Swing-up Control of a Single Inverted Pendulum on a Cart With Input and Output Constraints

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Abstract: In this paper we propose a new swing-up strategy for a single inverted pendulum. The proposed method has a feature that can handle the limitation of the pendulum-rail length and actuator constraints using both feedforward and feedback control. The feedforward trajectories are generated by solving an optimal control problem having two-point boundary conditions. The limitation of the rail length and the actuator constraints are taken into account in the problem formulation. Feedback control is combined with the feedforward control to compensate the deviation between the desired trajectories and actual trajectories. The experimental results of the proposed strategy show that it has a good swing-up performance while satisfying all the imposed constraints.

1 INTRODUCTION

An inverted pendulum system is one of the most popular experimental apparatuses used for control education. The control challenge of an inverted pendulum comes from the fact that it is nonlinear, unstable, and underactuated. Besides being used in linear and nonlinear control education, inverted pendulum systems are used to verify a designed control system. There are two different types of inverted pendulum systems. One consists of two links, only one of which is actuated (Spong and Block, 1995). The other one is a pendulum on a cart which moves horizontally (Chung and Hauser, 1995). Similar to an inverted pendulum on a cart, a rotary inverted pendulum which is called Furuta pendulum has an advantage of an unbounded rail length (Furuta et al., 1992). Since swinging up a pendulum successfully in short time on a short rail under some constraints of an actuator are very challenging, various nonlinear swing-up control methods have been developed. One of the most popular control approaches applied to an inverted pendulum systems is so-called energy-based method which is first proposed by (Wilklund et al., 1993). Some papers adopted a Lypunov function approach (Åström and Furuta, 2000) and (Yang et al., 2009). In their approaches, the stability is guaranteed irrespectively of the choice of design parameters, but a maximum cart displacement can be taken into account only by manually adjusting design parameters. The disadvantages of their approaches are that the rail length limitation is not taken into account systematically. As a result, the cart may collide with the wall. Furthermore, the swing-up time is obtained only after experiments.

Recently, another approach using a combination of feedforward and feedback control to solve the swing-up of an inverted pendulum problem was proposed in (Rubi et al., 2002). (Graichen et al., 2007) implemented that approach on a double inverted pendulum on a cart. The necessary swing-up maneuver of the double pendulum on the cart is determined by solving a two-point boundary value problem (BVP) for the internal dynamics of the pendulum. The two-point BVP is solved by providing free parameters in the desired trajectory without considering the input-output constraints. The results in (Graichen and Zeitz, 2005b) have been further elaborated in (Graichen and Zeitz, 2005a) such that it can incorporate input constraints directly within a feedforward control design for a non-linear SISO system. Later on, (Graichen and Zeitz, 2008) proposed a feedforward control design for a finite-time transition problem of nonlinear system input-output constraints. To incorporate constraints on outputs and its time derivatives, the input-output dynamics are replaced by a new system, which is systematically constructed by means of saturation functions. Disadvantages of this approach are that the feedforward control input needs to be pre-assumed to be of a form and the swing-up time must be predefined. A side-stepping of a single inverted pendulum is used as an example to illustrate their approach. The input trajectory to the system is a polynomial function with free parameters which is just one of the possible combinations, and the final transition time is predefined.

In this paper we also use the combination of feedforward and feedback control approach, but our feedforward control optimally generates the input trajectory for swing-up maneuver without assuming the input trajectory form and predefining the swing-up time. Moreover, it can handle the input-output constraints of the system. A typical feedforward control input trajectory is the transition within a finite-time interval between two stationary setpoints, one is at the downright position and the other one is at the upright position, which makes the system satisfy the boundary conditions (BCs). Feedback controller is used to correct the error along the nominal trajectory.

This paper is organized as follows. In Section 2, a mathematical model is provided and a swing-up problem is described. In Section 3, we describe boundary conditions of the system, the cost function to be optimized, and the feedforward control design. The experiment results, parameter optimization, and feedback control design will be presented in Section 4. Finally in Section 5, we make conclusions.

