\nu} = A\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right)$$

To determine A consider

$$G_{00} = A\left(R_{00} + \frac{1}{2}R\right) = -8\pi G_N T_{00}$$

We take the Newtonian limit. Then $T_{ij} \to 0$ giving

$$0 \simeq R_{ij} - \frac{1}{2}g_{ij}R$$
$$R_{ij} \simeq \frac{1}{2}g_{ij}R$$

And $g_{ij} \simeq \eta_{ij}$ giving

$$\begin{aligned} R &\simeq R_{ii} - R_{00} \simeq \frac{3}{2}R - R_{00} \\ R &\simeq 2R_{00} \end{aligned}$$

Then

$$G_{00} = 2AR_{00}$$

Similarly $\Gamma\Gamma \sim 0$ giving

$$R_{\lambda\mu\nu\kappa} \simeq \frac{1}{2}\left(\frac{\partial^2 g_{\lambda\nu}}{\partial x^\kappa \partial x^\mu} - \frac{\partial^2 g_{\mu\nu}}{\partial x^\nu \partial x^\lambda} - \frac{\partial^2 g_{\lambda\kappa}}{\partial x^\nu \partial x^\mu} + \frac{\partial^2 g_{\mu\kappa}}{\partial x^\nu \partial x^\lambda}\right)$$

g is $\sim$ static so

$$R_{0000} \simeq 0$$
$$R_{i0j0} \simeq \frac{1}{2}\frac{\partial^2 g_{00}}{\partial x^i \partial x^j}$$

Then

$$\begin{aligned} G_{00} &\simeq 2AR_{00} \\ &\simeq 2A(R_{i0i0} - R_{0000}) \\ &\simeq A\nabla^2 g_{00} \end{aligned}$$

implying $A \overset{!}{=} 1$.

The *Einstein field equations* are then

$$\underbrace{R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R}_{G_{\mu\nu}} = -8\pi G_N T_{\mu\nu}$$

where $G_{\mu\nu}$ is typically called the *Einstein tensor*. A more useful form is

$$R_{\mu\nu} = -8\pi G_N \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T^\alpha{}_\alpha\right)$$

We see $T_{\mu\nu} = 0$ gives $R_{\mu\nu} = 0$.

The EFE can be extended by including a *cosmological constant* Λ

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - \Lambda g_{\mu\nu} = -8\pi G_N T_{\mu\nu}$$

since $D_\mu g_{\mu\nu} = 0$! We can interpret $\Lambda g_{\mu\nu}$ as an energy density since we can move it to the RHS and absorb it in $T_{\mu\nu}$.

6. The Einstein-Hilbert action

6.1. Without matter

The simplest action we can write is

$$S = \int \mathrm{d}^4x \sqrt{-g} R$$

this is the *Einstein-Hilbert action*. We write $R = g^{\mu\nu} R_{\mu\nu}$ to find

$$\delta S = \int \mathrm{d}^4x \left[(\delta\sqrt{-g}) g^{\mu\nu} R_{\mu\nu} + \sqrt{-g}(\delta g^{\mu\nu}) R_{\mu\nu} + \sqrt{-g} g^{\mu\nu}\, \delta R_{\mu\nu}\right]$$

We claim

$$\begin{aligned}
\delta g^{\mu\nu} &= -g^{\mu\rho} g^{\nu\sigma}\, \delta g_{\rho\sigma} \\
\delta\sqrt{-g} &= -\frac{1}{2}\sqrt{-g} g_{\mu\nu}\, \delta g^{\mu\nu} \\
\delta R_{\mu\nu} &= D_\rho\, \delta\Gamma^\rho{}_{\mu\nu} - D_\nu\, \delta\Gamma^\rho{}_{\mu\rho}
\end{aligned}$$

with

$$\delta\Gamma^\rho{}_{\mu\nu} = \frac{1}{2} g^{\rho\sigma}\left(D_\mu\, \delta g_{\sigma\nu} + D_\nu\, \delta g_{\sigma\mu} - D_\sigma\, \delta g_{\mu\nu}\right)$$

and $\delta\Gamma$ being a tensor since it is the difference of two Christoffel symbols. Consider

$$\delta\Gamma^\rho{}_{\mu\nu} \overset{\text{trivially}}{=} \frac{1}{2} g^{\rho\sigma}\left(\partial_\mu\, \delta g_{\sigma\nu} + \partial_\nu\, \delta g_{\sigma\mu} - \partial_\sigma\, \delta g_{\mu\nu}\right)$$

and since $\delta\Gamma$ is a tensor it is valid for $\partial_\mu \to D_\mu$. Consider

$$R^\sigma{}_{\rho\mu\nu} \overset{\text{free fall}}{=} \partial_\mu \Gamma^\sigma{}_{\nu\rho} - \partial_\nu \Gamma^\sigma{}_{\mu\rho}$$

Then

$$
\begin{aligned}
\delta R^{\sigma}{}_{\rho\mu\nu} &= \partial_\mu\, \delta\Gamma^{\sigma}{}_{\nu\rho} - \partial_\nu\, \delta\Gamma^{\sigma}{}_{\mu\rho} \\
&\overset{\text{everywhere}}{=} D_\mu\, \delta\Gamma^{\sigma}{}_{\nu\rho} - D_\nu\, \delta\Gamma^{\sigma}{}_{\mu\rho}
\end{aligned}
$$

for $\mu = \sigma$

$$
\delta R_{\rho\nu} = D_\mu\, \delta\Gamma^{\mu}{}_{\nu\rho} - D_\nu\, \delta\Gamma^{\mu}{}_{\rho\mu}
$$

implying

$$
g^{\mu\nu}\, \delta R_{\mu\nu} = D_\mu X^\mu
$$

with

$$
X^\mu = g^{\rho\nu}\, \delta\Gamma^{\mu}{}_{\rho\nu} - g^{\mu\nu}\, \delta\Gamma^{\rho}{}_{\nu\rho}
$$

Then

$$
\delta S = \int \mathrm{d}^4x\, \sqrt{-g} \left[\left(R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \right) \delta g^{\mu\nu} + \underbrace{D_\mu X^\mu}_{\text{total derivative}} \right]
$$

Taking $\delta S = 0$ we find

$$
G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 0
$$

which are the vacuum EFE.

A slightly less trivial action is found by multiplying the volume form with a constant

$$
S = \frac{1}{16\pi G} \int \mathrm{d}^4x\, \sqrt{-g} (R - 2\Lambda)
$$

Taking $\delta S = 0$ we find

$$
R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = -\Lambda g_{\mu\nu}
$$

which are the vacuum EFE with a cosmological constant.

6.2. With matter

The action becomes

$$
S = \frac{1}{16\pi G} \int \mathrm{d}^4x\, \sqrt{-g} (R - 2\Lambda) + \underbrace{S_M}_{\text{matter}}
$$

and we define

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta S_M}{\delta g^{\mu\nu}}$$

Then

$$\delta S = \frac{1}{16\pi G} \int \mathrm{d}^4 x \, \sqrt{-g} \big(G_{\mu\nu} + \Lambda g_{\mu\nu} \big) \, \delta g^{\mu\nu} - \frac{1}{2} \int \mathrm{d}^4 x \, \sqrt{-g} T_{\mu\nu} \, \delta g^{\mu\nu}$$

Taking $\delta S = 0$ we find

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}$$

which are the EFE.

7. The geodesic as the minimal curve

7.1. The geodesic equation

Consider the line element $\mathrm{d}s^2 = g_{\mu\nu} \, \mathrm{d}x^\mu \, \mathrm{d}x^\nu$. Then

$$S = \int \mathrm{d}s = \int L \, \mathrm{d}\tau$$

where

$$L \equiv \sqrt{g_{\mu\nu} \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} \frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}}$$

We parametrize a curve by $x^\mu(\tau)$. Then the shortest curve between x_i^μ and x_f^μ is found by $\delta S = 0$. We use the *Euler-Lagrange equations*

$$\frac{\partial L}{\partial x^\mu} - \frac{\mathrm{d}}{\mathrm{d}\tau} \frac{\partial L}{\partial \dot{x}^\mu} = 0$$

The equations of motion from $\tilde{L} = L^2$ are the same as those obtained by using L. We let $L \equiv g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu$. Then

$$\frac{\partial L}{\partial \dot{x}^\mu} = 2 g_{\mu\nu} \dot{x}^\nu$$

$$\frac{\partial L}{\partial x^\mu} = \frac{\partial g_{\nu\lambda}}{\partial x^\mu} \dot{x}^\nu \dot{x}^\lambda$$

and we find

$$0 = \frac{\partial g_{\mu\nu}}{\partial x^\lambda} \frac{\mathrm{d}x^\lambda}{\mathrm{d}\tau} \frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} + g_{\mu\nu} \frac{\mathrm{d}^2 x^\nu}{\mathrm{d}\tau^2} - \frac{1}{2} \frac{\partial g_{\nu\lambda}}{\partial x^\mu} \frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} \frac{\mathrm{d}x^\lambda}{\mathrm{d}\tau}$$

$$0 \overset{g^{\sigma\mu}(\ldots)}{=} \frac{\mathrm{d}^2 x^\sigma}{\mathrm{d}\tau^2} + \frac{1}{2} g^{\sigma\mu} \left(2 \frac{\partial g_{\mu\nu}}{\partial x^\lambda} - \frac{\partial g_{\nu\lambda}}{\partial x^\mu} \right) \frac{\mathrm{d}x^\lambda}{\mathrm{d}\tau} \frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}$$

$$0 = \frac{\mathrm{d}^2 x^\sigma}{\mathrm{d}\tau^2} + \Gamma^\sigma{}_{\lambda\nu} \frac{\mathrm{d}x^\lambda}{\mathrm{d}\tau} \frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}$$

which is the geodesic equation!

7.2. A trick

We can easily find Γ by comparing the integrand of

$$\delta \int g_{\mu\nu} \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} \frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} \,\mathrm{d}\tau = 0$$

with the geodesic equation.

