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Brief paper

Swing-up of the double pendulum on a cart by feedforward and feedback control with experimental validation☆

Knut Graichen*, Michael Treuer, Michael Zeitz

Institut für Systemdynamik, Universität Stuttgart, Pfaffenwaldring 9, 70569 Stuttgart, Germany

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Abstract

The swing-up maneuver of the double pendulum on a cart serves to demonstrate a new approach of inversion-based feedforward control design introduced recently. The concept treats the transition task as a nonlinear two-point boundary value problem of the internal dynamics by providing free parameters in the desired output trajectory for the cart position. A feedback control is designed with linear methods to stabilize the swing-up maneuver. The emphasis of the paper is on the experimental realization of the double pendulum swing-up, which reveals the accuracy of the feedforward/feedback control scheme.

Keywords: Double pendulum swing-up; Nonlinear feedforward control; Boundary value problem; Input–output normal form; Linear feedback control; Experiment

1. Introduction

Pendulums are widely used in nonlinear control education and research as benchmark examples of underactuated mechanical systems. A vast range of contributions exists for the stabilization of different types of inverted pendulums, see e.g. Mori, Nishihara, and Furuta (1976), Furuta, Kajiwara, and Kosuge (1980), Anderson and Grantham (1989). Besides the stabilization aspect, the swing-up problem—especially of the classic single pendulum on a cart—has gained increasing attention during the recent past, see e.g. Wiklund, Kristenson, and Åström (1993), Åström and Furuta (2000). The swing-up of various types of double pendulums is also addressed in the literature, like the acrobot and pendubot (Fantoni, Lozano, & Spong, 2000; Graichen & Zeitz, 2005b; Spong, 1995), or the rotary double pendulum in Yamakati, Nonaka, and Furuta (1993), Yamakati, Iwashiro, Sugahara, and Furuta (1995).

A challenging problem is the swing-up of the double pendulum on a cart, which is less accounted for in the literature. This is mainly due to the limited rail length of the cart, in contrast e.g. to the rotary double pendulum. In Zhong and Röck (2001) and Huang and Fu (2003), a passivity-based approach is proposed in combination with partial feedback linearization. The swing-up of the double pendulum on a cart is accomplished in simulation studies, but no experimental results are provided in Zhong and Röck (2001) and Huang and Fu (2003).

Another approach utilizes a combined feedforward/feedback ("two-degree-of-freedom") control scheme to solve the swing-up problem (Rubí, Rubio, & Avello, 2002). The feedforward control and the nominal state trajectories for the swing-up are obtained by solving an optimization problem with the stationary downward and upward equilibria as boundary conditions. The underactuated dynamics of the double pendulum are taken into account by considering all links of the double pendulum to be active and minimizing the torques exerted at the unactuated links. Hence, the obtained trajectory is only an approximate solution for the swing-up problem since the torques acting at the free joints are not identically zero. A gain-scheduled feedback control is used to stabilize the system during the swing-up and in the upward position. To the authors knowledge, Rubí et al. (2002) is the only contribution so far

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* Corresponding author. Tel.: +49 711 685 66568; fax: +49 711 685 66371.
E-mail addresses: knut.graichen@isys.uni-stuttgart.de (K. Graichen), michael.treuer@isys.uni-stuttgart.de (M. Treuer), michael.zeitz@isys.uni-stuttgart.de (M. Zeitz).

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providing experimental results for the swing-up of the double pendulum on a cart.

In this paper, the inversion-based feedforward control design recently proposed in Graichen, Hagenmeyer, and Zeitz (2005) is applied to the swing-up maneuver of the double pendulum on a cart. Thereby, the nonlinear feedforward control is determined by solving a two-point boundary value problem (BVP) for the internal dynamics of the pendulum, i.e. the dynamics of the unactuated links, by providing free parameters in the desired output trajectory to solve the overdetermined BVP. Due to the high accuracy of the nonlinear feedforward control, the stabilizing feedback part can be designed by linear methods with the pendulum model linearized along the nominal trajectories. Experimental results for the swing-up maneuver illustrate the potential of the concept.

The paper is outlined as follows: the next section describes the model of the double pendulum experiment and formulates the transition problem for the swing-up maneuver. Section 3 is devoted to the feedforward control design and the BVP of the internal dynamics with free parameters. The swing-up trajectories are computed using a standard MATLAB BVP-solver. Section 4 addresses the experimental validation of the swing-up.

2. Problem statement

The double pendulum on a cart (see Fig. 1) consists of two links with the length l_i and the angles $\phi_i(t)$, $i = 1, 2$ to the vertical. The mechanical parameters are described in Table 1 together with their corresponding values, which have been measured and identified at the experimental device (see Section 4). Furthermore, the cart movement is subject to the constraints

$$|y| \leqslant 0.7\,\mathrm{m}, \quad |\dot{y}| \leqslant 2.2\,\mathrm{m/s}, \quad |\ddot{y}| \leqslant 20\,\mathrm{m/s^2} \tag{1}$$

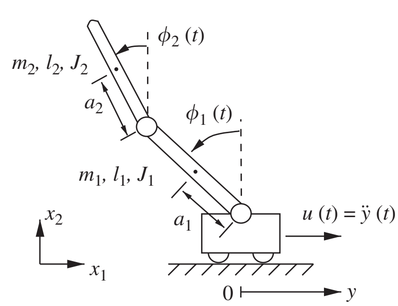


Fig. 1. Schematic of the double pendulum on a cart with the mechanical parameters in Table 1.

Table 1
Mechanical parameters of the double pendulum

Pendulum link	Inner $i=1$	Outer $i=2$
Length l_i (m)	0.323	0.480
Distance to center of gravity a_i (m)	0.2145	0.223
Mass m_i (kg)	0.853	0.510
Moment of inertia J_i (N m s^2)	0.0126	0.0185
Friction constant d_i (N m s)	0.005	0.005

Table 2
Equations of motion of the double pendulum in Fig. 1

$$\begin{aligned}
&(J_1 + a_1^2 m_1 + l_1^2 m_2)\ddot{\phi}_1 + a_2 l_1 m_2 \cos(\phi_1 - \phi_2)\ddot{\phi}_2 \\
&\quad = (a_1 m_1 + l_1 m_2) g \sin(\phi_1) - a_2 l_1 m_2 \sin(\phi_1 - \phi_2)\dot{\phi}_2^2 \\
&\qquad - d_1 \dot{\phi}_1 - d_2(\dot{\phi}_1 - \dot{\phi}_2) + (a_1 m_1 + l_1 m_2)\cos(\phi_1)\ddot{y} \\
&a_2 l_1 m_2 \cos(\phi_1 - \phi_2)\ddot{\phi}_1 + (J_2 + a_2^2 m_2)\ddot{\phi}_2 \\
&\quad = a_2 g m_2 \sin(\phi_2) + a_2 l_1 m_2 \sin(\phi_1 - \phi_2)\dot{\phi}_1^2 \\
&\qquad + d_2(\dot{\phi}_1 - \dot{\phi}_2) + a_2 m_2 \cos(\phi_2)\ddot{y}
\end{aligned}$$

due to the limited rail length and the physical limits of the cart actuator.