2 PROBLEM STATEMENT

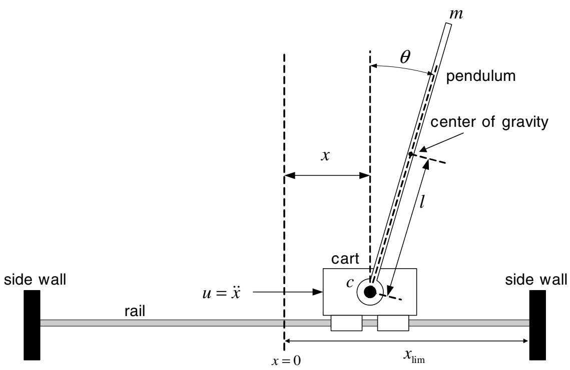


Figure 1: The conceptual diagram of a cart pendulum system.

Figure 1 shows the conceptual diagram of a cart pendulum system. A cart pendulum system consists of a pendulum and a cart that moves along the rail. The two main outputs are the angle of the pendulum and the position of the cart. The pendulum is swung by the cart and the cart is usually driven by an electric motor. Applying Lagrange's formulation, one can get the following differential equation which governs the movement of the pendulum system:

Table 1: Model parameters of the system.

Parameters	values
m	0.41 Kg
l	0.22 m
I	0.116 Nms^2
c	0.005 Nms^2

$$(I+ml^2)\ddot{\theta}+ml(\cos\theta)\ddot{x}=mgl(\sin\theta)-c\dot{\theta} \tag{1}$$

where I is the moment of inertia of a pendulum with respect to the center of gravity, l the length of the pendulum from the pivot to the center of gravity, m the mass of the pendulum, g the acceleration of gravity, c the rotational damping coefficient, θ the angular displacement of the pendulum, x the displacement of the cart from the center of the rail to any positions on the rail. Equation (1) shows that the motion of the pendulum is directly governed by the cart acceleration. If we assume that the acceleration $\ddot{x}$ of the cart serves as an input u to the system, then the overall model of the pendulum can be described as follows:

$$\begin{aligned} \ddot{x} &= u && (2)\\ \ddot{\theta} &= \frac{mgl(\sin\theta)-c\dot{\theta}}{I+ml^2}-\frac{ml(\cos\theta)}{I+ml^2}u && (3) \end{aligned}$$

It is noted that the system described by equation (2) and (3) is of 4th order. In this paper, we use a lab-built inverted pendulum system shown in Figure 2 for the verification of the proposed results. The model parameters of our lab-built system are given in Table 1. The cart movement is subject to the constraints due to the limited rail length and the physical limits of a DC motor used for actuation as follows:

$$|x|\le 0.3\text{m},\ |\dot{x}|\le 1.3\text{m/s},\ |\ddot{x}|\le 22\text{m/s}^2 \tag{4}$$

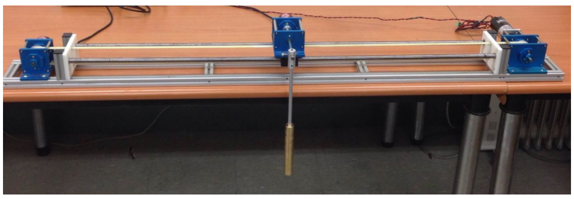


Figure 2: A lab-built cart inverted pendulum system used for experiments.

The purpose of this paper is to develop a swing-up control method for a cart inverted pendulum that satisfies the constraints given in equation (4) by extending the idea presented in (Graichen et al., 2007) through the optimal control.

The approach presented in (Graichen et al., 2007) for the swing-up of a double-pendulum within a finite time interval $t \in [0,T]$ generates feedforward trajectories for the cart acceleration to steer the pendulum from the initial downward equilibrium to the terminal upward equilibrium by solving a two-point BVP. Then the cart is controlled such that it follows the pre-computed feedforward trajectory and this will erect the double pendulum. For the robust control performance, the authors combine the feedback control together with feedforward control. However, this approach has several things to be improved. Firstly, it confines the feedforward trajectory to be of some pre-assumed form. Secondly, input and output constraints are not systematically taken into account. Finally, it does not use any optimality criterion for the generation of the feedforward trajectory.