8. The time-dependent spherically symmetric metric

8.1. The metric

$\mathrm{d}\tau^2$ can only depend on rotationally invariant quantities. These are

$$\{t, \mathrm{d}t, r, r\,\mathrm{d}r = \boldsymbol{x}\,\mathrm{d}\boldsymbol{x}, \mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2) = \mathrm{d}\boldsymbol{x}^2\}$$

We define $\mathrm{d}\Omega^2 \equiv \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2$. Then the metric has the form

$$\mathrm{d}\tau^2 = A\,\mathrm{d}t^2 - B\,\mathrm{d}r^2 - C\,\mathrm{d}r\,\mathrm{d}t - Dr^2\,\mathrm{d}\Omega^2$$

with $A, B, \ldots$ being functions of t and r. We absorb D into r by redefining $r \to r' = r\sqrt{D}$

$$\mathrm{d}\tau^2 = A\,\mathrm{d}t^2 - B\,\mathrm{d}r^2 - C\,\mathrm{d}r\,\mathrm{d}t - r^2\,\mathrm{d}\Omega^2$$

with new $A, B, \ldots$ and C. Similarly to absorb C we redefine $t \to t'$ by

$$\mathrm{d}t' = \underbrace{\eta(r,t)}_{\text{independent}} \left[A\,\mathrm{d}t - \frac{1}{2} C\,\mathrm{d}r \right]$$

Then

$$\frac{1}{A\eta^2}\,\mathrm{d}t'^2 = A\,\mathrm{d}t^2 - C\,\mathrm{d}t\,\mathrm{d}r + \frac{C^2}{4A}\,\mathrm{d}r^2$$

implying

$$\mathrm{d}\tau^2 = \frac{1}{\eta^2 A}\,\mathrm{d}t^2 - \left(B + \frac{C^2}{4A} \right) \mathrm{d}r^2 - r^2\,\mathrm{d}\Omega^2$$

$$\mathrm{d}\tau^2 = E(r,t)\,\mathrm{d}t^2 - F(r,t)\,\mathrm{d}r^2 - r^2\,\mathrm{d}\Omega^2$$

We find

$$g_{rr} = F; \quad g_{\theta\theta} = r^2; \quad g_{\varphi\varphi} = r^2\sin^2\theta; \quad g_{tt} = -E$$
$$g^{rr} = \frac{1}{F}; \quad g^{\theta\theta} = \frac{1}{r^2}; \quad g^{\varphi\varphi} = \frac{1}{r^2\sin^2\theta}; \quad g^{tt} = -\frac{1}{E}$$

8.2. The Christoffel symbols

We compute the Christoffel symbols using

$$\delta\int \mathrm{d}\tau\left[E\dot{t}^2 - F\dot{r}^2 - r^2\dot{\theta}^2 - r^2\sin^2\theta\dot{\varphi}^2\right] = 0$$

Consider the $\mu = 0$ component of

$$\frac{\partial L}{\partial x^\mu} = \frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\partial L}{\partial \dot{x}^\mu}$$

We find

$$\begin{aligned}
\dot{t}^2\frac{\partial E}{\partial t} - \frac{\partial F}{\partial t}\dot{r}^2 &= \frac{\mathrm{d}}{\mathrm{d}\tau}\big(2E\dot{t}\big) \\
&= 2E\ddot{t} + 2\dot{t}\frac{\mathrm{d}E}{\mathrm{d}\tau} \\
&= 2E\ddot{t} + 2\dot{t}\left(\frac{\partial t}{\partial\tau}\frac{\partial E}{\partial t} + \frac{\partial r}{\partial\tau}\frac{\partial E}{\partial r}\right) \\
&= 2E\ddot{t} + 2\dot{t}^2\frac{\partial E}{\partial t} + 2\dot{t}\dot{r}\frac{\partial E}{\partial r}
\end{aligned}$$

simplifying

$$\begin{aligned}
0 &= \frac{\partial F}{\partial t}\dot{r}^2 + 2E\ddot{t} + \dot{t}^2\frac{\partial E}{\partial t} + 2\dot{t}\dot{r}\frac{\partial E}{\partial r} \\
0 &= \ddot{t} + \frac{1}{2E}\frac{\partial E}{\partial t}\dot{t}^2 + \frac{1}{E}\frac{\partial E}{\partial r}\dot{t}\dot{r} + \frac{1}{2E}\frac{\partial F}{\partial t}\dot{r}^2
\end{aligned}$$

We compare with

$$0 = \ddot{t} + \Gamma^0{}_{\mu\nu}\dot{x}^\mu\dot{x}^\nu$$

giving

$$\Gamma^t{}_{tt} = \frac{1}{2E}\frac{\partial E}{\partial t}$$

$$
\begin{aligned}
\Gamma^t{}_{rr} &= \frac{1}{2E}\frac{\partial F}{\partial t} \\
\Gamma^t{}_{rt} = \Gamma^t{}_{tr} &= \underbrace{\frac{1}{2}\times}_{\text{symmetric}} \frac{1}{E}\frac{\partial E}{\partial r}
\end{aligned}
$$

Similarly the r-component gives

$$
\begin{aligned}
\Gamma^r{}_{tr} = \Gamma^r{}_{rt} &= \frac{1}{2F}\frac{\partial F}{\partial t} \\
\Gamma^r{}_{rr} &= \frac{1}{2F}\frac{\partial F}{\partial r} \\
\Gamma^r{}_{tt} &= \frac{1}{2F}\frac{\partial E}{\partial r} \\
\Gamma^r{}_{\theta\theta} &= -\frac{r}{F} \\
\Gamma^r{}_{\varphi\varphi} &= -\frac{r\sin^2\theta}{F}
\end{aligned}
$$

And the θ-component gives

$$
\begin{aligned}
\Gamma^\theta{}_{r\theta} = \Gamma^\theta{}_{\theta r} &= \frac{1}{r} \\
\Gamma^\theta{}_{\varphi\varphi} &= -\sin\theta\cos\theta
\end{aligned}
$$

And the φ-component gives

$$
\begin{aligned}
\Gamma^\varphi{}_{r\varphi} = \Gamma^\varphi{}_{\varphi r} &= \frac{1}{r} \\
\Gamma^\varphi{}_{\theta\varphi} = \Gamma^\varphi{}_{\varphi\theta} &= \frac{\cos\theta}{\sin\theta}
\end{aligned}
$$

8.3. The Ricci tensor

We have

$$
\Gamma^\mu{}_{\mu\nu} = \frac{1}{2}g^{\mu\sigma}\frac{\partial g_{\mu\sigma}}{\partial x^\nu} = \frac{1}{2}\frac{\partial \ln g}{\partial x^\nu}
$$

so

$$
\begin{aligned}
R_{\mu\kappa} &= \partial_\kappa \Gamma^\sigma{}_{\sigma\mu} - \partial_\sigma \Gamma^\sigma{}_{\mu\kappa} + \Gamma^\lambda{}_{\mu\sigma}\Gamma^\sigma{}_{\kappa\lambda} - \Gamma^\lambda{}_{\mu\kappa}\Gamma^\sigma{}_{\sigma\lambda} \\
&= \frac{1}{2}\partial_\kappa\partial_\mu \ln g - \partial_\sigma \Gamma^\sigma{}_{\mu\kappa} + \Gamma^\lambda{}_{\mu\sigma}\Gamma^\sigma{}_{\kappa\lambda} - \frac{1}{2}\Gamma^\lambda{}_{\mu\kappa}\partial_\lambda \ln g
\end{aligned}
$$

Then

$$
\begin{aligned}
R_{rr} &= \frac{1}{2E}\frac{\partial^2 E}{\partial r^2} - \frac{1}{4E^2}\left(\frac{\partial E}{\partial r}\right)^2 - \frac{1}{4EF}\frac{\partial E}{\partial r}\frac{\partial F}{\partial r} \\
&\quad - \frac{1}{rF}\frac{\partial F}{\partial r} - \frac{1}{2E}\frac{\partial^2 F}{\partial t^2} + \frac{1}{4E^2}\frac{\partial E}{\partial t}\frac{\partial F}{\partial t} + \frac{1}{4EF}\left(\frac{\partial F}{\partial t}\right)^2 \\
R_{\theta\theta} &= -1 + \frac{1}{F} - \frac{r}{2F^2}\frac{\partial F}{\partial r} + \frac{r}{2EF}\frac{\partial E}{\partial r} \\
R_{tt} &= -\frac{1}{2F}\frac{\partial^2 E}{\partial r^2} + \frac{1}{4F^2}\frac{\partial E}{\partial r}\frac{\partial F}{\partial r} - \frac{1}{rF}\frac{\partial E}{\partial r} \\
&\quad + \frac{1}{4EF}\left(\frac{\partial E}{\partial r}\right)^2 + \frac{1}{2F}\frac{\partial^2 F}{\partial t^2} - \frac{1}{4F^2}\left(\frac{\partial F}{\partial t}\right)^2 - \frac{1}{4EF}\frac{\partial E}{\partial t}\frac{\partial F}{\partial t} \\
R_{tr} &\overset{\text{simplified}}{=} -\frac{1}{rF}\frac{\partial F}{\partial t} \\
R_{\varphi\varphi} &= \sin^2\theta R_{\theta\theta}
\end{aligned}
$$

The trace-inverted EFE then give equations for E and F in terms of $T_{\mu\nu}$. As an example since $g_{tr} = 0$ we have

$$\frac{1}{rF}\frac{\partial F}{\partial t} = 8\pi G T_{tr}$$

9. The Schwarzschild solution

9.1. The solution

Consider a point mass with mass M at the origin. We assume $m_{\mathrm{obs}} \ll M$ so we can ignore any backreaction. We use the metric we just derived above but assume E and F are time-independent (this is justified below)

$$\mathrm{d}\tau^2 = E(r)\,\mathrm{d}t^2 - F(r)\,\mathrm{d}r^2 - r^2\,\mathrm{d}\Omega^2$$

