5

THE CONCEPT OF "PARTIAL" INDUCTANCE

In the preceding chapters we discussed the meaning and calculation of the "loop" inductance of various conducting structures that support a closed loop of current. This "loop" inductance is calculated fundamentally for steady (dc) currents which we showed in Section 2.9 must form closed loops. If we open the loop at a point with a small gap, the loop inductance of that current loop is seen as an inductance L at these input terminals. When we pass a time-varying current around the loop via these terminals a voltage, $V(t) = LdI(t)/dt$, is developed across the terminals. This voltage is essentially the Faraday's law voltage induced into the loop. For electrically small loop dimensions, this lumped inductance and the voltage across its terminals can be represented as a lumped voltage source and placed *anywhere* in the loop perimeter (see Fig. 4.1). It is important, however, to remember that *neither this lumped inductance nor the equivalent voltage source it represents can be placed in a unique position in the loop*! This loop inductance is a property of the entire loop and its use is valid only at the input terminals of the loop. Hence, it is not possible to associate the loop inductance with any particular segment of the loop.

However, there are numerous situations, some of which were described in Chapter 1, where it is useful to develop a lumped-circuit model of a closed current loop wherein the segments of the perimeter of the loop are represented

---

Inductance: Loop and Partial, By Clayton R. Paul

with a self inductance as well as mutual inductances between that segment and other segments of this and other *adjacent current loops*. The concept of "partial" inductance allows us to do that in a unique way.

It has been said that "you cannot ascribe the properties of inductance to an isolated piece of wire." Of course you can't because an isolated piece of wire is not capable of supporting a dc current, which must form a closed loop (i.e., it must return to its source). This is therefore a misleading statement. The proper question is: Can you ascribe the properties of inductance *uniquely* to a segment of a *closed loop of current*? The answer to this question is yes, and the method for doing so is with "partial" inductances.

There are three significant references regarding partial inductance. Those by Grover [14] and Ruehli [15] are excellent general references, and the paper by Hoer and Love [16] gives results for the partial inductances of conductors of rectangular cross section [e.g., printed circuit board (PCB) lands].

5.1 GENERAL MEANING OF PARTIAL INDUCTANCE

Consider a closed physical loop constructed of a conductor such as a wire, PCB land, and so on, that supports a dc current I. The "loop" inductance of this current loop is defined fundamentally in previous chapters as

$$L = \frac{\psi}{I} \tag{5.1a}$$

where

$$\psi = \int_s \mathbf{B} \cdot d\mathbf{s} \tag{5.1b}$$

is the total magnetic flux that penetrates the open surface s that is surrounded by the closed contour of the loop, c, and $\mathbf{B}$ is the magnetic flux density (caused by current I) through the surface s. In Chapter 2 we calculated $\mathbf{B}$ for various configurations of loop shapes. In Chapter 4 we calculated the flux ψ and hence the inductance according to (5.1) for various loop shapes. Faraday's fundamental law of induction (Chapter 3) gives the induced voltage appearing at the terminals of the loop as

$$\begin{aligned} V &= \frac{d\psi}{dt} \\ &= L\frac{dI}{dt} \end{aligned} \tag{5.2}$$

where the current I is now allowed to be "slowly varying with time," as demonstrated in Section 3.4. Essentially, the condition "slowly varying with time" is satisfied approximately as long as the physical dimensions of the loop are much less than a wavelength (e.g., $< \lambda/10$, where the wavelength is $\lambda = v/f$,

f is the highest significant frequency in the waveform of the current I, and v is the velocity of propagation of the current.

In Chapter 4 we developed an alternative means of calculating the inductance by using the vector magnetic potential **A**, which is defined by

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{5.3}$$

Hence, the total magnetic flux through the surface s is

$$\begin{aligned}\psi &= \int_s \mathbf{B} \cdot d\mathbf{s} \\ &= \int_s (\nabla \times \mathbf{A}) \cdot d\mathbf{s} \\ &= \oint_c \mathbf{A} \cdot d\mathbf{l}\end{aligned} \tag{5.4}$$

where we have used Stokes's theorem (see the Appendix) to convert the surface integral over surface s to a line integral around contour c that encloses the surface. This gives an alternative way of calculating the flux through the loop, ψ, in terms of **A**. Hence, an alternative way of calculating the inductance of the current loop is