In this paper, we borrow the idea of using feedforward control to swing up the pendulum from (Graichen et al., 2007). However, we extend the results through the optimal control so that the three limitations mentioned above can be removed.

3 OPTIMAL FEEDFORWARD CONTROL

The main purpose of the feedforward control is to generate the input control which makes the cart move such that the pendulum swings up from the downright to the upright position. It is noted that the cart tracking performance must be good so that the cart can follow the pre-computed trajectory well. The PD controller is used as a position controller in the experiment. Figure 3 shows the experimental results of the PD position controller for two sinusoidal references with different frequencies. It is clearly shown in (a) of Figure 3 that the cart velocity is limited to $1.55m/s$. When the feedforward control input acceleration requires the cart to move faster than $1.55m/s$, the cart cannot follow the trajectory well. If the required cart velocity is within constraint, a PD controller can make the cart follow the desired trajectory well as shown in (b) of Figure 3. The actual constraints of the system is shown in equation (5) but in the feedforward control input generation process, we set the values of the constraints smaller than the actual ones which is shown in equation (4) because we want less error in position control and to prevent a cart from colliding with the wall of the pendulum rail.

$$|x| \leq 0.4\text{m},\ |\dot{x}| \leq 1.55\text{m/s},\ |\ddot{x}| \leq 40\text{m/s}^2 \quad (5)$$

3.1 Feedforward Control Input Generation

The maneuver of the cart acceleration within a finite-time interval $t \in [0,T]$ is required to make the system meet the BCs (6) and (7). The BCs of the system are the conditions of the position and the velocity of the cart and the pendulum at downward and upward equilibrium. The displacement of the cart $x(t)$ and other states $\dot{x}(t)$, $\theta(t)$, and $\dot{\theta}(t)$ can be obtained by solving the differential equation (2) and (3).

$$x(0) = 0,\ \dot{x}(0) = 0,\ \theta(0) = -\pi,\ \dot{\theta}(0) = 0 \quad (6)$$

$$x(T) = 0,\ \dot{x}(T) = 0,\ \theta(T) = 0,\ \dot{\theta}(T) = 0 \quad (7)$$

$$u(0) = 0, \quad u(T) = 0 \quad (8)$$

We are interested in starting with zero acceleration and in forcing it to zero at the final transition time because, in our lab-built pendulum system, the cart driven by a DC motor is always accelerated from zero to any value within constraint, and it will help to reduce the bump of the error when we switch from nonlinear swing-up controller to linear controller at the final transition time. Suppose that the desired control input acceleration is

$$u^*(t) = \ddot{x}^*(t) \quad (9)$$

which makes the pendulum swing up successfully. The input acceleration $u^*(t)$ makes the cart move $x^*(t)$ within limited rail length with velocity $\dot{x}^*(t)$ which is within the physical constraint of the cart actuator. $\theta^*(t)$ and $\dot{\theta}^*(t)$ are the angle and the angular velocity of the pendulum respectively which result from the feedforward control input acceleration $u^*(t)$. Let's represent the dynamic equation (2) and (3) into a state space form as follows:

$$\begin{aligned}
\dot{\xi}_1 &= \xi_2 && (10)\\
\dot{\xi}_2 &= u && (11)\\
\dot{\xi}_3 &= \xi_4 && (12)\\
\dot{\xi}_4 &= \frac{mgl(\sin\xi_1) - c\xi_2}{I+ml^2} - \frac{ml(\cos\xi_1)}{I+ml^2}u && (13)
\end{aligned}$$

where

$$\xi_1 = x,\ \xi_2 = \dot{x},\ \xi_3 = \theta,\ \xi_4 = \dot{\theta}. \quad (14)$$

The strategy in generating the feedforward control input presented in (Graichen et al., 2007) is to construct the control input such that it makes the system meet the BCs by solving a two-point BVP without including the constraints of the system and optimality. In (Graichen and Zeitz, 2008), the designed feedforward