Since $T_{\mu\nu} = 0$ for all $r \neq 0$ we have $R_{\mu\nu} = 0$. Using $R_{rr} = R_{tt} = 0$ we find

$$
\begin{aligned}
0 &= \frac{R_{rr}}{F} + \frac{R_{tt}}{E} \\
0 &\overset{\text{time-independent}}{=} \frac{1}{rF}\underbrace{\left(\frac{1}{F}\frac{\partial F}{\partial r} + \frac{1}{E}\frac{\partial E}{\partial r}\right)}_{\overset{!}{=}0}
\end{aligned}
$$

implying

$$\frac{\partial \ln F}{\partial r} = -\frac{\partial \ln E}{\partial r}$$

so $E(r)F(r) = \text{const}$. As $r \to \infty$ the metric should reduce to $\eta_{\mu\nu}$ where $E = F = 1$. Then

$$E = \frac{1}{F}$$

for all r. Consider

$$\begin{aligned} R_{\theta\theta} &= 0 \\ -1 + E + r\frac{\partial E}{\partial r} &= 0 \end{aligned}$$

implying

$$\frac{\partial}{\partial r}(rE) = 1$$

Then

$$E = 1 + \frac{C}{r}$$

We take the Newtonian limit

$$\begin{aligned} g_{00} &= -(1 + 2\Phi) \\ &\overset{\text{point mass}}{=} -1 + \frac{2G_N M}{r} \end{aligned}$$

Using $E = -g_{00}$ we find $C = -2G_N M$ so

$$E = 1 - \frac{2MG_N}{r}; \quad F = \frac{1}{E}$$

We have found *the Schwarzschild metric*

$$\mathrm{d}\tau^2 = \left(1 - \frac{2MG_N}{r}\right)\mathrm{d}t^2 - \frac{\mathrm{d}r^2}{1 - \frac{2MG_N}{r}} - r^2\,\mathrm{d}\Omega^2$$

which is very important! Due to *Birkhoff's theorem* this describes any spherically symmetric mass distribution.

9.2. Birkhoff's theorem

Theorem 1 (Birkhoff's theorem): A spherically symmetric gravitational field in empty space must be static with a metric given by the Schwarzschild metric.

Proof: We assumed E and F were time-independent. This can be shown to always be the case. From $R_{tr} = 0$ we have

$$\frac{\partial F}{\partial t} = 0$$

which is nice. We can write

$$E = f(t)\left(1 - \frac{2MG_N}{r}\right) \neq E(r)$$

and redefine the time-coordinate to absorb $f(t)$ by

$$t' = \int \sqrt{f(t)}\,\mathrm{d}t \Rightarrow \mathrm{d}t' = \sqrt{f(t)}\,\mathrm{d}t$$

So E and F are always time-independent! Then everything done before is valid when $T_{\mu\nu} = 0$. We can even have weird oscillatory mass distributions if they are spherically symmetric. □

This explains why gravitational waves are rare since we need to break spherical symmetry for the metric to be changing in time. This also makes it trivial to analyze cavities since these are spherically symmetric and $T_{\mu\nu} = 0$. But $M = 0$ so in cavities we just have $\eta_{\mu\nu}$!

9.3. Killing vectors

Consider the Lagrangian (with $\theta = \pi/2$)

$$L = g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = -\left(1 - \frac{R}{r}\right)\dot{t}^2 + \frac{1}{1 - \frac{R}{r}}\dot{r}^2 + r^2\dot{\varphi}^2$$

where we define the *Schwarzchild radius* $R \equiv 2MG_N$.

The Schwarzschild metric is time-independent. Generally if $g_{\mu\nu}$ is independent of x^0 we have

$$\frac{\partial L}{\partial x^0} = 0$$

Then

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\underbrace{\left(\frac{\partial L}{\partial \dot{x}^0}\right)}_{\text{constant}} \stackrel{!}{=} 0$$

so

$$\frac{\partial L}{\partial \dot{x}^0} = \frac{\partial L}{\partial \dot{t}} = 2g_{\mu 0}\dot{x}^\mu$$

is conserved. To ensure covariance we define the *time-like Killing vector*

$$K^{\mu}_{(t)} \equiv (1,0,0,0)$$

Then

$$\gamma_{(t)} = g_{\mu\nu}\dot{x}^{\mu}K^{\nu}_{(t)}$$

is covariant and conserved. For other Killing vectors $K^{\nu}_{(\alpha)}$ the object

$$\gamma_{(\alpha)} = g_{\mu\nu}\dot{x}^{\mu}K^{\nu}_{(\alpha)}$$

is covariant and conserved if $g_{\mu\nu}$ is independent of x^{α}.

The Schwarzchild metric only depends on r so we have conserved quantities related to both t and φ defined by

$$\kappa \equiv -g_{\mu\nu}\dot{x}^{\mu}K^{\nu}_{(t)} = -g_{00}\dot{t} = \left(1-\frac{R}{r}\right)\dot{t}$$

$$l \equiv m g_{\mu\nu}\dot{x}^{\mu}K^{\nu}_{(\varphi)} = m g_{\varphi\varphi}\dot{\varphi} = mr^2\varphi$$

We recognize κ as an *energy* and l as the angular momentum.

9.4. Precession of orbits

We use the expressions for κ and l to rewrite L. With $L=-1$ we find

$$-\frac{\kappa^2}{1-\frac{R}{r}} + \frac{\dot{r}^2}{1-\frac{R}{r}} + \frac{l^2}{m^2r^2} = -1$$

We multiply by $\frac{1}{2}m\big(1-\frac{R}{r}\big)$ and define

$$\frac{E}{m} \equiv \frac{\kappa^2-1}{2}$$

Then

$$\underbrace{\frac{1}{2}mr^2}_{E_{\text{kin}}} + \underbrace{\left(1-\frac{R}{r}\right)}_{\text{GR corr.}}\underbrace{\frac{l^2}{2mr^2}}_{E_l} - \underbrace{\frac{G_N mM}{r}}_{E_g} = E$$

This expresses conservation of energy with a relativistic correction. We can solve this to find the equation for an orbit

$$r = \frac{\alpha}{1+e\cos[(1-\epsilon)\varphi]}$$

with e being the eccentricity and

$$
\begin{aligned}
\alpha &= \frac{l^2}{GMm^2} = (1+e)r_{\min} \\
\epsilon &= \frac{3R}{2\alpha}
\end{aligned}
$$

The orbit returns to $r_{\min}$ at

$$
\varphi = \frac{2\pi}{1-\epsilon} \simeq 2\pi + \frac{3\pi R}{\alpha}
$$

so the orbit precesses! Einstein used this to compute the precession of Mercury's orbit which helped show the *correctness* of general relativity.

10. Black holes

We write the Schwarzschild metric as

$$
\mathrm{d}\tau^2 = f(r)\,\mathrm{d}t^2 - \frac{\mathrm{d}r^2}{f(r)} - r^2\,\mathrm{d}\Omega^2
$$

where we define

$$
f(r) \equiv 1 - \frac{R}{r}
$$

At $r = R$ we find a singularity. Usually this is not a problem since $R < R_{\text{mass}}$. But what about a point mass or some object with $R_{\text{mass}} < R$? We call such objects black holes for reasons we discuss now.

10.1. The geodesic equation

We consider a free-falling observer. Then

$$
\frac{\mathrm{d}^2 x^\alpha}{\mathrm{d}\tau^2} + \Gamma^\alpha{}_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} = 0
$$

or

$$
\begin{aligned}
0 &= \delta \int \left[g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu\right] \mathrm{d}\tau \\
&= \delta \int \underbrace{\left[-f(r)\dot{t}^2 + f^{-1}\dot{r}^2\right]}_{L} \mathrm{d}\tau
\end{aligned}
$$

where we assume $\dot{\theta} = \dot{\varphi} = 0$. This only depends on r so we have the conserved quantity

$$
C = g_{\mu\nu}\dot{x}^\mu K^\nu_{(t)} = f(r)\dot{t}
$$

We fix C by considering $r \to \infty$ where $\dot{t} \to 1$ and $f(r) \to 1$ so $C = 1$ meaning $\dot{t} = f^{-1}(r)$. Then

$$L = -\frac{1}{f(r)} + f^{-1}(r)\dot{r}^2 = -1$$

implying

$$\frac{1}{f(r)} - f^{-1}(r)\dot{r}^2 = 1$$

Then

$$\begin{aligned} \dot{r}^2 &= 1 - f(r) \\ &= \frac{R}{r} \end{aligned}$$

which is nice and simple!

10.2. Crossing the horizon

We want to know what happens when crossing $r = R$. By the above

$$\sqrt{r}\dot{r} = \underbrace{-\sqrt{2MG}}_{\text{since infalling}}$$

which has solution

$$\frac{2}{3\sqrt{2MG}}\left(r^{3/2} - r_0^{3/2}\right) = \tau_0 - \tau$$

We see that nothing special at $r = R$. We even find for $r = 0$

$$\tau = \tau_0 + \frac{2}{3\sqrt{2MG}} r_0^{3/2} < \infty$$

So the free-falling observer reaches the center in finite time! To find t we do

$$\begin{aligned} \frac{\mathrm{d}r}{\mathrm{d}t} &= \frac{\dot{r}}{\dot{t}} \\ &= -\sqrt{\frac{2MG}{r}}\left(1 - \frac{2MG}{r}\right) \end{aligned}$$

Assuming $r \gg 2MG$ we find

$$\frac{\mathrm{d}r}{\mathrm{d}t} = -\sqrt{\frac{2MG}{r}}$$

so $t \sim \tau$. Assuming $r = 2MG$ we find

$$\frac{\mathrm{d}r}{\mathrm{d}t} = 0$$

so the free-falling observer appears to stop moving. We write

$$\frac{\mathrm{d}r}{\mathrm{d}t} = -\frac{1}{r}\sqrt{\frac{2MG}{r}}(r - 2MG)$$

and assume $r \simeq 2MG$. Then to first order in $r - 2MG = \delta r$ we have

$$\frac{\mathrm{d}r}{\mathrm{d}t} = -\frac{1}{2MG}(r - 2MG)$$

Then

$$r = R + C\exp\left(-\frac{t}{2MG}\right)$$

so as $t \to \infty$ we find $r \to 2MG$. The free-falling observer appears to stop at $r = R$ but taking infinite time to do so!