$$L = \frac{\oint_c \mathbf{A} \cdot d\mathbf{l}}{I} \tag{5.5}$$

where c is the closed contour that bounds the open surface s. Hence, we can compute the inductance of a loop by integrating, with a line integral, the product of the differential path lengths around the contour c that surrounds the open surface s and the components of the vector magnetic potential **A** that are tangent to that closed path. But (5.5) can be decomposed into the line integral *along unique segments of the closed loop* as

$$\boxed{\begin{aligned}L &= \frac{\oint_c \mathbf{A} \cdot d\mathbf{l}}{I} \\ &= \frac{\int_{c_1} \mathbf{A}_1 \cdot d\mathbf{l}}{I} + \frac{\int_{c_2} \mathbf{A}_2 \cdot d\mathbf{l}}{I} + \cdots + \frac{\int_{c_n} \mathbf{A}_n \cdot d\mathbf{l}}{I}\end{aligned}} \tag{5.6}$$

where the closed path c is segmented into *n contiguous segments* c_i so that $c = c_1 + c_2 + \cdots + c_n$ and $\mathbf{A}_i$ is the *total* **A** along contour c_i that is due to the current of that segment as well as the currents of the other segments of c or of some other current loop. This allows us to *uniquely associate an inductance contribution to each segment of the closed loop as*

$$\boxed{L_i = \frac{\int_{c_i} \mathbf{A}_i \cdot d\mathbf{l}}{I}} \tag{5.7a}$$

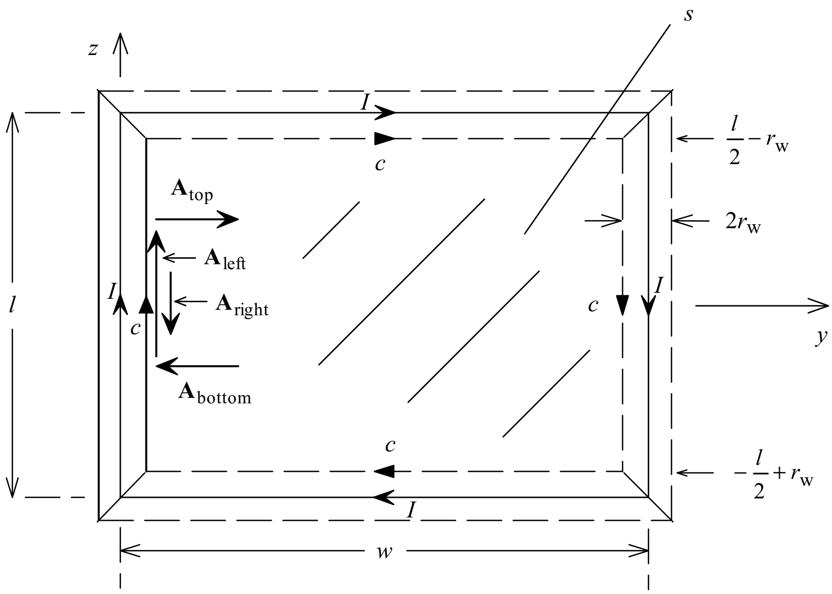


FIGURE 5.1. Rectangular loop.

so that the *total* loop inductance is the sum of these parts:

$$\boxed{L = L_1 + L_2 + \cdots + L_n} \tag{5.7b}$$

For example, in Fig. 5.1 we have shown Fig. 4.10, where in Section 4.3.1 we detailed the calculation of the inductance of a rectangular loop using the vector magnetic potential according to (5.5) Essentially, we are indirectly computing the total magnetic flux threading the loop, which is the region surrounded by the interior surfaces of the wires whose radii are r_w. This contour surrounding the open surface s is denoted as contour c in Fig. 5.1. As discussed in Sections 4.5 and 4.6, two important assumptions in computing the **B** field (and the subsequent calculation of the **A** field) are that (1) the current I is *distributed uniformly over the wire cross section so that the current I can be represented as a filament on the wire axis* (as it is for dc currents), and (2) there are no other currents in close enough proximity to this wire to upset this uniform current distribution over its cross section (i.e., the “proximity effect” is not pronounced). The total vector magnetic potential along the left side of the loop, $\mathbf{A}_1$, is the sum of the vector magnetic potentials along that side that are due to the current of that side, $\mathbf{A}_{\text{left}}$, and those that are due to the currents of the other three sides of the loop, $\mathbf{A}_{\text{right}}$, $\mathbf{A}_{\text{top}}$, and $\mathbf{A}_{\text{bottom}}$:

$$\mathbf{A}_1 = \mathbf{A}_{\text{left}} + \mathbf{A}_{\text{right}} + \mathbf{A}_{\text{top}} + \mathbf{A}_{\text{bottom}} \tag{5.8a}$$