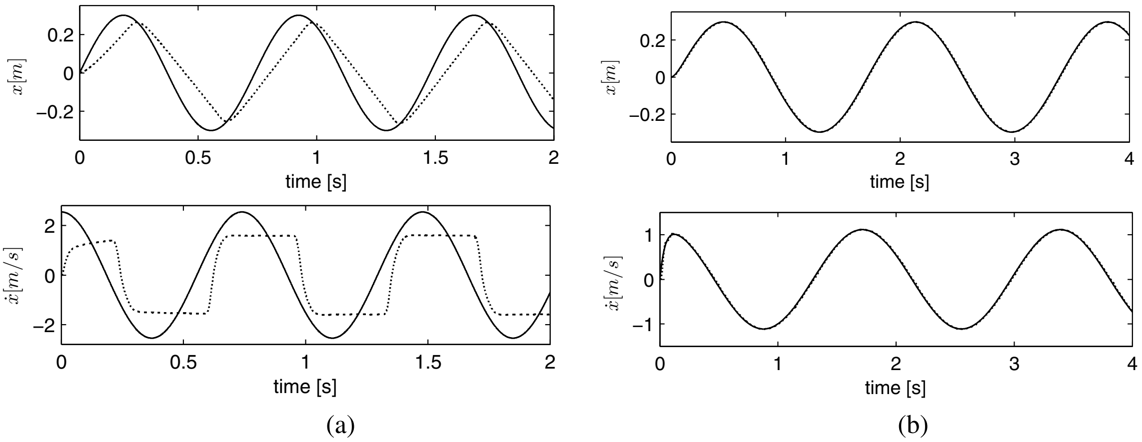


Figure 3: Tracking performances of a PD position controller for different references: reference(solid) and actual measurement(dotted).

control can deal with input-output constraints by incorporating the constraints with pre-assumed input-output form. The pre-assumed form is provided with n free parameters if the systems has $2n$ BCs, and the transition time is predefined. Due to these disadvantages, the approach cannot generate a feedforward control trajectory in a flexible form. Furthermore, it does not use any optimality criterion. In this paper, the feedforward control input is generated by solving the following optimal control problem:

Minimize	$J(\xi, u, t_f)$
subject to	constraint (4),
	dynamic equations $(10),(11),(12),(13)$,
	and boundary conditions $(6),(7),(8)$.

In this paper, we choose to optimize both energy consumption and the cart displacement by taking the cost function as follows:

$$J = \int_0^T (a\xi_1^2 + bu^2)dt \tag{15}$$

where $a+b=1$, $a,b \in R^+$. The parameters a and b are used to weigh the importance in the optimization. If a is bigger than b, the optimization will focus on the cart displacement more than the cart acceleration and vice versa.

We solve the above nonlinear optimal control problem with boundary conditions using a newly developed solver known as GPOPS-II (Patterson and Rao, 2014).

Remark 1. *We can minimize the transition time by taking the cost function as follows:*

$$J = \int_0^T 1dt.$$

3.2 Feedforward Control Input Trajectories

GPOPS-II solver yields a large variation in the resulting control input trajectories depending on the cost function to be minimized and the value of constraint in (4). A swing-up of an inverted pendulum in short time with the short displacement of the cart are very challenging. Therefore, there are many possible cost functions which may be used in the optimization for various purposes. Figure 4 shows the feedforward control and state trajectories for different cost functions. The swing-up time is reduced to $1.3s$ when the swing-up time is the objective in the optimization (dash-dotted lines). The swing-up time is $T = 1.4s$ when we optimize the cart displacement (dotted lines). It shows that when we want the swing-up time to be as short as possible, the cart velocity goes up to the boundary of constraint, and the cart displacement is longer than that of other cost functions. Moreover, the control input acceleration is much faster than other control input acceleration. When the cart displacement is optimized, the feedforward control generated the shortest cart displacement trajectory. Using a cost function (15) yields a different input acceleration and a swing-up motion (solid lines). Remarkably, the swing-up time was not predefined in the generation of feedforward trajectories, and all the output states remain within the constraints of the system.