For light $\mathrm{d}\tau = 0$

$$\left(1 - \frac{2MG}{r}\right)^2 \mathrm{d}t^2 - \mathrm{d}r^2 = 0$$

The velocity of light as seen by some far-away observer is then

$$\left|\frac{\mathrm{d}r}{\mathrm{d}t}\right| = 1 - \frac{2MG}{r}$$

for $r \geq 2MG$. At $r = 2MG$

$$\left|\frac{\mathrm{d}r}{\mathrm{d}t}\right| = 0$$

so light can not escape the black hole! This is why we call R the *horizon*. Also $g_{00} \to 0$ as $r \to 2MG$ meaning a free-falling observer will appear more and more redshifted.

10.3. Kruskal-Szekeres coordinates

The singularity at $r = R$ is a coordinate artifact. The only *true* singularity is at $r = 0$ where curvature explodes to ∞. We can use *Kruskal-Szekeres coordinates* defined by

$$X^2 - T^2 = \left(\frac{r}{R} - 1\right)\exp\left(\frac{r}{R}\right) \quad \text{and} \quad \frac{T}{X} = \tanh\left(\frac{t}{2R}\right)$$

to *remove* the singularity at $r = R$. With these the Schwarzschild metric takes the form

$$\mathrm{d}\tau^2 = \frac{32R^3}{r^2}\exp\left(-\frac{r}{R}\right)\underbrace{\left(\mathrm{d}T^2 - \mathrm{d}X^2\right)}_{\text{light travels at }45^\circ} - r^2\,\mathrm{d}\Omega^2$$

where $r \equiv r(X, T)$. Things about these coordinates are nicely summarized by a *Kruskal-Szekeres chart* as in Figure 1:

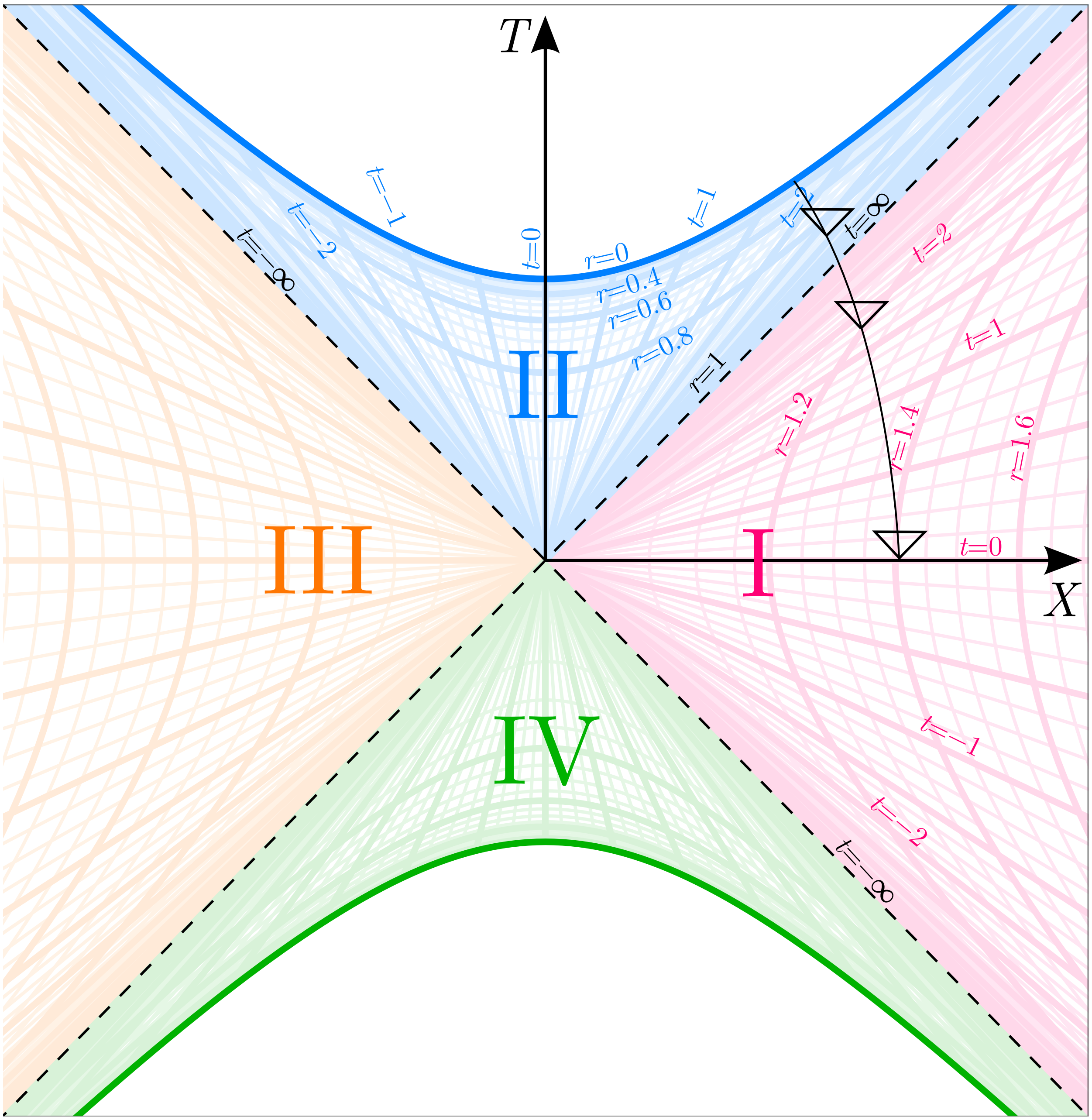

Figure 1: A Kruskal-Szekeres chart with $R = 1$.

We see as $t \to \pm\infty$ we have $T = \pm X$ corresponding to the dashed lines. We see that for $r = R$ we have $X^2 = T^2$ so again $X = \pm T$. The dashed line seperating I and II is then the horizon. The various lines drawn for constant r are all hyperbolic since $X^2 - T^2 =$ constant with their direction depending on if $r < R$ or $r > R$. The limit is $r = 0$ where $X^2 - T^2 = -1$. Then for $X = 0$ we have $T = 1$ meaning the singularity is a hyperbola with the minimum at $T = 1$. The two respective mirror regions are denoted by III and IV. When $t = 0$ then $T = 0$ so this corresponds to the X-axis. Since light travels at 45° it is not possible to travel from I to III. Similarly light can never escape II or enter IV.

But when $T = 0$ region I and III appear connected? This is what we call a *wormhole*.

10.4. Wormholes

Consider the slice $T = 0$

$$X = \left(\frac{r}{R} - 1\right)^{1/2} \exp\left(\frac{r}{2R}\right)$$

Then

$$\mathrm{d}T = 0; \quad \mathrm{d}X = \frac{r^{1/2}}{2R^{3/2}}\left(1 - \frac{R}{r}\right)^{-1/2} \exp\left(\frac{r}{2R}\right) \mathrm{d}r$$

Let $\theta = 0$ then

$$\mathrm{d}s^2 = \left(1 - \frac{R}{r}\right)^{-1} \mathrm{d}r^2 + r^2 \,\mathrm{d}\varphi^2$$

this describes a two-dimensional surface!

We want to embed this surface in $\mathbb{R}^3$. We do this by adding a coordinate $(r, \varphi) \to (r, \varphi, z)$. These are just cylindrical coordinates with the metric

$$\begin{aligned} \mathrm{d}s^2 &= \mathrm{d}z^2 + \mathrm{d}r^2 + r^2 \,\mathrm{d}\varphi^2 \\ &= \left[\left(\frac{\mathrm{d}z}{\mathrm{d}r}\right)^2 + 1\right] \mathrm{d}r^2 + r^2 \,\mathrm{d}\varphi^2 \end{aligned}$$

By comparison we find

$$\left(\frac{\mathrm{d}z}{\mathrm{d}r}\right)^2 + 1 = \left(1 - \frac{R}{r}\right)^{-1}$$

implying

$$\mathrm{d}z = \pm\sqrt{\frac{R}{r - R}}\,\mathrm{d}r$$

Then

$$z^2(r) = 4R(r - R)$$

plotting this gives an *embedding diagram* as in Figure 2:

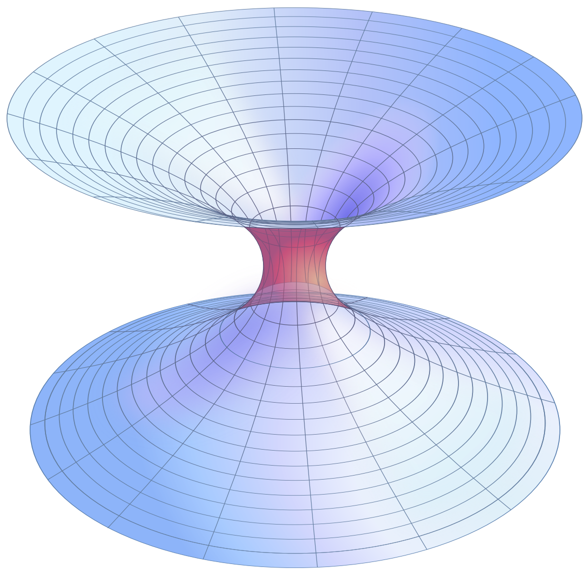

Figure 2: The embedding diagram of a wormhole.

At the throat $z = 0$ where $r = R$. So the surface at $T = 0$ connects regions I and III!

We would like to determine the lifetime of the wormhole. The embedding can be done for $T \neq 0$ in the same way. All slices $T = \text{constant}$ give a surface with

$$r_{\text{throat}} = r_{\text{min}} = r(X = 0, T)$$

For $T = 1$ we find $r_{\text{min}} = 0$ meaning the wormhole closes and for $T > 1$ it pinches off with $r_{\text{min}} \in \mathbb{C}$. This happens on a timescale

$$\delta t \sim \frac{R}{c}$$

which is smaller than the time it takes for light to travel from I $\to$ III.

11. Friedmann-Robertson-Walker metric

11.1. Basic assumptions

At *large enough* scales clusters of galaxies etc. can be treated like pointlike particles in a continuous cosmic fluid. This combined with the assumption that there is no preferred point in the Universe leads to the cosmological principle:

The universe is homogeneous and isotropic.