2.1. Equations of motion

The model of the double pendulum can be derived via the Lagrangian method. The absolute position $\boldsymbol{x}^i = [x_1^i, x_2^i]^\mathsf{T}$, $i=1, 2$ of the center of mass of each link i is given by

$$\text{Link 1}: \quad \boldsymbol{x}^1 = \begin{bmatrix} y - a_1 \sin\phi_1 \\ a_1 \cos\phi_1 \end{bmatrix},$$

$$\text{Link 2}: \quad \boldsymbol{x}^2 = \begin{bmatrix} y - l_1 \sin\phi_1 - a_2 \sin\phi_2 \\ l_1 \cos\phi_1 + a_2 \cos\phi_2 \end{bmatrix}.$$

The kinetic and potential energies are determined to be

$$T = \frac{1}{2} m_c \dot{y}^2 + \frac{1}{2} \sum_{i=1}^{2} [m_i |\dot{\boldsymbol{x}}^i|^2 + J_i \dot{\phi}_i^2],$$

$$V = \sum_{i=1}^{2} m_i g x_2^i.$$

The non-conservative friction forces in the links are modeled by the linear expressions

$$F_1 = -d_1 \dot{\phi}_1 + d_2(\dot{\phi}_2 - \dot{\phi}_1), \quad F_2 = d_2(\dot{\phi}_1 - \dot{\phi}_2), \tag{2}$$

whereby the parameters d_i denote the damping coefficient at the respective link i. The Lagrangian $L = T - V$ yields the equations of motion

$$\frac{\partial}{\partial t} \frac{\partial L}{\partial \dot{\phi}_i} - \frac{\partial L}{\partial \phi_i} = F_i, \quad i = 1, 2, \tag{3}$$

which are given in Table 2. Furthermore, in the double pendulum experiment, the acceleration $\ddot{y}$ of the cart serves as input $u = \ddot{y}$ to the system. The overall model of the double pendulum can be formally written as a system of second-order ODEs

$$\ddot{y} = u, \tag{4}$$

$$\ddot{\boldsymbol{\phi}} = \boldsymbol{\beta}(\boldsymbol{\phi}, \dot{\boldsymbol{\phi}}, u), \tag{5}$$

with $\boldsymbol{\phi} = [\phi_1, \phi_2]^\mathsf{T}$, $\boldsymbol{\beta} = [\beta_1, \beta_2]^\mathsf{T}$, and the system order $n = 6$. Note that the ODEs (5) are independent of the cart displacement y and velocity $\dot{y}$.

2.2. Swing-up problem

The swing up within a finite time interval $t \in [0, T]$ requires to steer the double pendulum from the initial downward equilibrium

$$y(0)=0, \quad \dot{y}(0)=0, \quad \boldsymbol{\phi}(0)=[-\pi, -\pi]^{\mathsf{T}}, \quad \dot{\boldsymbol{\phi}}(0)=\mathbf{0} \tag{6}$$

to the terminal upward equilibrium

$$y(T)=0, \quad \dot{y}(T)=0, \quad \boldsymbol{\phi}(T)=\mathbf{0}, \quad \dot{\boldsymbol{\phi}}(T)=\mathbf{0}. \tag{7}$$

The internal dynamics (5) is weakly asymptotically stable in the downward equilibrium and unstable in the upward position.

The ODEs (4)–(5) together with the boundary conditions (BCs) (6)–(7) form a nonlinear two-point BVP for the states $y(t)$, $\dot{y}(t)$ and $\boldsymbol{\phi}(t)$, $\dot{\boldsymbol{\phi}}(t)$ that depends on the input trajectory $u(t)$. Its determination is the main objective of the feedforward control design.

The swing-up time T is an important parameter and mainly depends on the constraints (1) and the system dynamics (4)–(5). If T is chosen too small, the cart may violate the constraints. On the other hand, the swing up is not possible arbitrarily slowly due to the fact that no quasi-stationary connection exists between the downward and upward equilibria, i.e. they are not connected by a set of equilibria in between. Hence, the swing-up time T has to be determined appropriately in course of the feedforward control design.

3. Nonlinear feedforward control design

The inversion-based feedforward control design (Devasia, Chen, & Paden, 1996; Graichen et al., 2005) uses the input–output coordinates of the considered system. In case of the double pendulum, the system (4)–(5) is already given in input–output normal form (Isidori, 1995) with the cart position y as the output and the relative degree $r=2$. The respective cart ODE (4) represents the input–output dynamics, and the ODEs (5) of the angles ϕ_1, ϕ_2 form the internal dynamics of order $n-r=4$.

The feedforward control is obtained by inverting the input–output dynamics (Devasia et al., 1996; Graichen et al., 2005). In view of (4), the feedforward control[1]

$$u^*(t)=\ddot{y}^*(t) \tag{8}$$

is simply the second time derivative of the desired output trajectory $y^*(t)$.

3.1. BVP of the internal dynamics

In order to determine the trajectories $\boldsymbol{\phi}^*(t)$ and $\dot{\boldsymbol{\phi}}^*(t)$ of the angles, the internal dynamics (5) can be rewritten by inserting the feedforward control (8), i.e.

$$\ddot{\boldsymbol{\phi}}^* = \boldsymbol{\beta}(\boldsymbol{\phi}^*, \dot{\boldsymbol{\phi}}^*, \ddot{y}^*) \tag{9}$$

subject to the respective BCs in (6)–(7):

$$\boldsymbol{\phi}^*(0)=[-\pi, -\pi]^{\mathsf{T}}, \quad \boldsymbol{\phi}^*(T)=\mathbf{0}, \quad \dot{\boldsymbol{\phi}}^*|_{t=0,T}=\mathbf{0}. \tag{10}$$

Note that the second time derivative $\ddot{y}^*(t)$ of the output trajectory serves as input to (9). Obviously, the BVP (9)–(10) of the internal dynamics is overdetermined by eight BCs for two second-order ODEs.