To clearly show the difference from the approach presented in (Graichen et al., 2007), we generated feedforward trajectories using the method given therein. Figure 5 shows the obtained feedforward trajectories for the displacement and the velocity of the cart. Trajectories were generated for the three

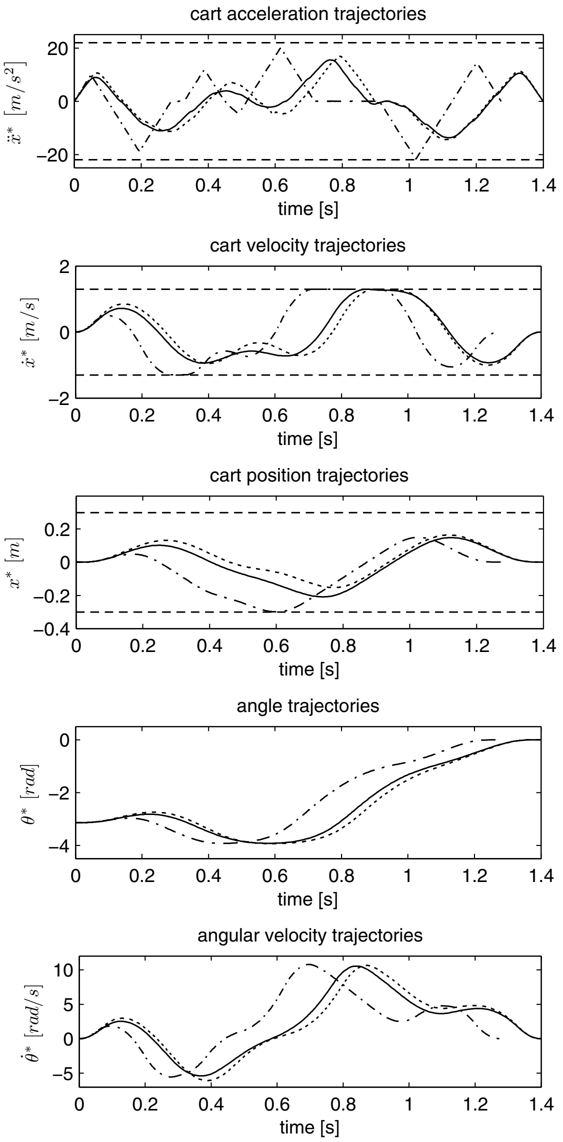


Figure 4: The possible feedforward trajectories for swing-up maneuver of the inverted pendulum system for different cost functions.

swing-up times, which are $T = 1.2$(dash-dot), $T = 1.3$(solid), and $T = 1.4$(dotted). It is shown that the trajectories generated for the swing-up time $T = 1.3$ and $T = 1.4$ satisfy neither the displacement constraint nor the velocity constraint. Trajectories for $T = 1.2$ satisfy the displacement constraint. However, they go beyond the velocity constraint. Violation of the cart displacement constraint will result in the cart's collision with the wall during swing-up. Furthermore, dissatisfaction of the velocity constraint means that the swing-up control is not possible using the given actuator, which also means that we have to replace the actuator with better ones for the swing-up.

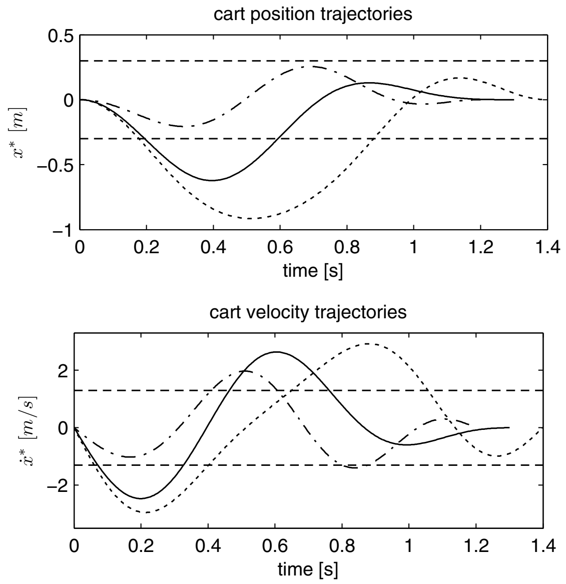


Figure 5: Feedforward trajectories generated by the method of (Graichen et al., 2007) for different values of T: $T = 1.2$(dash-dot), $T = 1.3$(solid), $T = 1.4$(dotted).