We also assume there exists some cosmological time implying the Universe has a rest frame. So at some fixed time t matter is at rest. The motion of the cosmic fluid would then follow this cosmological time.

11.2. The metric

We describe the above by considering Gaussian coordinates (the synchronous gauge). Consider two spatial slices at fixed cosmic times t denoted by S_0 and S_*. We let every point $P_0(x^i)$ evolve along a geodesic perpendicular to S_0 to become points $P_*(x^i)$. Along such a geodesic the x^i are fixed (comoving) and we take $t = \tau$, meaning

$$\mathrm{d}\tau^2 = \mathrm{d}t^2$$

implying $g_{00} = -1$. We also require $g_{i0} = g_{0i} = 0$ if the geodesics are to remain perpendicular. Then

$$\mathrm{d}\tau^2 = \mathrm{d}t^2 - g_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j \text{ with } x^i \in S_0$$

We check this is consistent. Consider

$$\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\tau^2} + \Gamma^\mu{}_{\lambda\rho}\frac{\mathrm{d}x^\lambda}{\mathrm{d}\tau}\frac{\mathrm{d}x^\rho}{\mathrm{d}\tau} = 0$$

For a point on S_0

$$\Gamma^i{}_{00} = 0$$

implying

$$2\frac{\partial g_{0i}}{\partial t} = \underbrace{\frac{\partial g_{00}}{\partial x^i}}_{0}$$

so g_{0i} will be zero for all time. Consider a spatial slice ($\mathrm{d}t^2 = 0$)

$$\mathrm{d}s^2 = g_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j$$

Since g_{ij} can depend on t then $\mathrm{d}s^2$ can change over time but the x^i are still fixed. This implies a time-dependent rescaling of $\mathrm{d}s^2$.

Using isotropy we require

$$\mathrm{d}\tau^2 = \mathrm{d}t^2 - f_1(t,r)\,\mathrm{d}r^2 - f_2(t,r)r^2\,\mathrm{d}\Omega^2$$

we can decompose f_1 and f_2

$$\mathrm{d}\tau^2 = \mathrm{d}t^2 - f(t)L(r)\,\mathrm{d}r^2 - g(t)H(r)r^2\,\mathrm{d}\Omega^2$$

Proof: Consider two points at the same t with $\mathrm{d}\theta = \mathrm{d}\varphi = 0$. The line elements of these points are $f_1(t,r_1)\,\mathrm{d}r^2$ and $f_1(t,r_2)\,\mathrm{d}r^2$. The ratio of these is the same at all times since only the overall scale can change

$$f_1(t,r_1) = f_1(t,r_2)F(r_1,r_2)$$

or taking $r_2 = \text{constant}$ we find

$$f_1(t,r) = f(t)L(r)$$

Similarly f_2 can be decomposed. □

By redefining r we can absorb H

$$\mathrm{d}\tau^2 = \mathrm{d}t^2 - f(t)L(r)\,\mathrm{d}r^2 - g(t)r^2\,\mathrm{d}\Omega^2$$

To determine f, L and g we use the EFE. We assume the Universe is a perfect fluid

$$T_{\mu\nu} = pg_{\mu\nu} + (p+\rho)U_\mu U_\nu$$

The comoving frame has $U^i = 0$ and $U^t = 1$. Then $R_{tr} = 0$ implies

$$\frac{\dot{g}}{g} = \frac{\dot{f}}{f}$$

so $g = f$ up to a constant. Then

$$\mathrm{d}\tau^2 = \mathrm{d}t^2 - f(t)[L(r)\,\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2]$$

To determine $L(r)$ we consider $R^r{}_r$. This can not depend on r due to homogeneity meaning the spatial part must be constant

$$-\frac{L'}{rL^2} = 2k$$

implying

$$L = \frac{1}{\tilde{k} - kr^2}$$

Similarly for $R^\theta{}_\theta$

$$\frac{1}{r^2} - \frac{1}{r^2 L} + \frac{L'}{2rL^2} = 2k$$

Then

$$\frac{1-\tilde{k}}{r^2} + 2k = 2k$$

so $\tilde{k} = 1$. We define the *scale factor* $a \equiv \sqrt{f}$ giving the *Friedmann-Robertson-Walker metric*

$$\mathrm{d}\tau^2 = \mathrm{d}t^2 - a^2(t)\left[\frac{1}{1-kr^2}\,\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2\right]$$

11.3. Geometric interpretation of k

To see the meaning of k consider the spatial curvature

$$R^i{}_i = g^{ik} R_{ki} = -\frac{6k}{a^2} = -6K$$

where $K = ka^{-2}$ is the *Gaussian curvature*. So k determines the spatial curvature!

To determine the shape we construct embedding diagrams. We embed it using

$$\mathrm{d}s^2 = (\mathrm{d}\boldsymbol{x})^2 + (\mathrm{d}z)^2$$

Consider a hypersphere

$$\boldsymbol{x}^2 + z^2 = R^2$$

implying

$$\mathrm{d}(z^2) = 2z\,\mathrm{d}z = -\,\mathrm{d}(\boldsymbol{x}^2)$$

Then

$$\begin{aligned}
\mathrm{d}s^2 &\overset{\text{remove } z}{=} (\mathrm{d}\boldsymbol{x})^2 + \frac{\left(\mathrm{d}(\boldsymbol{x}^2)\right)^2}{4(R^2 - \boldsymbol{x}^2)} \\
&\overset{\text{polar}}{=} \mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2 + \frac{r^2\,\mathrm{d}r^2}{R^2 - r^2} \\
&= \frac{1}{1 - r^2/R^2}\,\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2
\end{aligned}$$

defining $\rho \equiv r/R$

$$\mathrm{d}s^2 = R^2\left(\frac{\mathrm{d}\rho^2}{1-\rho^2} + \rho^2\,\mathrm{d}\Omega^2\right)$$

And if we define $R \equiv a(t)$ for some fixed t then this is the FRW metric if $\rho^2 = kr^2$ meaning $k > 0$. So for $k > 0$ the FRW describes a sphere! A similar calculation shows that using

$$-\boldsymbol{x}^2 + z^2 = R^2$$

then $k \overset{!}{<} 0$. And for $k = 0$ the FRW metric is spatially flat.

11.4. Hubble's law

Consider an observer standing far away from a galaxy. Light has $\mathrm{d}\tau^2 = 0$ (with $\mathrm{d}\Omega^2 = 0$) meaning

$$\mathrm{d}t^2 = a^2(t)\frac{\mathrm{d}r^2}{1-kr^2}$$

implying

$$\int \frac{\mathrm{d}t}{a(t)} = \underbrace{-}_{\substack{\text{incoming}\\\text{light}}}\int \frac{\mathrm{d}r}{\sqrt{1-kr^2}}$$

We assume the light travels from (t_1, r_1) to $(t_0, 0)$ with $t_0 > t_1$. Then

$$\int_{t_1}^{t_0} \frac{\mathrm{d}t}{a(t)} = -\int_{r_1}^{0} \frac{\mathrm{d}r}{\sqrt{1-kr^2}} = \begin{cases} \sin^{-1} r_1 & \text{for } k = +1 \\ r_1 & \text{for } k = 0 \\ \sinh^{-1} r_1 & \text{for } k = -1 \end{cases}$$

Consider two crests of light emitted at (t_1, r_1) and $(t_1 + \delta t_1, r_1)$. The observer would see these at $(t_0, 0)$ and $(t_0 + \delta t_0, 0)$ respectively. Then

$$\int_{t_1+\delta t_1}^{t_0+\delta t_0} \frac{\mathrm{d}t}{a(t)} = \int_{t_1}^{t_0} \frac{\mathrm{d}t}{a(t)}$$

We assume δt_i is very small meaning $a(t_i) \sim$ constant. Then

$$\frac{t_0 + \delta t_0}{a(t_0)} - \frac{t_1 + \delta t_1}{a(t_1)} = \frac{t_0}{a(t_0)} - \frac{t_1}{a(t_1)}$$

implying

$$\frac{\delta t_0}{a(t_0)} = \frac{\delta t_1}{a(t_1)}$$

so we have a redshift! With frequencies

$$\frac{\lambda_1}{\lambda_0} = \frac{\nu_0}{\nu_1} = \frac{\delta t_1}{\delta t_0} = \frac{a(t_1)}{a(t_0)}$$

We define the redshift z by

$$\begin{aligned} z \equiv \frac{\lambda_0 - \lambda_1}{\lambda_1} &= \frac{a(t_0)}{a(t_1)} - 1 \\ &\overset{\text{small time}}{\simeq} \left[1 + \frac{\dot{a}(t_0)}{a(t_0)}(t_1 - t_0)\right]^{-1} \\ &\overset{\text{Taylor}}{\simeq} \underbrace{\frac{\dot{a}(t_0)}{a(t_0)}}_{H_0}(t_0 - t_1) \end{aligned}$$

with H_0 being the *Hubble rate* at t_0.

Consider the *comoving distance*

$$r \equiv \int_{t_1}^{t_0} \frac{\mathrm{d}t}{a(t)} \simeq \frac{t_0 - t_1}{a(t_0)}$$

The *proper distance* at t_0 is defined by $d \equiv a(t_0)r$ so

$$z \simeq H_0 d$$

Taking the derivative of d we find

$$v_r = \dot{a}(t_0)r = \frac{\dot{a}(t_0)}{a(t_0)}\underbrace{a(t_0)r}_{t_0 - t_1} = z$$

This is *Hubble's law*

$$v_r = H_0 d = z$$

which *proves* the Universe is expanding!