The basic idea of the approach presented in Graichen et al. (2005) is to provide a sufficient number of four free parameters in the internal dynamics (9), which are required for its solvability. Thereby, the parameters $\boldsymbol{p}=(p_1, \ldots, p_4)$ are provided in a setup function $\Upsilon(t, \boldsymbol{p})$ for the output trajectory $y^*(t)=\Upsilon(t, \boldsymbol{p})$.

3.2. Output trajectory setup with free parameters

The output trajectory $y^*(t)=\Upsilon(t, \boldsymbol{p})$ has to satisfy the four BCs (6)–(7), which implies that the output trajectory must be at least once differentiable,[2] i.e. $y^*(t) \in \mathscr{C}^1$. The setup function $\Upsilon(t, \boldsymbol{p})$ is constructed using the cosine series

$$\Upsilon(t, \boldsymbol{p}) = a_0 + a_1 \cos\left(\frac{\pi t}{T}\right) + \sum_{i=2}^{5} p_{i-1} \cos\left(\frac{i\pi t}{T}\right), \tag{11}$$

with the free parameters $\boldsymbol{p}=(p_1, \ldots, p_4)$ as the coefficients for the cosine terms with the highest frequencies. The remaining coefficients $a_0=-p_1-p_3$ and $a_1=-p_2-p_4$ follow from solving the equations stemming from the BCs $\Upsilon(0, \boldsymbol{p})=0$ and $\Upsilon(T, \boldsymbol{p})=0$. Note that the cosine series directly satisfies $\dot{\Upsilon}(0, \boldsymbol{p})=\dot{\Upsilon}(T, \boldsymbol{p})=0$ due to the sine terms occuring in the first time derivative $\dot{\Upsilon}(t, \boldsymbol{p})$. Other possible choices for the setup of $\Upsilon(t, \boldsymbol{p})$ are e.g. polynomials or spline functions (Graichen, Treuer, & Zeitz, 2005).

Remark 1. The swing-up time T is directly affected by the choice of the setup function $y^*(t)=\Upsilon(t, \boldsymbol{p})$. For instance, the number of times that the output $y^*(t)$ passes through zero ("swinging" of the cart) is limited by the highest frequency of $\Upsilon(t, \boldsymbol{p})$ in (11). This corresponds to certain regions of T where solutions for the swing-up problem exist. Alternatively, the swing-up time can also be treated as a free parameter (via time transformation) with the remaining three parameters $\boldsymbol{p}=(p_1, p_2, p_3)$ in the setup function $\Upsilon(t, \boldsymbol{p})$, see Graichen and Zeitz (2005a,b).

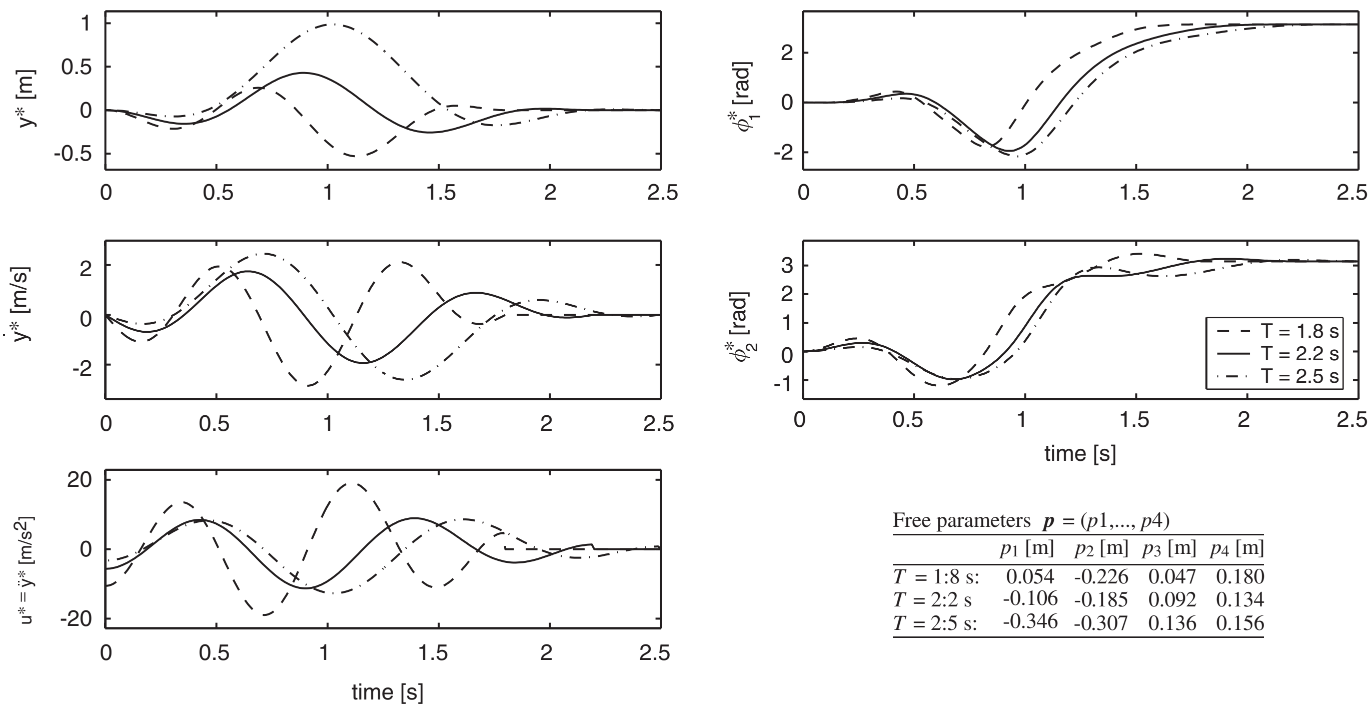
3.3. Numerical results

The numerical solution of the BVP (9)–(11) is a standard task in numerics. A particularly convenient way is to use the MATLAB function bvp4c[3] designed for the solution of two-point BVPs with unknown parameters. The initial guesses

[1] The asterisk "$*$" signifies the feedforward variables.