Hence, the portion of the loop inductance *uniquely attributable to the left side* is

$$
\begin{aligned}
L_1 &= \frac{\int_{\substack{\text{left}\\\text{side}}} \mathbf{A}_1 \cdot d\mathbf{l}}{I} \\
&= \frac{\int_{\substack{\text{left}\\\text{side}}} \mathbf{A}_{\text{left}} \cdot d\mathbf{l}}{I} + \frac{\int_{\substack{\text{left}\\\text{side}}} \mathbf{A}_{\text{right}} \cdot d\mathbf{l}}{I} + \frac{\int_{\substack{\text{left}\\\text{side}}} \mathbf{A}_{\text{top}} \cdot d\mathbf{l}}{I} \\
&\quad + \frac{\int_{\substack{\text{left}\\\text{side}}} \mathbf{A}_{\text{bottom}} \cdot d\mathbf{l}}{I}
\end{aligned}
\tag{5.8b}
$$

Observe that $\mathbf{A}_{\text{top}}$ and $\mathbf{A}_{\text{bottom}}$ are in the directions of the currents of those sides and hence are both orthogonal to the left side and do not contribute to the line integral for L_1 along the left side. In a similar fashion we obtain the inductances attributable to the other three sides, L_2, L_3, and L_4. Hence, the rectangular loop can be represented *uniquely* by the lumped equivalent circuit shown in Fig. 5.2.

Observe that the *total* vector magnetic potential *along the left side*, $\mathbf{A}_1$, in (5.8a) has contributions due to its own current as well as the currents of the other three sides. So this leads us to break the inductance of the left side, L_1, into four distinct pieces according to (5.8b):

$$L_1 = L_{p1} + M_{p12} + M_{p13} + M_{p14} \tag{5.9}$$

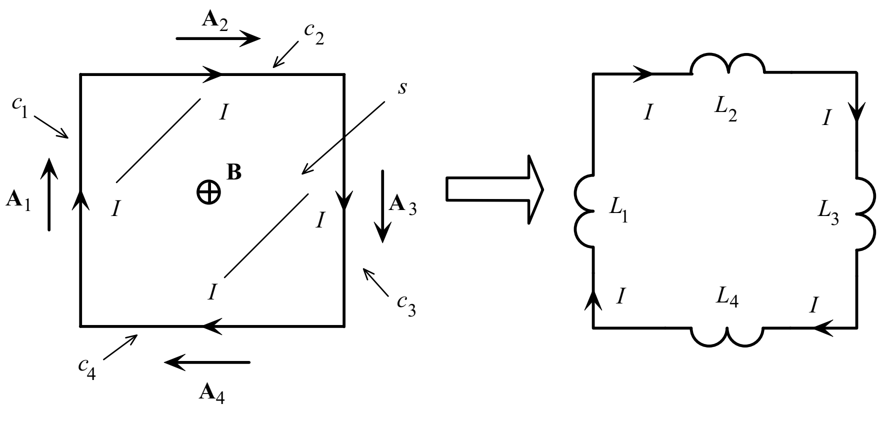


FIGURE 5.2. Uniquely attributing inductances to the sides of the rectangular loop of Fig. 5.1.

The first contribution is the *self partial inductance* of the left side:

$$L_{p1} = \frac{\int_{\substack{\text{left}\\\text{side}}} \mathbf{A}_{\text{left}} \cdot d\mathbf{l}}{I} \tag{5.10a}$$

which is due to the current of the left side. The other three contributions are due to the currents of the other three sides and are referred to as the *mutual partial inductances* between the other three sides and the left side:

$$M_{p12} = \frac{\int_{\substack{\text{left}\\\text{side}}} \mathbf{A}_{\text{top}} \cdot d\mathbf{l}}{I} \tag{5.10b}$$

$$M_{p13} = \frac{\int_{\substack{\text{left}\\\text{side}}} \mathbf{A}_{\text{right}} \cdot d\mathbf{l}}{I} \tag{5.10c}$$

$$M_{p14} = \frac{\int_{\substack{\text{left}\\\text{side}}} \mathbf{A}_{\text{bottom}} \cdot d\mathbf{l}}{I} \tag{5.10d}$$

Hence, the more complete equivalent circuit of the rectangular loop in terms of the *partial inductances* is shown in Fig. 5.3.