12. The Friedmann equations

12.1. The equations

Consider the EFE

$$R^\mu{}_\nu = 8\pi G\left(T^\mu{}_\nu - \frac{1}{2}T\delta^\mu{}_\nu\right)$$

with

$$T_{\mu\nu} \overset{\text{perfect fluid}}{=} pg_{\mu\nu} + (p+\rho)U_\mu U_\nu$$

where $U^t = 1$ and $U^i = 0$. We compute

$$T^t{}_t = -\rho; \quad T^i{}_i = p$$

meaning

$$T = -\rho + 3p$$

Similarly we compute

$$R^t{}_t = 3\frac{\ddot{a}}{a}; \quad R^r{}_r = \frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2} + \frac{2k}{a^2}$$

The tt-component gives

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p)$$

this is the *acceleration equation* or *second Friedmann equation*! The rr-component gives

$$4\pi G(\rho - p) = \frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2} + \frac{2k}{a^2}$$

We rewrite this using the acceleration equation

$$H^2 \equiv \frac{\dot{a}^2}{a^2} = \frac{8\pi G\rho}{3} - \frac{k}{a^2}$$

which is the *first Friedmann equation*! We have defined the *Hubble rate* $H = \dot{a}/a$. These are very important and can be solved to find $a(t)$. Then solving these give the metric meaning we do not need to solve the full EFE!

We can also use $D_\mu T^{\mu\nu} = 0$ to find

$$\begin{aligned}
\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + p) &= 0 \\
a^3\dot{\rho} + 3a^2\dot{a}(\rho + p) &= 0 \\
\frac{\mathrm{d}}{\mathrm{d}t}(\rho a^3) &= -3pa^2\dot{a} \\
\frac{\mathrm{d}}{\mathrm{d}a}(\rho a^3) &= -3pa^2
\end{aligned}$$

which is the *fluid equation*.

We also need an equation of state $p = p(\rho)$. We assume $p = w\rho$ as typical for an *ideal fluid*. Then by the above

$$\begin{aligned}\frac{\dot\rho}{\rho} &= -3\left(1+\frac{p}{\rho}\right)\frac{\dot a}{a}\\ &= -3(1+w)\frac{\dot a}{a}\end{aligned}$$

implying

$$\rho \propto a^{-3(1+w)}$$

1. $p \ll \rho$ (pressure-less matter). Then $w \sim 0$ giving

$$\rho_m \propto \frac{1}{a^3}$$

this is also called non-relativistic matter.

2. $p = \frac{1}{3}\rho$ (radiation). Then $w = \frac{1}{3}$ giving

$$\rho_r \propto \frac{1}{a^4}$$

this is also called relativistic matter.

3. $p = -\rho$. Then $w = -1$ giving

$$\rho_\Lambda \propto \text{constant}$$

as we would expect for a cosmological constant.

12.2. Consequences

Consider $p \geq 0$. Then

$$\frac{\ddot a}{a} = -\frac{4\pi G}{3}(\rho + 3p) < 0; \quad \frac{\dot a}{a} \overset{\text{today}}{>} 0$$

This implies that $a = 0$ at some point. We define the beginning of the Univeres by $a(t = 0) \overset{!}{=} 0$. This is called the *Big Bang*.

The age of the Universe can be computed by

$$t_0 = \int \frac{\mathrm{d}t}{\mathrm{d}a}\,\mathrm{d}a = \int \frac{\mathrm{d}a}{\dot a} < \int_0^{a(t_0)} \frac{\mathrm{d}a}{\dot a(t_0)}$$

We find

$$t_0 < H_0^{-1}$$

for $p \geq 0$. Using modern values we have $t_0 \lesssim 13.8$ Gyr.

Consider $p \geq 0$. Then

$$\frac{\mathrm{d}}{\mathrm{d}a}(\rho a^3) = -3pa^2$$

implies ρa^3 is decreasing. Then for $a \to \infty$ we have $\rho a^2 \to 0$. We have

$$\dot{a}^2 = \frac{8\pi G}{3}\rho a^2 - k$$

so $\dot{a}^2 \to -k$ as $a \to \infty$. For $k = -1$ the Universe expands forever. For $k = 0$ the expansion slows down asymptotically. For $k = +1$ we have $\dot{a} = 0$ for some $a = a_{\text{max}}$ given by

$$\rho a_{\text{max}}^2 = \frac{3}{8\pi G}$$

after a_{max} the Universe would begin collapsing.

Assume $k = 0$. Then

$$H^2 = \frac{8\pi G}{3}\rho$$

We define the *critical density* by

$$\rho_c \equiv \frac{3H^2}{8\pi G}$$

so this is the energy density in the Universe if it were flat. We define the *density parameter*

$$\Omega = \frac{\rho}{\rho_c}$$

Then assuming $k \neq 0$ we can write the curvature as

$$\frac{k}{a^2} = H^2(\Omega - 1)$$

meaning $k = 0$ implies $\Omega = 1$.

Using the density parameter we can write the first Friedmann equation as

$$\frac{H^2}{H_0^2} = \frac{\Omega_{m,0}}{a^3} + \frac{\Omega_{r,0}}{a^4} + \Omega_{\Lambda,0} + \frac{1 - \Omega_0}{a^2}$$

As an example of why we would use this form consider a *radiation-dominated* Universe. Then (assuming $k = 0$)

$$\frac{H^2}{H_0^2} = \frac{\Omega_{r,0}}{a^4}$$

or

$$a\,\mathrm{d}a \propto \mathrm{d}t$$

integrating gives

$$a \propto t^{1/2}$$

Similarly a *matter-dominated* Universe would have $a \propto t^{2/3}$.

13. Big Bang cosmology

13.1. The Big Bang

Above we found

$$\begin{aligned}\rho_m &\propto a^{-3}\\ \rho_r &\propto a^{-4}\\ \rho_\Lambda &\propto \text{const}\end{aligned}$$

These are different implying there are periods where different ρ_i dominate. As an example when at early times $\rho_r \sim \rho_{\text{tot}}$ we say the Universe was radiation-dominated. At some early time $\rho_r \sim \rho_m$ and at some later time $\rho_m \sim \rho_\Lambda$. We call these *periods of equality*. At very late times we expect a Universe completely dominated by ρ_Λ since all other $\rho_i \sim 0$ eventually.

Travelling backwards in time ρ clearly increases. This implies the Universe has expanded from an initially very hot and dense state. We call this the *hot Big Bang*. We assume particles were in thermal equilibrium due to inter-particle interactions in the early Universe. Then the number density n of bosons $(+)$ and fermions $(-)$ would be given by

$$\mathrm{d}n = \frac{4\pi g}{h^3c^3}\frac{E^2\,\mathrm{d}E}{e^{E/k_BT} \pm 1}$$

We compute

$$\begin{aligned}n &= \frac{4\pi g}{h^3c^3}\int \frac{E^2\,\mathrm{d}E}{e^{E/k_BT} \pm 1}\\ n &\propto \begin{cases}\frac{3}{2}T^3\zeta(3) & \text{for fermions}\\ 2T^3\zeta(3) & \text{for bosons}\end{cases}\end{aligned}$$

implying

$$n_b = \frac{4}{3}n_f = 2.404\frac{g}{2\pi^2}\left(\frac{k_BT}{hc}\right)^3$$

Then in the non-relativistic case with $\rho \propto n$ we find $T \propto a^{-1}$. We compute

$$\rho = \int E \, \mathrm{d}n \propto T^4$$

Then in the relativistic case we also find $T \propto a^{-1}$. With $a_r \propto t^{1/2}$ we find

$$k_B T \simeq 0.46 E_{pl} \left(\frac{t}{t_{pl}} \right)^{-1/2}$$

At these scales general relativity breaks down and the formalism we have developed only makes sense for $t > t_{pl}$. So being pedantic the point $t = t_{pl}$ and not $t = 0$ is the Big Bang.

13.2. Big Bang nucleosynthesis

At temperatures $T >$ MeV we had equilibrium

$$p^+ + n \longleftrightarrow D^+ + ...$$

At temperatures $T <$ MeV the formed nuclei could no longer be ripped apart so we had reactions like

$$p^+ + n \longrightarrow D^+ \longrightarrow \text{He} \longrightarrow \text{Li}$$

This is *Big Bang nucleosynthesis*.

13.3. The cosmic microwave background

At temperatures $T \gtrsim 13.6$ eV we had equilibrium

$$e^- + p^+ \longleftrightarrow H + \gamma$$

At temperatures $T \lesssim$ eV free electrons and protons form neutral hydrogen

$$e^- + p^+ \longrightarrow H + \gamma$$

This is called *recombination*. Before recombination photons scatter with free electrons and protons forming a coupled plasma. At this point the Universe is *opaque*. After recombination photons can move freely since they do not scatter with hydrogen making the Universe *transparent*. At some point we had *last scattering*. These photons constitute the *cosmic microwave background* or CMB. The temperature of CMB photons today is $T_{\text{mm, CMB}} = 2.73$ K found by measurement. We can also compute T_{CMB} since we know z_{rec} and ρ_{rec}. These values agree to high precision!

To summarize recombination happened at $z_{\text{rec}} \sim 1370$ after which photons decoupled leading to last scattering at $z_{\text{LS}} \sim 1100$.

13.4. CMB anisotropies

When observing the *CMB* we observe the *last scattering surface*. The primary observation are temperature anisotropies on the scale of

$$\frac{\delta T}{T} \sim 10^{-5}$$

These are very small confirming the isotropy and homogeneity of the Universe.

Consider a physical distance ℓ on the last scattering surface. Then

$$\delta\theta \simeq \frac{\ell}{d_A}$$

with d_A being the *angular diameter distance* and $\delta\theta$ being the *angular separation*. Using the FRW metric we have

$$\mathrm{d}s = a(t_e)r\delta\theta \simeq \ell$$

Then

$$\begin{aligned}\ell &= \frac{a(t_e)}{a(t_0)}a(t_0)r\delta\theta \\ &= \frac{a(t_0)r\delta\theta}{1+z}\end{aligned}$$

We define the *particle horizon* d_H by

$$d_H = a(t_0)\underbrace{\int_0^{t_0}\frac{\mathrm{d}t}{a(t)}}_{\text{comoving horizon}} = a(t_0)r$$

Then

$$\ell = \frac{d_H\delta\theta}{1+z}$$

implying

$$d_A = \frac{d_H}{z+1} \simeq \frac{d_H}{z}$$

We can compute

$$d_H \overset{\text{last scattering}}{\simeq} 1.4 \times 10^4 \text{ Mpc}$$

Then

$$\ell \simeq \frac{d_H \delta\theta}{z} \overset{\text{last scattering}}{\simeq} 0.22 \text{ Mpc} \left(\frac{\delta\theta}{1°}\right)$$

We have observed anisotropies with $\delta\theta \sim 7°$. These correspond to $\ell_{\text{LS}} \sim 1.6$ Mpc and $\ell_{\text{today}} \sim 1700$ Mpc $\sim$ size of the Universe. This implies the anisotropies are the initial conditions of the Universe.