Since the trajectories generated by the proposed approach consider the given constraints systematically, we can make the most of the performance of the actuator and protect the system from collision. This will give much better freedom in choosing the actuator for the system.

4 EXPERIMENTAL RESULTS

Swing-up maneuver is experimentally realized with a lab-built pendulum shown in Figure 2. Two incremental encoders with the resolution of 5000 pulses/rev are used, one for measuring the displacement of the cart and the other one for measuring the angular displacement of pendulum. All the measurement information is transmitted to a controller board with a sample time of 1 ms. The nominal trajectories $\theta^*(t)$, $\dot{\theta}^*(t)$, $x^*(t)$, and feedforward control acceleration $u^*(t) = \ddot{x}^*$, $t \in [0, T]$ are stored in lookup tables. We use a PD position controller to generate the required acceleration $u^*(t)$ by tracking the position reference $x^*(t)$, which is obtained by double integration of $u^*(t)$. Coefficients for the PD controller are chosen to be $K_p = 700$ and $K_d = 20$.

4.1 Model Parameter Estimation

Using only nonlinear feedforward control to swing up the pendulum is hard to succeed because the uncertainty of the model parameters yields the infeasi-

Table 2: Estimated model parameters of the system.

Parameters	values
m	0.4 Kg
l	0.24 m
I	0.016 Nms^2
c	0.005 Nms^2

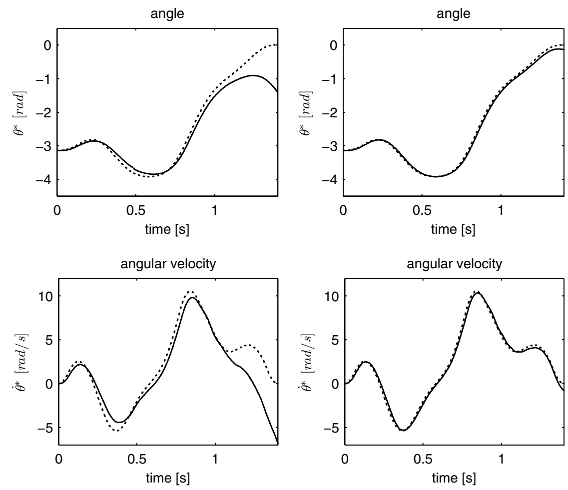
ble feedforward trajectories. The external disturbance and the poor cart tracking performance are also the factors which make the control unsuccessful in real experiments. If the feedforward has a big error, the feedback controller cannot correct the error properly. Therefore the accurate feedforward control is necessary. (a) of Figure 6 shows the experimental results of the angle and the angular velocity of the pendulum in open loop corresponding to the model parameters given in Table 1. The simulation and experiment result are much different. This is because model parameter values used in the simulation and the actual ones of the real system are different. To enhance the accuracy of the model parameter values, the optimization-based adjustment is needed to find a set of correct values of model parameters by minimizing the following cost function:

$$J = \int_0^T (\theta^*(t) - \theta(t))^2 + (\dot{\theta}^*(t) - \dot{\theta}(t))^2 dt \qquad (16)$$

where $\theta^*(t)$ and $\dot{\theta}^*(t)$ are simulation trajectories which are generated by using the model parameter values in Table 1. $\theta(t)$ and $\dot{\theta}(t)$ are the actual measured angle and angular velocity respectively in the open loop experiment. We use 'fmincon', the optimization function provided in Matlab toolbox to solve the optimization problem. We regenerated feedforward trajectories after getting a new set of the model parameters. It is noted that the new set of model parameters are not the actual values of the real system. Instead, it is just a possible combination that verifies the dynamic equation (1). The new set of model parameters are shown in Table 2. (b) of Figure 6 is the experimental results of new model parameters.