[2] If the feedforward control (8) has to be continuous at the transition bounds $t=0$ and T, two further BCs $\ddot{y}(0)=\ddot{y}(T)=0$ have to be satisfied by the output trajectory $y^*(t) \in \mathscr{C}^2$, see e.g. Graichen et al. (2005).

[3] http://www.mathworks.com/bvp_tutorial



Free parameters $\boldsymbol{p} = (p1,\ldots,p4)$

	p_1 [m]	p_2 [m]	p_3 [m]	p_4 [m]
T = 1:8 s:	0.054	-0.226	0.047	0.180
T = 2:2 s	-0.106	-0.185	0.092	0.134
T = 2:5 s:	-0.346	-0.307	0.136	0.156

Fig. 2. Nominal trajectories and parameters $\boldsymbol{p}$ for the swing-up of the double pendulum in three different times T.

for the trajectories $\boldsymbol{\phi}^*(t_k)$ and $\dot{\boldsymbol{\phi}}^*(t_k)$ are linear interpolations between the BCs (10) on a uniform time mesh with 30 points $t_k \in [0, T]$, $k = 1, \ldots, 30$. The initial guess of the free parameters $\boldsymbol{p}$ is $p_i = 0$, $i = 1, \ldots, 4$. bvp4c returns the trajectories $\boldsymbol{\phi}^*(t) = [\phi_1^*(t), \phi_2^*(t)]^\top$ and the parameter set $\boldsymbol{p}$, which yields the output trajectory $y^*(t) = \Upsilon(t, \boldsymbol{p})$ and the feedforward control (8), i.e. $u^*(t) = \ddot{\Upsilon}(t, \boldsymbol{p})$.

Fig. 2 shows the nominal trajectories for the swing-up maneuver for different swing-up times $T \in \{1.8, 2.2, 2.5\}$ s. The significant influence of the swing-up time T is particularly apparent in the cart trajectories $y^*(t)$, $\dot{y}^*(t)$, and $\ddot{y}^*(t)$. For $T = 1.8$ s, the cart movement $y^*(t)$ is very limited, but the maximum acceleration $\max_t \ddot{y}^*(t) = 19\,\mathrm{m/s^2}$ almost hits the respective constraint in (1). In contrast to this, the swing-up time $T = 2.5$ s leads to a different swing-up motion with the large cart displacement $\max_t y^*(t) = 1$ m. A good trade-off between the maximum amplitudes of the trajectories $y^*(t)$, $\dot{y}^*(t)$, $\ddot{y}^*(t)$ with respect to the constraints (1) is obtained for $T = 2.2$ s, which is therefore chosen as swing-up time for the experimental implementation.

Fig. 3 shows snapshots of the pendulum for the swing-up time $T = 2.2$ s to illustrate its motion. It is interesting to mention that both arms of the pendulum are in a hinged position during the swing-up (see sequences 2 and 3 in Fig. 3) and only stretch close to the upward "inverted" position.

4. Experimental validation

The experimental realization of the swing-up maneuver requires a stabilization of the double pendulum by a feedback controller. In the context of the two-degree-of-freedom control scheme in Fig. 4, the feedforward control Σ_{FF} is supported by a state feedback control Σ_{FB} with an observer $\widehat{\Sigma}$ in order to stabilize the system Σ along the nominal trajectories $\boldsymbol{x}^*(t)$ provided by the signal generator Σ^*. Thereby, a highly accurate feedforward control Σ_{FF} is necessary in order to minimize the demands on the feedback part during the swing-up. The accuracy of the nonlinear feedforward control can be enhanced by an optimization-based adjustment of the mechanical parameters in Table 1 with respect to the open-loop experimental results for the nominal feedforward control $u^*(t)$. The experimental setup and the above mentioned points are addressed in the following subsections.

4.1. Experimental construction of the double pendulum

The swing-up maneuver is experimentally realized with the double pendulum in Fig. 5 corresponding to the model parameters in Table 1 and the cart constraints (1).[4] The incremental angle encoders at the two joints have a resolution of $2\pi/8192$ rad and transmit their information through optical links in the joints to reduce friction. The cart is driven by a toothed belt connected to a synchronous motor. The control algorithm is implemented on a 933 MHz computer with real-time Linux and the sampling

[4] The construction of the pendulum was a joint project of the *Max Planck Institute for Dynamics of Complex Technical Systems* (www.mpi-magdeburg.mpg.de) and the company (*Hasomed GmbH*). The pendulum can also be employed with three links, see e.g. Graichen et al. (2005) for the side-stepping of the triple inverted pendulum.