As is typical when we have some signal with noise we can do a Fourier transform. When done on a sphere the Fourier basis are the *spherical harmonics* Y_l^m. Then

$$\frac{\delta T}{T}(\theta, \varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{m=l} a_{lm} Y_{lm}(\theta, \varphi)$$

We want the *correlation function*

$$C(\theta) = \left\langle \frac{\delta T}{T}(\hat{n}_1) \frac{\delta T}{T}(\hat{n}_1) \right\rangle \Bigg|_{\hat{n}_1 \cdot \hat{n}_2 = \cos\theta}$$

This is the average correlation between all pairs of points seperated by the angle θ. Assuming Gaussian fluctuations we have

$$\langle a_{lm} \rangle = 0; \quad \langle a_{lm}^* a_{l'm'} \rangle = \underbrace{C_l}_{\text{power}} \delta_{ll'} \delta_{mm'}$$

Then

$$C(\theta) = \frac{1}{4\pi} \sum_{l=0}^{\infty} (2l+1) C_l P_l(\cos\theta)$$

with P_l being the *Legendre polynomials*. This leads to the *power-spectrum* as in ...

The *multipole* l is related to length since we can probe smaller angles with higher multipoles. As an example for $l = 2$ (dipole) we split the sky in two and for $l = 1$ (monopole) we simply measure the average T_{mean}. This also leads to *cosmic variance*.

$l < 100$ corresponds to large $\delta\theta$ and is called the *Sachs-Wolfe plateau*. These scales are larger than the *sound horizon* at last scattering. We define the sound horizon by

$$\lambda = \int_0^{t_0} \frac{c_s}{a(t)} \, \mathrm{d}t$$

Then

$$\alpha \simeq \frac{\lambda}{d_A}$$

is the corresponding angular separation. Assuming a matter-dominated Universe with $a \propto t^{2/3}$ we find

$$\alpha \overset{\text{last scattering}}{=} \frac{c_s\left(1+z_\gamma\right)^{-1/2}}{c\left[(1+z_0)^{-1/2}-\left(1+z_\gamma\right)^{-1/2}\right]} \simeq 1^\circ$$

which corresponds to $l \sim 200$! We used

$$c_s \simeq \sqrt{\frac{p}{\rho}} = \frac{1}{\sqrt{3}}$$

as the speed of sound. We see $l \sim 200$ corresponds to the first peak of the power-spectrum. We say this is the *fundamental mode*. This implies that for $l < 200$ the initial density and pressure fluctuations are fixed since they have no way to reach equilibrium. At larger l initial overdensities lead to potential wells with higher ρ_{baryon} and ρ_{dm}. Then gravity tries to pull matter together. But baryons and photons are tightly coupled in the early Universe giving rise to radiation pressure. Then the photons try to escape the potential wells pulling baryons along with them. The dark matter resists the baryons escaping and acts to pull them back. This leads to damped oscillations and the production sound waves. The modes of these oscillations, the standing waves, become peaks in the power-spectrum. At very large l we see exponential damping due to the last scattering surface having a *thickness*.

14. Inflation

14.1. Problems with the Big Bang

Big Bang cosmology has multiple problems. The first problem is the *flatness problem*. The model assumes the Universe is flat. We see from the first Friedmann equation

$$H^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2}$$

that $\rho_{\text{curvature}} \propto a^{-2}$. We should be able to measure this but we simply do not. This is explainable if $\rho_{\text{curvature}}$ is very small.

The second problem is the *horizon problem* which is unexplainable. We previously defined the particle horizon

$$d_H = a(t)\int_0^t \frac{\mathrm{d}t'}{a(t')}$$

We also define the *Hubble radius* H^{-1}. Taking H^{-1} as constant then any light from $d > H^{-1}$ would never reach us since space would recede faster that the speed of light. Then each volume $\left(H^{-1}\right)^3$ would be *causally disconnected*. Assuming $a \propto t^p$ with $0 < p < 1$ then

$$d_H \sim H^{-1}$$

for a matter-dominated Universe we would have $d_H = 2H^{-1}$. Consider

$$\begin{aligned}\lambda_H(t_{\text{LS}}) &= d_H(t_0)\left(\frac{a_{\text{LS}}}{a_0}\right) \\ &= d_H(t_0)\left(\frac{T_0}{T_{\text{LS}}}\right)\end{aligned}$$

This is the size of the observable Universe at last scattering if we scaled the current size. Consider also

$$H_{\text{LS}}^{-1} \overset{\text{matter-dominated}}{\simeq} H_0^{-1}\left(\frac{a_{\text{LS}}}{a_0}\right)^{3/2} \sim d_H(t_0)\left(\frac{T_0}{T_{\text{LS}}}\right)^{\frac{3}{2}}$$

This is the Hubble radius at last scattering. We compute

$$\frac{\lambda_H^3}{d_H^3} \sim 10^6$$

so the Universe consisted of $\sim 10^6$ causally disconnected volumes at last scattering! This is a problem since observations of the CMB imply isotropy and homogeneity. There needs to be some mechanism allowing the Universe to *thermalize*. This is where *inflation* comes in. We can plot (λ_H, a) as in ...

Assuming the Universe is radiation-dominated we have

$$H^{-1} \propto a^2$$

while

$$\lambda_H \propto a$$

by definition. We see today $\lambda_H < H^{-1}$ but at early times $\lambda_H > H^{-1}$ which is the problem described above. We need $\lambda_H < H^{-1}$ at some even earlier time. To do this inflation assumes a period with $H^{-1} \simeq$ constant and $a \propto e^{Ht}$. This is the same as requiring a period with

$$0 < \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\lambda}{|H|^{-1}}\right) = \frac{\mathrm{d}}{\mathrm{d}t}\left(a\left|\frac{\dot{a}}{a}\right|\right) = \frac{\mathrm{d}}{\mathrm{d}t}\ |\dot{a}|$$

Assuming $\dot{a} > 0$ this implies $\ddot{a} > 0$.

14.2. Slow-roll inflation

Consider a scalar field described by the action

$$
\begin{aligned}
S &= \int \mathrm{d}^4x \overbrace{\sqrt{-g}}^{\text{minimal coupling}} \mathcal{L} \\
&= \int \mathrm{d}^4x \sqrt{-g}\left[\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi + V(\varphi)\right]
\end{aligned}
$$

Then solving the Euler-Lagrange equations give

$$
\ddot{\varphi} + \underbrace{3H\dot{\varphi}}_{\text{friction}} - \frac{\nabla^2\varphi}{a^2} + V_\varphi(\varphi) = 0
$$

Assuming homogeneity the energy-momentum tensor is

$$
T_{\mu\nu} = \partial_\mu\varphi\partial_\nu\varphi - g_{\mu\nu}\mathcal{L}
$$

implying

$$
\rho = \frac{\dot{\varphi}^2}{2} + V(\varphi); \quad p = \frac{\dot{\varphi}^2}{2} - V(\varphi)
$$

When $V(\varphi) \gg \dot{\varphi}^2$ then $p = -\rho$ which is just the condition for a cosmological constant. This implies an accelerated expansion! We define the number of *e-folds* by $a(t_R) = a(t_i)e^N$ or

$$
\underbrace{N}_{\text{number of } e\text{-folds}} = \ln\left(\frac{a(t_R)}{a(t_i)}\right) \gtrsim \underbrace{70}_{\text{length of inflation}}
$$

with t_R being the time at *reheating*. The ~ 70 comes from requiring $\lambda_H(t_i) \lesssim H_{\text{inf}}^{-1}$

$$
\begin{aligned}
\lambda_H(t_i) &= d_H(t_0)\left(\frac{a_i}{a_R}\right)\left(\frac{a_R}{a_0}\right) \\
&\simeq H_0^{-1}e^{-N}\frac{T_0}{T_R} \lesssim H_{\text{inf}}^{-1}
\end{aligned}
$$

so

$$
e^N \gtrsim \frac{T_0}{H_0}\frac{H_{\text{inf}}}{T_R} \Rightarrow N \gtrsim \ln\left(\frac{T_0}{H_0}\right) - \ln\left(\frac{T_R}{H_{\text{inf}}}\right) \sim 70
$$

This restricts $V(\varphi)$ since φ has to be *stuck for long enough* determined by $+3H\dot{\varphi}$.

With *slow-roll inflation* we assume $\dot{\varphi}^2 \ll V(\varphi)$ and small $\ddot{\varphi}$. This is enough for the above assumptions to hold. Then

$$\underbrace{H^2 \simeq \frac{8\pi G}{3} V(\varphi)}_{\text{first Friedmann equation}} ; \quad 3H\dot{\varphi} = -V_\varphi$$

which are the *slow-roll equations*. We define the *slow-roll parameters*

$$\epsilon = 4\pi G \frac{\dot{\varphi}^2}{H^2}; \quad \eta = \frac{1}{8\pi G} \left(\frac{V''}{V} \right)$$

both are $\ll 1$ since

$$\eta - \epsilon = -\frac{\ddot{\varphi}}{H\varphi}$$

Then

$$\begin{aligned} N &= \int_{t_i}^{t_R} H \, \mathrm{d}t \\ &\sim 8\pi G \int_{\varphi_R}^{\varphi_i} \frac{V}{V_\varphi} \, \mathrm{d}\varphi \overset{!}{\gtrsim} 70 \end{aligned}$$

which is now a condition on $V(\varphi)$!

From the slow-roll equations we see $H \sim$ constant if $V(\varphi) \sim$ constant. When $V(\varphi)$ eventually begins *rolling quickly*, H stops being constant, and inflation stops. This is quantified by $\epsilon \sim 1$ or equivalently $\dot{\varphi}^2 \sim V(\varphi)$. When this happens $p \sim 0$ and $\rho > 0$ implying $\ddot{a} < 0$ by the fluid equation! While $V(\varphi)$ is *rolling slowly* the *effective cosmological constant* Λ_{eff} dominates leading to a large repulsive pressure causing rapid expansion with $a \propto e^{Ht}$. After slow-roll inflation we have $\Lambda_{\text{eff}} \to \Lambda_{\text{vacuum}}$ with Λ_{vacuum} being dominated by ρ_r and ρ_m.