According to the dot convention described in Section 4.8.1, the total voltage across the left conductor is

$$\begin{aligned} V_1 &= L_{p1}\frac{dI}{dt} + M_{p12}\frac{dI}{dt} + M_{p13}\frac{dI}{dt} + M_{p14}\frac{dI}{dt} \\ &= \underbrace{\left(L_{p1} + M_{p12} + M_{p13} + M_{p14}\right)}_{L_1}\frac{dI}{dt} \end{aligned} \tag{5.11}$$

Observe that because sides 2 and 4 are orthogonal to side 1, $\mathbf{A}_{\text{top}}$ and $\mathbf{A}_{\text{bottom}}$ are orthogonal to the left side, so that $M_{p12} = M_{p14} = 0$. Also, because the direction of $\mathbf{A}_{\text{right}}$ is opposite the direction of the contour c along the left side,

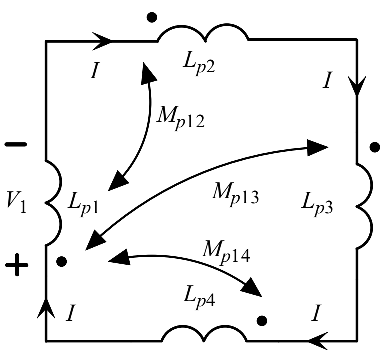


FIGURE 5.3. Rectangular loop equivalent circuit in terms of the partial inductances.

M_{p13} in (5.10c) is *negative*. The effective inductance of the left side of the loop, L_1, in (5.9) is referred to as the *net partial inductance* but has little value or use. Separating the net partial inductance into its constituent parts as in (5.9) gives more information about the contributions of all the other side currents.

5.2 PHYSICAL MEANING OF PARTIAL INDUCTANCE

The self partial inductance of the *i*th segment of a current loop is

$$\boxed{L_{pi} = \frac{\int_{c_i} \mathbf{A}_i \cdot d\mathbf{l}}{I_i}} \tag{5.12a}$$

and $\mathbf{A}_i$ is the portion of $\mathbf{A}$ along c_i that is produced by the current I_i of that segment. The voltage developed across that self partial inductance is

$$V_i = L_{pi}\frac{dI_i}{dt} \tag{5.12b}$$

as shown in Fig. 5.4.

Although (5.12a) gives the mathematical definition of self partial inductance, we now investigate the physical meaning of self partial inductance. Consider a segment c_i of a current loop carrying current I_i as shown in Fig. 5.5(a). Draw a surface extending from the segment to infinity with sides that are perpendicular to the current segment. Now determine the magnetic flux through that surface:

$$\begin{aligned}
\frac{\psi_\infty}{I_i} &= \frac{\int_s \mathbf{B}\cdot d\mathbf{s}}{I_i} = \frac{\oint_c \mathbf{A}\cdot d\mathbf{l}}{I_i} \\
&= \underbrace{\frac{\int_{c_i} \mathbf{A}_i \cdot d\mathbf{l}}{I_i}}_{c_i} + \underbrace{\frac{\int_c \mathbf{A}\cdot d\mathbf{l}}{I_i}}_{\text{left side}} + \underbrace{\frac{\int_c \mathbf{A}\cdot d\mathbf{l}}{I_i}}_{\text{right side}} + \underbrace{\frac{\int_c \mathbf{A}\cdot d\mathbf{l}}{I_i}}_{\infty} \\
&= \underbrace{\frac{\int_{c_i} \mathbf{A}_i \cdot d\mathbf{l}}{I_i}}_{c_i} \\
&= L_{pi}
\end{aligned} \tag{5.13}$$

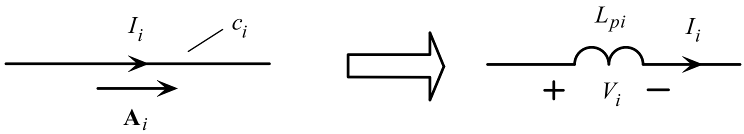


FIGURE 5.4. Self partial inductance of the *i*th segment of a current loop.