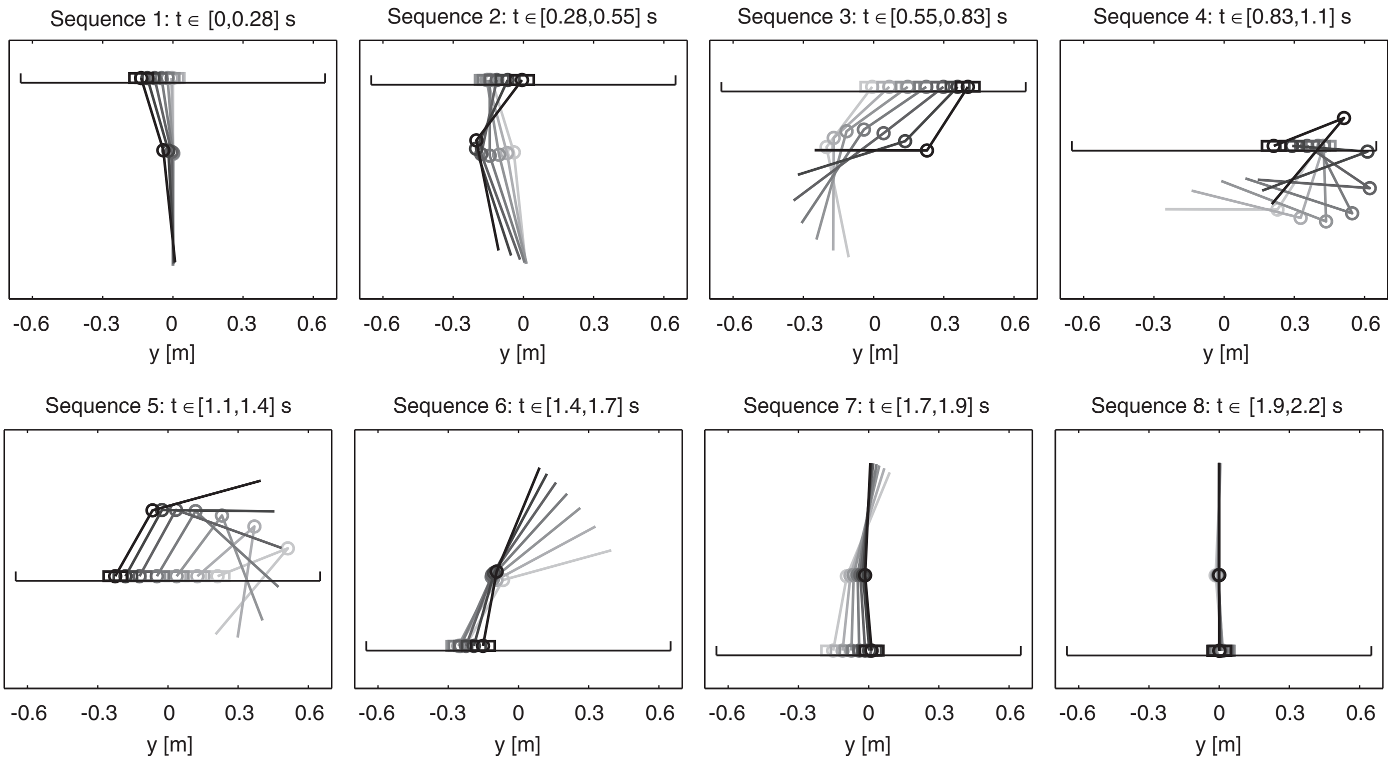


Fig. 3. Snapshots of the nominal swing-up maneuver for $T = 2.2$ s (see Fig. 2) depicted in eight sequences with increasing darkness of the snapshots as time increases during the respective sequence.

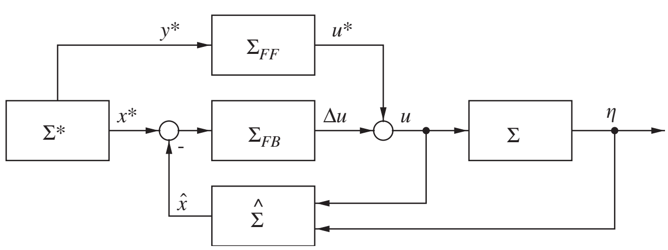


Fig. 4. Two-degree-of-freedom control scheme with system Σ, signal generator Σ^*, feedforward control Σ_{FF}, feedback control Σ_{FB}, observer $\widehat{\Sigma}$, and measurement vector $\boldsymbol{\eta} = [y, \phi_1, \phi_2]^{\mathrm{T}}$.

time 1 ms. The nominal trajectories $\boldsymbol{\phi}^*(t), \dot{\boldsymbol{\phi}}^*(t), y^*(t), \dot{y}^*(t)$, and the feedforward control $u^*(t) = \ddot{y}^*(t), t \in [0, T]$ are calculated offline and stored in look-up tables.

Remark 2. The experimental setup uses an underlying fast PI control for the velocity $\dot{y}$ of the cart instead of controlling the acceleration $u = \ddot{y}$ in the pendulum model (4)–(5). Therefore, the input u is integrated before it is used as setpoint for the cascaded velocity controller. This justifies the use of the output trajectory (11) which leads to discontinuities of the feedforward control $u^*(t)$ at the time instants $t = 0$ and $t = T$. Due to the internal integration of $u = \ddot{y}$, a step at the time instants $t = 0$ and $t = T$ results in a continuous velocity $\dot{y}$ which the cascaded PI controller is able to follow.

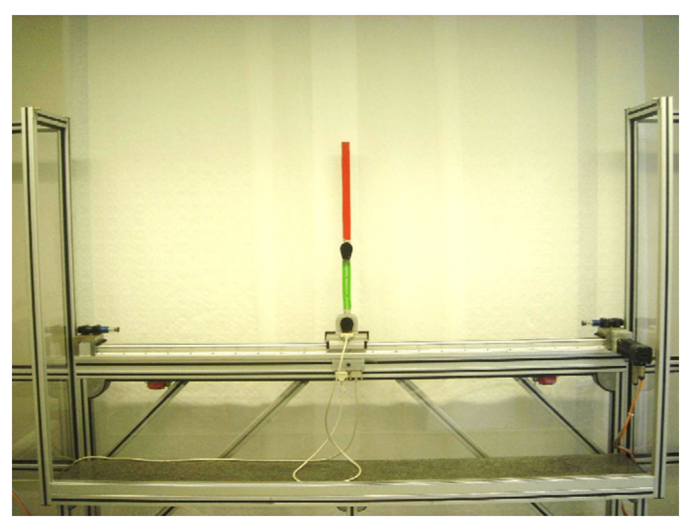


Fig. 5. Experimental construction of the double pendulum on a cart (Hasomed GmbH) with model parameters in Table 1 and the cart constraints (1).

4.2. Adjustment of model parameters

The feedforward control $u^*(t)$ must be highly accurate in order to steer the double pendulum along the nominal swing-up trajectories close to the unstable upward equilibrium. If the model is too inaccurate, the pendulum drifts away from the nominal trajectories too early, and the feedback control has to correct the tracking error. As a result, the deviations of the cart

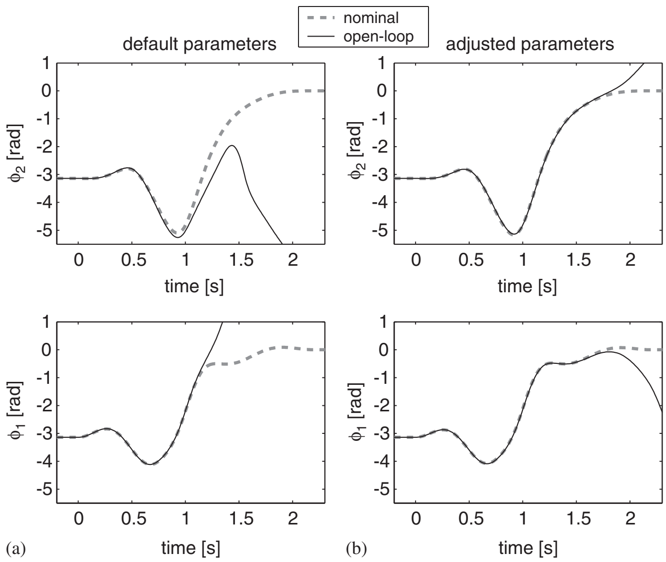


Fig. 6. Nominal and experimental trajectories $\boldsymbol{\phi}^*(t)$ and $\boldsymbol{\phi}(t)$ for the swing-up in open-loop mode with (a) nominal and (b) adjusted model parameters.

position $y(t)$ from its nominal trajectory $y^*(t)$ might exceed the maximum rail length (1). In order to enhance the accuracy of the feedforward control $u^*(t)$, the mechanical parameters (Table 1) of the pendulum model (4)–(5) are adjusted by solving an optimization problem with respect to the experimental swing-up maneuver in open-loop.