14.3. Quantum fluctuations

Assuming the field is inhomogeneous we have

$$\varphi = \varphi_c(t) + \underbrace{\delta\varphi(t, \boldsymbol{x})}_{\text{small}}$$

We can find an equation for $\delta\varphi$ by

$$\delta\varphi_k = \int \frac{\mathrm{d}^3 x}{(2\pi)^{3/2}} \, \delta\varphi \, e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}$$

We find

$$\delta\ddot{\varphi}_k + 3H\,\delta\dot{\varphi}_k + \frac{k^2}{a^2}\,\delta\varphi_k + V_{\varphi\varphi}\,\delta\varphi_k = 0$$

Assuming $k < aH$ then

$$\delta\varphi_k \sim \frac{H}{2\pi}\left(\frac{k}{k_*}\right)^{n_s-1};\quad n_s - 1 = -6\epsilon + 2\eta$$

We can show $\delta\varphi_k$ gives rise to $\delta T/T$. So they determine the initial conditions of the Universe! Using the CMB for $l < 100$ we can measure these perturbations.

15. Gravitational waves

15.1. Linearized field equations

We write the metric as

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

with $\left|h_{\mu\nu}\right| \ll 1$. The Riemann tensor is

$$R_{\alpha\mu\beta\nu} \propto \partial^2 g \underbrace{+ \ldots + \Gamma^2}_{\text{vanishing}}$$

with the linearized metric we find (trivially $R^{(0)} = 0$)

$$R^{(1)}_{\alpha\mu\beta\nu} = \frac{1}{2}\left(\partial_\alpha\partial_\nu h_{\mu\beta} + \partial_\mu\partial_\beta h_{\alpha\nu} - \partial_\alpha\partial_\beta h_{\mu\nu} - \partial_\mu\partial_\nu h_{\alpha\beta}\right)$$

since $\partial\eta = 0$. Then the Ricci tensor becomes

$$R^{(1)}_{\mu\nu} = \eta^{\alpha\beta} R^{(1)}_{\alpha\mu\beta\nu} = \frac{1}{2}\left(\partial_\alpha\partial_\nu h^\alpha{}_\mu + \partial_\mu\partial_\alpha h^\alpha{}_\nu - \partial^2 h_{\mu\nu} - \partial_\mu\partial_\nu h\right)$$

with h being the trace of $h_{\mu\nu}$. Then the Ricci scalar becomes

$$R^{(1)} = \eta^{\mu\nu} R^{(1)}_{\mu\nu} = \partial_\mu\partial_\nu h^{\mu\nu} - \partial^2 h$$

The Einstein field equations become

$$R^{(1)}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} R^{(1)} = -8\pi G T^{(0)}_{\mu\nu}$$

we take the trace

$$R^{(1)} = -8\pi G {T^{(0)}}^\mu{}_\mu$$

and we find

$$\partial_\mu \partial_\nu h^{\mu\nu} - \partial^2 h = 8\pi G T^{(0)\mu}{}_\mu$$

which looks like a wave equation!

15.2. Gauge freedom

The above h has 10 degrees of freedom. We want to reduce this number.

Consider

$$x^\mu \to x'^\mu = x^\mu + \xi^\mu(x)$$

We know how the metric transforms

$$\begin{aligned}
g'_{\alpha\beta} &= \frac{\partial x^\mu}{\partial x'^\alpha} \frac{\partial x^\nu}{\partial x'^\beta} g_{\mu\nu} \\
&= \frac{\partial x^\mu}{\partial x'^\alpha} \frac{\partial x^\nu}{\partial x'^\beta} \big(\eta_{\mu\nu} + h_{\mu\nu}\big) \\
&= \Big(\delta^\mu{}_\alpha \delta^\nu{}_\beta - \delta^\mu{}_\alpha \partial_\beta \xi^\nu - \partial_\alpha \xi^\mu \delta^\nu{}_\beta + \mathcal{O}(\xi^2)\Big)\big(\eta_{\mu\nu} + h_{\mu\nu}\big) \\
&= \eta_{\alpha\beta} + h_{\alpha\beta} - \partial_\beta \xi_\alpha - \partial_\alpha \xi_\beta
\end{aligned}$$

we see

$$h_{\alpha\beta} \to h'_{\alpha\beta} = h_{\alpha\beta} - \partial_\alpha \xi_\beta - \partial_\beta \xi_\alpha$$

This gives us gauge freedom. We can pick

$$\partial_\mu h^{\mu\nu} = \frac{1}{2} \partial^\nu h$$

we can do this since we have four degrees of freedom in ξ^μ. This is called the *Lorentz gauge* or the *de Donder gauge* if written in the form

$$g^{\mu\nu} \Gamma^\lambda{}_{\mu\nu} = 0$$

Using this we have in the Lorentz gauge

$$\partial^2 h = -16\pi G T^{(0)\mu}{}_\mu$$

We define

$$\overline{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h$$

then

$$\partial^\mu \overline{h}_{\mu\nu} = 0$$

giving

$$\partial^2 \overline{h}_{\mu\nu} = -16\pi G T^{(0)}_{\mu\nu}$$

Consider

$$h \to h' = h - 2\partial_\mu \xi^\mu$$

then

$$\begin{aligned}\partial^\mu h'_{\mu\nu} &= \partial^\mu h_{\mu\nu} - \partial^2 \xi_\nu - \partial_\mu \partial_\nu \xi^\mu \\ &\overset{\text{Lorentz}}{=} \frac{1}{2}\partial_\nu h' = \frac{1}{2}\partial_\nu h - 2\partial_\nu \partial_\mu \xi^\mu\end{aligned}$$

so

$$\frac{1}{2}\partial_\nu h = \partial^\mu h_{\mu\nu} \underbrace{-\partial^2 \xi_\nu}_{\overset{!}{=}0}$$

meaning if $\partial^2 \xi_\nu = 0$ the Lorentz gauge is satisfied! This is called residual gauge degree of freedoms. And we have now fixed 8 out of 10 degrees of freedom! So we are left with two dynamical degrees of freedom.

15.3. Plane waves in vacuum

We consider

$$T_{\mu\nu} = 0$$

meaning

$$\partial^2 h = 0; \quad \partial^2 \overline{h}_{\mu\nu} = 0$$

implying

$$\partial^2 h_{\mu\nu} = 0$$

with the condition

$$\partial_\mu h^{\mu\nu} = \frac{1}{2}\partial^\nu h$$

We make the ansatz

$$h_{\mu\nu}(\lambda) = \underbrace{\varepsilon_{\mu\nu}}_{\text{polarization}} e^{ik_\alpha x^\alpha}$$

and $k^\alpha = (\omega, \boldsymbol{k})$. We find

$$k^2 \varepsilon_{\mu\nu} e^{ik_\alpha x^\alpha} = 0$$

meaning

$$k^2 = -\omega^2 + \boldsymbol{k}^2 = 0$$

So the ansatz works given the normal dispersion relation for a plane wave holds. We also find

$$\partial_\mu h^{\mu\nu} = 0 \Rightarrow k^\mu \varepsilon_{\mu\nu} = 0$$

which is called the *transverse gauge condition.*

The condition $k^\mu \varepsilon_{\mu\nu} = 0$ still has degrees of freedom we need to fix. We impose

$$\varepsilon_{\mu 0} = \varepsilon_{0\mu} = 0$$

and

$$\varepsilon_\mu{}^\mu = 0$$

As an example take

$$k^\alpha = \left(\omega, 0, 0, \underbrace{k_z}_{\stackrel{!}{=}\omega}\right)$$

then

$$k^\mu \varepsilon_{\mu\nu} = 0 \Rightarrow \omega \varepsilon_{3\nu} = 0$$

giving (with $\varepsilon_{\nu 0} = 0$ and $\varepsilon^\mu{}_\mu = 0$ and $\varepsilon_{\mu\nu} = \varepsilon_{\nu\mu}$)

$$h_{\mu\nu}(z,t) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_\times & 0 \\ 0 & h_\times & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} e^{i\omega(z-t)}$$

We can define the basis

$$\varepsilon_+^{\mu\nu} = h_+ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}; \quad \varepsilon_\times^{\mu\nu} = h_\times \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Consider a test particle at rest with $U^\mu = (1,0,0,0)$. The geodesic equation is

$$\frac{\mathrm{d}U^\mu}{\mathrm{d}\tau} + \Gamma^\mu{}_{\nu\lambda} U^\nu U^\lambda = 0 \Rightarrow \frac{\mathrm{d}U^\mu}{\mathrm{d}\tau} = -\Gamma^\mu{}_{00}$$

where

$$\Gamma^{\mu}{}_{00} = \frac{1}{2}\eta^{\mu\nu}(\partial_0 h_{\nu 0} + \partial_0 h_{0\nu} - \partial_\nu h_{00}) = 0$$

So the effect on a test particle is

$$\frac{\mathrm{d}U^\mu}{\mathrm{d}\tau} = 0$$

meaning there is no effect!

Consider two test particles with $x^\mu = (0,0,0,0)$ and $x^\mu + \mathrm{d}x^\mu = (0,\xi,0,0)$. Then

$$\begin{aligned}
\mathrm{d}s &= \sqrt{g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu} \\
&= \sqrt{g_{11}}\xi \\
&\overset{\text{binom}}{=} \left(\eta_{11} + \frac{1}{2}h_{11}\right)\xi
\end{aligned}$$

for $+$ polarization we find

$$\mathrm{d}s = \left(1 + \frac{1}{2}h_+ \sin(t-z)\right)\xi$$

for separation along y we find

$$\mathrm{d}s = \left(1 - \frac{1}{2}h_+ \sin(t-z)\right)\xi$$

So we find a stretch along x and a squeeze along y (initially) with it oscillating as one would expect.

For $\times$ polarization one would find a *diagonal* stretching and squeezing.