4.2 Linear Feedback Controller

The nonlinearity and instability in dynamics make the pendulum sensitive to the perturbation which is caused by the external disturbance and the delay in the system. As a result, the pendulum may not follow the nominal trajectory. The realization of the swing-up of an inverted pendulum requires a feedback controller to correct the derivation between the actual states $(x, \dot{x}, \theta, \dot{\theta})$ and desired states $(x^*, \dot{x}^*, \theta^*, \dot{\theta}^*)$. Figure 7 is a two-degree-of-freedom control scheme



(a) (b)

Figure 6: The open-loop experimental results for the angle and the angular velocity using (a) default parameters and (b) the estimated parameters.

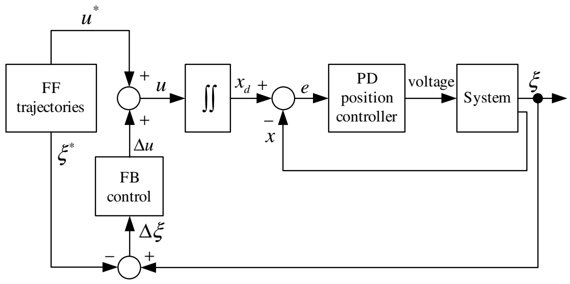


Figure 7: The 2-DOF control scheme for the swing-up.

of feedforward and feedback control. In order to compensate the possible steady state error in the cart position, the inverted pendulum model is dynamically extended by the disturbance model $\dot{\tilde{x}} = x$ as in (Graichen et al., 2007). The new desired system becomes

$$\dot{\xi}^* = f(\xi^*, u^*) \qquad (17)$$

where

$$\xi^* = [x^*,\ \dot{x}^*,\ \theta^*,\ \dot{\theta}^*,\ \tilde{x}^*] \qquad (18)$$

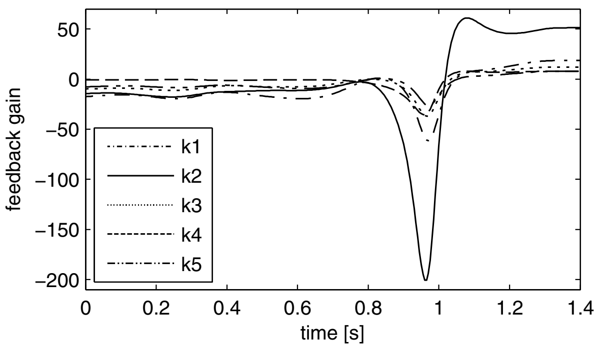


Figure 8: Time-varying LQ feedback gain.

and

$$\tilde{x}^*(t) = \int_0^t x^*(\tau)d\tau. \qquad (19)$$

We design feedback control for nonlinear system based on linear system theory as in (Graichen et al., 2007). This leads to linear time-varying system and the system is linearized as follow:

$$\Delta\dot{\xi} = A(t)\Delta\xi + B(t)\Delta u \qquad (20)$$

where

$$A(t) = \frac{\partial f}{\partial \xi}\bigg|_{\xi^*(t),u^*(t)}, \qquad (21)$$

$$B(t) = \frac{\partial f}{\partial u}\bigg|_{\xi^*(t),u^*(t)} \qquad (22)$$

where $k(t)$ is the time varying feedback gain, ξ^* is the desired state trajectories, and u^* is the desired control trajectory. Time-varying feedback gain $k(t)$ can be obtained by solving an optimal LQ (linear quadratic) control which minimize the objective function as follows:

$$J = \int_0^T (\Delta\xi^T Q \Delta\xi + \Delta u^T R \Delta u)dt$$

$$+\Delta\xi^T(T)S\Delta\xi(T) \qquad (23)$$

where Q and $S \in R^{5\times5}$ are the symmetric positive semi-definite matrices and R is a positive scalar. Feedback gain $k(t)$, $t \in [0,T]$ is determined by

$$k(t) = R^{-1}B^T(t)P(t), \qquad (24)$$

where $P(t)$ is the solution on $[0,T]$ of the matrix Riccati differential equation (RDE) as follows:

$$\dot{P} = PB(t)R^{-1}B^T(t)P - PA(t) - A^T(t)P - Q. \quad (25)$$

It is noted that $P(T) = S$, where S is determined by solving the algebraic Riccati equation (25) with $\dot{P} = 0$. The weighting matrix Q is chosen to be the diagonal matrix $(800, 3000, 0, 0, 100)$ and the $R = 50$.