Fig. 6a shows the open-loop measurements of the angles $\phi_1(t)$ and $\phi_2(t)$ (solid lines) with respect to the nominal swing-up trajectories (dashed lines), also see Fig. 2. The angle $\phi_1(t)$ of the inner arm stays close to the nominal trajectory $\phi_1^*(t)$ almost up to the unstable upward equilibrium, but the outer arm angle $\phi_2(t)$ follows $\phi_2^*(t)$ only at the beginning of the swing up. For instance at $t = 1.3$ s, $\phi_2(t)$ is 0.9 rad $\approx$ 50 deg away from $\phi_2^*(t)$. The inaccuracy is mainly due to the time-delayed response of the PI controller for the cart velocity (see Remark 2) and due to unmodeled effects like nonlinear friction.

In order to meet the measured angle profiles $\phi_1(t)$, $\phi_2(t)$ in Fig. 6a as close as possible, the parameters $\boldsymbol{\theta} = (a_1, a_2, m_1, m_2, J_1, J_2, d_1, d_2)$ occurring in the pendulum dynamics (5) (also see Table 2) are adjusted by solving the optimization problem

$$\min_{\theta} \quad I = \int_0^{T_0} (\phi_{\theta,1}(t) - \phi_1(t))^2 + (\phi_{\theta,2}(t) - \phi_2(t))^2 \, \mathrm{d}t \tag{12}$$

$$\text{s.t.} \quad \ddot{\boldsymbol{\phi}}_\theta = \boldsymbol{\beta}(\boldsymbol{\phi}_\theta, \dot{\boldsymbol{\phi}}_\theta, u^*), \quad \boldsymbol{\phi}_\theta(0) = -[\pi, \pi]^\top, \dot{\boldsymbol{\phi}}_\theta(0) = \mathbf{0}. \tag{13}$$

The feedforward control $u^*(t)$ is based on the default parameter values $\boldsymbol{\theta}_{\text{nom}}$ (see Table 1 and Fig. 2) and serves as input to the pendulum dynamics (13) with the states $\boldsymbol{\phi}_\theta = [\phi_{\theta,1}, \phi_{\theta,2}]^\top$ and $\dot{\boldsymbol{\phi}}_\theta = [\dot{\phi}_{\theta,1}, \dot{\phi}_{\theta,2}]^\top$. The cost function I in (12) rates the deviation of the angles $\phi_{\theta,1}(t)$, $\phi_{\theta,2}(t)$ with respect to the measured open-loop trajectories $\phi_1(t)$, $\phi_2(t)$ in Fig. 6a. Thereby, I is evaluated over the time interval $t \in [0, T_0]$ with $T_0 = 1.6$ s being smaller

Table 3
Adjusted mechanical parameters for the open-loop swing-up of the double pendulum in Fig. 6.

Pendulum link	Inner $i=1$	Outer $i=2$
Distance to center of gravity a_i (m)	0.186	0.195
Mass m_i (kg)	0.881	0.551
Moment of inertia J_i (N m s^2)	0.0141	0.0177
Friction constant d_i (N m s)	0.0034	0.0016

than the swing-up time $T = 2.2$ s, because the dynamics (13) turn unstable at the end of the swing-up leading to an unstable numerical integration close to the upward equilibrium. Note that the solution of the optimization problem (12)–(13) cannot be interpreted as identification of the mechanical parameters $\boldsymbol{\theta}$, but rather serves to fit the pendulum dynamics(13) to the measurement results of the open-loop swing-up maneuver.

The optimization problem is solved with the MATLAB function fmincon of the optimization toolbox. Table 3 lists the adjusted parameters $\boldsymbol{\theta}$, which are used together with the remaining default ones in Table 1 to recalculate the feedforward trajectories according to Section 3. Fig. 6b shows the open-loop experimental results (solid lines) for the swing up with the nominal feedforward trajectories (dashed lines) based on the adjusted parameters. The accuracy of the second angle $\phi_2(t)$ is clearly improved and both angles follow the nominal trajectories $\phi_1^*(t)$, $\phi_2^*(t)$ close to the upward position. Although the difference between the nominal trajectories $\phi_1^*(t)$ and $\phi_2^*(t)$ in Fig. 6a and b is almost negligible, the enhanced accuracy of the second angle ϕ_2 shows the high sensitivity of the swing-up maneuver with respect to the model parameters.

4.3. Linear feedback control design

The experimental realization of the swing-up maneuver requires a feedback control which stabilizes the double pendulum along the nominal trajectories. In order to compensate for a possible steady state error in the cart position y, the pendulum model (4)–(5) is dynamically extended by the disturbance model $\dot{\tilde{y}} = y$ with the new state $\tilde{y}$, which yields the overall state vector $\boldsymbol{x} = [y, \dot{y}, \boldsymbol{\phi}^\top, \dot{\boldsymbol{\phi}}^\top, \tilde{y}]^\top \in \mathbb{R}^7$. Hence, the system (4)–(5) together with the additional ODE $\dot{\tilde{y}} = y$ can be written in the nonlinear form $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, u)$.