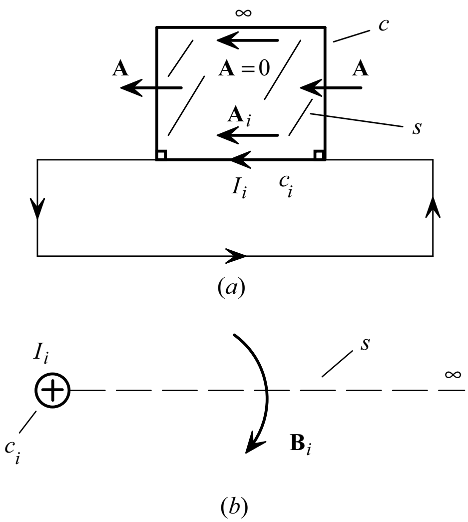


FIGURE 5.5. Physical meaning of self partial inductance.

The line integrals along the left and right sides are zero since the vector magnetic potential **A** is parallel to the current I_i that produces it and is therefore perpendicular to the left and right sides of the closed contour. The vector magnetic potential from a line current goes to zero at infinity [see (2.57)], so that the line integral along this portion of the closed contour at infinity is also zero. Hence, we are left with the partial inductance given in (5.12a) and the observation that:

> *The self partial inductance of a segment of a current loop is the ratio of the magnetic flux between the current segment and infinity and the current of that segment.*

This is illustrated in cross section in Fig. 5.5(b).

The mutual partial inductance between two segments c_i and c_j (which may be parts of the same current loop or different current loops) is defined by

$$\boxed{M_{pij} = \frac{\int_{c_i} \mathbf{A}_{ij} \cdot d\mathbf{l}}{I_j}} \tag{5.14a}$$

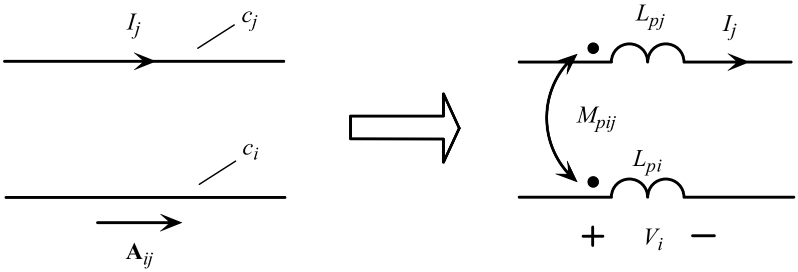


FIGURE 5.6. Mutual partial inductance between two current loop segments c_i and c_j.

where $\mathbf{A}_{ij}$ is along contour c_i and is due to the current of another segment, I_j. The voltage developed across that self partial inductance is

$$V_i = M_{pij}\,\frac{dI_j}{dt} \tag{5.14b}$$

as shown in Fig. 5.6.

The physical meaning of mutual partial inductance is illustrated in Fig. 5.7. Consider a current loop and two segments of that loop, c_i and c_j, as shown in Fig. 5.7(a). Again draw a surface s extending from the jth segment (carrying the current) to infinity with sides that are perpendicular to that current

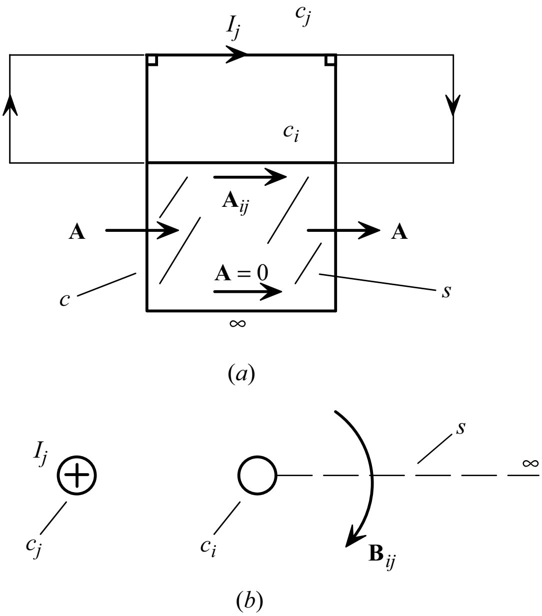


FIGURE 5.7. Physical meaning of mutual partial inductance.