Figure 8 shows the time-varying feedback gain $k_i(t)$, $i = 1,\cdots,5$ in the time interval $t \in [0,T]$. In the time interval $t \in [0.8, 1]$ the feedback gain has a big oscillation because in that time interval, the pendulum is about to lie down on the horizontal axis and its controllability is very weak and lose its controllability when the pendulum is at the 90 degree position.

4.3 Experimental Results of Combined Control

We performed two control experiments. One used only generated control input trajectory, which is actually called open-loop control. The other one used the combination of feedforward and feedback control as shown in the control scheme in Figure 7. Figure 9 shows the experimental results of the cart displacement $x(t)$ and angular displacement $\theta(t)$. The open-loop control gives a good performance only at the beginning. The pendulum tracked smoothly along the nominal trajectory almost up to the upright position. When we switch from the swing-up control to the stabilizing linear controller at the upright position, the stabilizing linear controller could handle the small error and steer the pendulum to the upright position. However, relatively big bump is encountered after switching. The dash-dot line in the Figure 9 denotes the open-loop experiment results. The dotted line represents a tracking performance of pendulum when we performed the closed-loop control. It shows the effectiveness of feedback controller which compensates the deviation between the actual states and the desired states. As a result, the pendulum tracked the nominal trajectory well up to the upright position and then smoothly switched from the nonlinear controller to the linear controller.

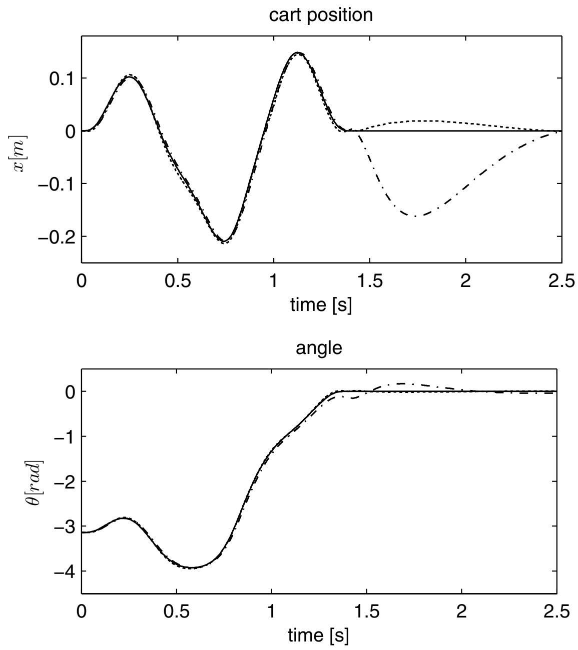


Figure 9: Comparison of the experimental results: open loop trajectory(dash-dot), closed loop trajectory(dotted), desired trajectory(solid).

5 CONCLUSIONS

The presented approach in this paper extends the previous results such that it can handle the rail length limitation and actuator constraints systematically. In this approach, the swing-up maneuver of an inverted pendulum from downward equilibrium to upward equilibrium is accomplished within a two-degrees of freedom control scheme consisting of nonlinear optimal feedforward controller and the optimal feedback controller. The feedforward control input trajectory is generated by the newly developed optimal control solver that can handle the input and output constraints of the system. Simulation and experimental results showed close resemblance, which shows that the proposed method is quite practical. The swing-up of the inverted pendulum through the proposed method turned out to be always successful. The proposed approach enables one to make the most of performance of the given actuator. The presented approach can be extended to the swing-up control of a double or triple inverted pendulum without much of modification.

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