Due to the accuracy of the nonlinear feedforward control, the feedback part is designed with linear methods by linearizing the overall system $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, u)$ along the nominal trajectories $\boldsymbol{x}^* = [y^*, \dot{y}^*, \boldsymbol{\phi}^{*\top}, \dot{\boldsymbol{\phi}}^{*\top}, \tilde{y}^*]^\top$ (with $\tilde{y}^*(t) = \int_0^t y^*(\tau)\,\mathrm{d}\tau$) and u^*. This leads to the linear time-varying system

$$\Delta\dot{\boldsymbol{x}} = A(t)\Delta x + \boldsymbol{b}(t)\Delta u, \tag{14}$$

with

$$A(t) = \left.\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}\right|_{\boldsymbol{x}^*(t), u^*(t)} \quad \text{and} \quad \boldsymbol{b}(t) = \left.\frac{\partial \boldsymbol{f}}{\partial u}\right|_{\boldsymbol{x}^*(t), u^*(t)}.$$

According to the two-degree-of-freedom control scheme in Fig. 4, the control

$$u = u^* + \boldsymbol{k}^\top(t)(\boldsymbol{x}^* - \hat{\boldsymbol{x}}) \qquad (15)$$

comprises the feedforward control $u^*(t)$ and the feedback part $\Delta u = \boldsymbol{k}^\top(t)(\boldsymbol{x}^* - \hat{\boldsymbol{x}})$. The calculation of the time-varying feedback gains $\boldsymbol{k}(t)$ is based on an optimal LQ (linear quadratic) feedback design which minimizes the objective functional

$$I = \Delta\boldsymbol{x}^\top(T) M \Delta\boldsymbol{x}(T) + \int_0^T (\Delta\boldsymbol{x}^\top Q \Delta\boldsymbol{x} + \Delta u R \Delta u)\, \mathrm{d}t, \qquad (16)$$

with the symmetric positive semidefinite matrices $M \in \mathbb{R}^{7\times 7}$, $Q \in \mathbb{R}^{7\times 7}$ and the positive scalar $R > 0$. The solution $P(t)$, $t \in [0, T]$ of the Riccati ODE, see e.g. Kwakernaak and Sivan (1972), Bertsekas (2000),

$$\dot{P} = -PA(t) - A(t)^\top P + P\boldsymbol{b}(t)R^{-1}\boldsymbol{b}(t)^\top P - Q,$$
$$P(T) = M \qquad (17)$$

determines the feedback gains

$$\boldsymbol{k}(t) = R^{-1}\boldsymbol{b}(t)^\top P(t). \qquad (18)$$

The weighting matrices in (16) are chosen to $Q = \mathrm{diag}(50, 0, 500, 500, 0, 0, 10)$ and $R = 5$ (with consistent units). The choice of the terminal condition $P(T) = M$ for the reverse-time integration of the Riccati equation (17) is a degree-of-freedom in the LQ-design. Thereby, the matrix $M \in \mathbb{R}^{7\times 7}$ is determined by solving the algebraic Riccati equation following from (17) with $\dot{P} = 0$. The MATLAB function lqr of the control system toolbox is used to calculate M, whereas the reverse-time integration of (17) is performed with a standard ODE solver of MATLAB.

Fig. 7 shows the time-varying feedback gains $k_i(t)$, $i = 1, \ldots, 7$ in the time interval $t \in [0, T]$ for the swing-up maneuver with $T = 2.2$ s. During the time interval $t \in [0.5, 1.2]$ s, the gains $k_i(t)$ oscillate and change the signs several times. Although the LQ design provides optimal feedback gains $k_i(t)$ for minimizing the cost functional (16) over the time interval $t \in [0, T]$, the large gradients and magnitudes of the gains $k_i(t)$ for $t \in [0.5, 1.2]$ s pose significant demands on the closed-loop control of the double pendulum leading to large displacements of the cart position y. Moreover, the linear time-varying system (14) looses its controllability (see e.g. Silverman & Meadows, 1967; Kailath, 1980) several times in this time interval. Due to these reasons, the feedback control is turned off for $t \in [0.6, 1.1]$ s by setting the gain vector $\boldsymbol{k}(t)$ to zero. In the bordering intervals $t \in [0.5, 0.6]$ s and $t \in [1.1, 1.2]$ s, the gains $k_i(t)$ are linearly interpolated between zero and the respective gains values at $t = 0.5$ and 1.2 s in order to smoothly switch on/off the feedback control. Hence, during the time interval $t \in [0.6, 1.1]$ s, the pendulum is steered along the nominal trajectories $\boldsymbol{x}^*(t)$ by the feedforward control without a stabilizing feedback control.

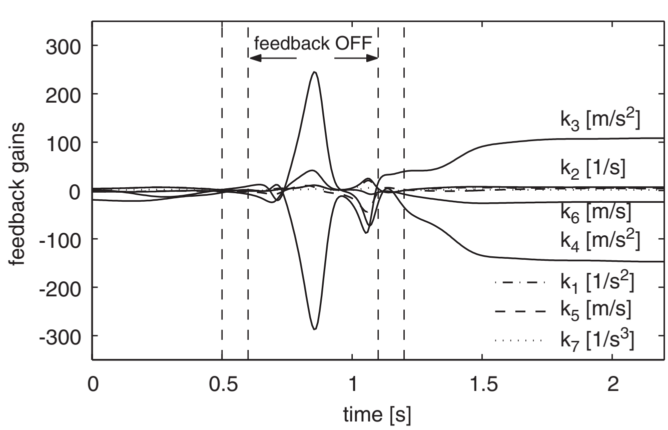


Fig. 7. Time-varying LQ feedback gains $k_i(t)$, $i = 1, \ldots, 7$ for the swing-up maneuver.

4.4. Experimental results

The implementation of the closed-loop control (15) requires full state information of the double pendulum. A Luenberger observer (O'Reilly, 1983) based on the nonlinear model (4)–(5) is used for the state estimation $\hat{\boldsymbol{x}}$, see Fig. 4. The error dynamics of the observer is designed by eigenvalue assignment point-wise in time with the linearized pendulum model.

Fig. 8 shows the experimental and nominal trajectories of the angles $\boldsymbol{\phi}(t)$, the cart $y(t)$, $\dot{y}(t)$, and the input $u(t) = \ddot{y}(t)$ for open-loop (also see Fig. 6b) and closed-loop control of the swing-up maneuver. As pointed out in Section 4.2, the open-loop trajectories reveal the good accuracy of the designed feedforward control $u^*(t)$, but the pendulum drifts away at-approximately $t = 1.5$ s when it approaches the unstable upward position. In closed-loop mode, the feedback control is smoothly turned on at $t = 1.1$ s and stabilizes the pendulum along the nominal trajectories. The correction Δu of the stabilizing feedback in (15) is less than $3\,\mathrm{m/s^2}$, which reveals the quality of the feedforward control $u^*(t)$ and the effectiveness of the parameter adjustment by solving the optimization problem (12)–(13).

The experimental swing up is easily repeatable without nameable performance loss after several successive swing-up and swing-down maneuvers (along the accordingly calculated swing-down trajectories). Although the double pendulum is a highly sensitive system, simulations and experiments have shown that the control scheme is robust enough to deal with minor deviations in the parameters and initial conditions.

5. Conclusions

The swing-up of the double pendulum on a cart is used to illustrate the inversion-based feedforward control design for finite-time transition problems of nonlinear systems recently proposed in Graichen et al. (2005). The resulting two-point BVP of the internal dynamics, i.e. the pendulum dynamics, requires free parameters for its solvability, which are provided in the setup of the output trajectory, i.e. the cart position. The

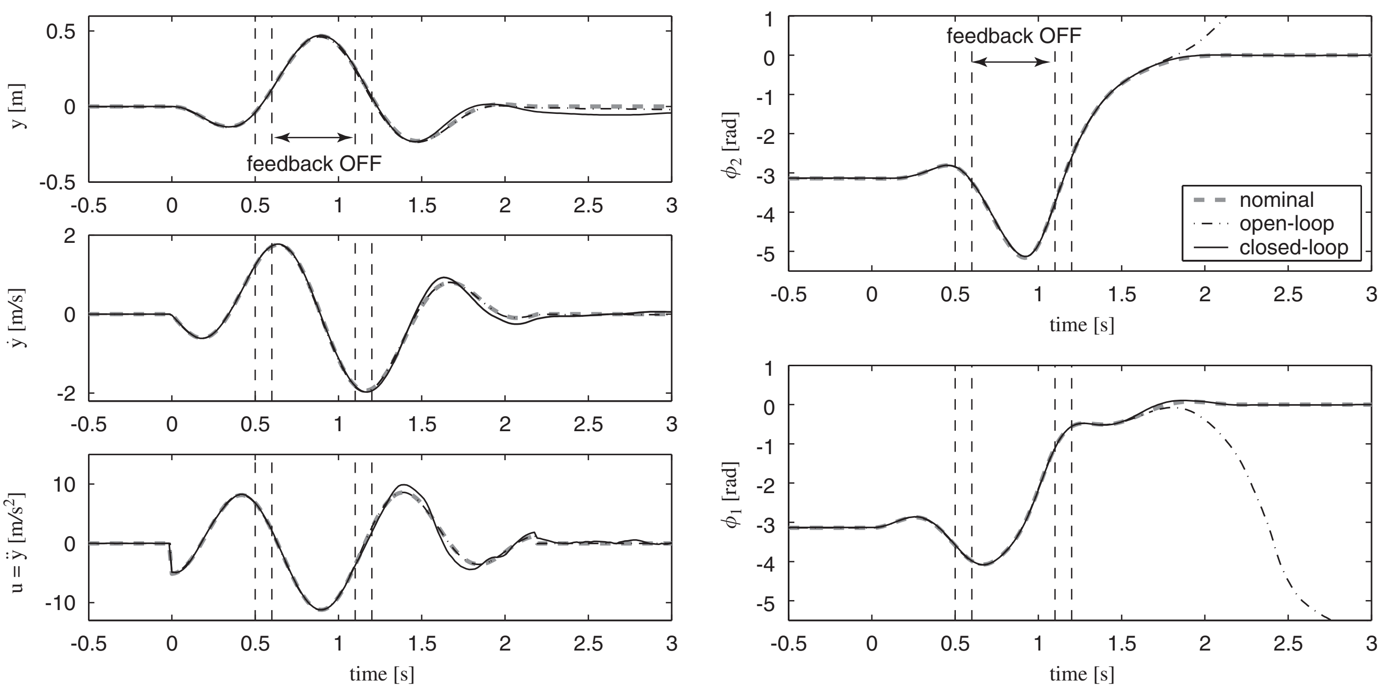


Fig. 8. Measured and nominal trajectories of the cart $y(t)$, $\dot{y}(t)$, $u=\ddot{y}(t)$ and the angles $\phi_1(t)$, $\phi_2(t)$ for the swing-up of the double pendulum.

two-point BVP with free parameters is solved numerically with the MATLAB function bvp4c. The mechanical parameters of the double pendulum are adjusted with respect to the measured open-loop trajectories of the swing up, in order to increase the accuracy of the feedforward trajectories. Experimental results of the swing-up maneuver reveal the high performance and accuracy of the tracking control with the nonlinear feedforward and linear feedback control. The applied feedforward control design also allows to account for input constraints (Graichen & Zeitz, 2005a), which is of importance e.g. for mechatronic systems with physical limitations of the actuators.

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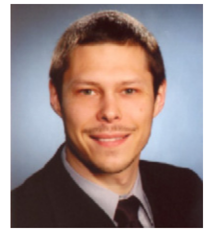
Knut Graichen (1977) received his Diploma degree in Engineering Cybernetics from Universität Stuttgart (Germany) in 2002. During the preparation of his Diploma thesis he was with Robert Bosch Corp., Research and Technology Center North America (RTC) in Palo Alto, CA (USA).
Since graduation he is a research assistant and Ph.D. student at Institut für Systemdynamik, Universität Stuttgart. His main research interests are in nonlinear feedforward/feedback control design and differential flatness with applications to mechatronic systems and chemical engineering.



Michael Treuer (1978) received his Master of Engineering from the University of Auckland, New Zealand in 2004 and his Diploma degree in Engineering Cybernetics from Universität Stuttgart (Germany) in 2005. For his Diploma thesis on constrained feedforward control design for multiple-link pendulums he received the Award of the University's Alumni Foundation ("Freunde der Universität Stuttgart"). Since graduation he is a research assistant and Ph.D. student at the Institute of Process Engineering and Power Plant Technology, Universität Stuttgart.
His main research interests are nonlinear modeling, simulation and control design with applications to power plants and power systems.

Michael Zeitz (1940) received his Diploma degree in Electrical Engineering from Technische Hochschule Darmstadt in 1967, and the Doctor and Habilitation degrees from Universität Stuttgart in 1973 and 1977. After holding a professorship for Simulation Engineering at Ruhr-Universität Bochum (Germany) from 1977 until 1979, he was Professor of Control Engineering at Institut für Systemdynamik, Universität Stuttgart, and is retired since 2005. He obtained the Research Prize of Baden-Württemberg in 1992 and the Honorary Doctorate of Technical University Donetsk (Ukraine) in 1999. His main research interests are in feedforward/feedback control and observer design for nonlinear and distributed parameter systems.