segment. Now determine the magnetic flux through the surface s between the ith segment and infinity. Carrying through a development similar to that in (5.13) we see that the line integrals along the left and right sides are zero since the vector magnetic potential $\mathbf{A}$ is parallel to the current I_j that produces it and is therefore perpendicular to the left and right sides of the contour c that surrounds surface s. Also, the vector magnetic potential from a line current goes to zero at infinity so that the line integral along the portion of the contour at infinity is also zero. Hence, we are left with the mutual partial inductance given in (5.14a) and the observation that

> *The mutual partial inductance between two segments of the same or different current loops is the ratio of the magnetic flux (produced by the current of the first segment) that penetrates the surface between the second segment and infinity and the current of the first segment.*

This is illustrated in cross section in Fig. 5.7(b).

Although Fig. 5.7 shows the result for the mutual partial inductance of two *parallel* conductors, the result also obtains for two conductors at any angle to each other as shown in Fig. 5.8. Again draw two lines to infinity

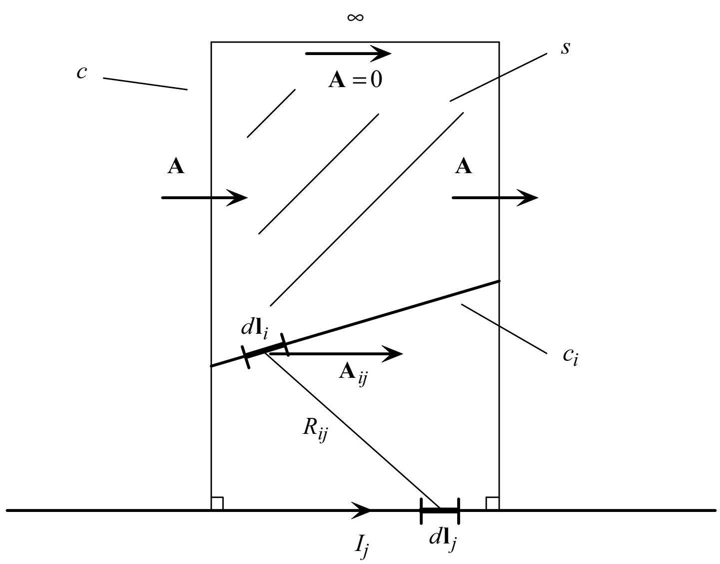


FIGURE 5.8. Mutual partial inductance for conductors at any angle to each other.

that are perpendicular (shown as small rectangles) to current I_j and which enclose the open surface s that lies between those parallel lines and between the skewed conductor and infinity. Integrating the line integral of the vector magnetic potential around the closed contour c surrounding this surface to infinity again gives

$$\begin{aligned} M_{pij} &= \frac{\psi_\infty}{I_j} \\ &= \frac{\oint_c \mathbf{A} \cdot d\mathbf{l}}{I_j} \\ &= \frac{\int_{c_i} \mathbf{A}_{ij} \cdot d\mathbf{l}}{I_j} \end{aligned} \tag{5.15}$$

This is obtained again since $\mathbf{A}$ is parallel to I_j at all points in space, so that $\mathbf{A}$ is perpendicular to the left and right sides of s and contribute nothing to the line integral along those sides, and $\mathbf{A}$ goes to zero at infinity. Observe that *the same result is obtained even if the two conductors do not lie in the same plane*, since $\mathbf{A}$ will still be orthogonal to the two sides of the open surface because they were constructed perpendicular to the current I_j and will also go to zero at infinity. (Again draw two lines for the sides of s that are perpendicular to conductor c_j.)

The mutual partial inductance can also be obtained from the Neumann integral by substituting the explicit equation for $\mathbf{A}_{ij}$ into (5.15):

$$\boxed{M_{pij} = \frac{\mu_0}{4\pi} \int_{c_i} \int_{c_j} \frac{1}{R_{ij}} \, d\mathbf{l}_i \cdot d\mathbf{l}_j} \tag{5.16}$$

where c_j is the contour along the conductor carrying current I_j, and R_{ij} is the distance between differential segments $d\mathbf{l}_i$ along contour c_i and $d\mathbf{l}_j$ along contour c_j, as shown in Fig. 5.8.

5.3 SELF PARTIAL INDUCTANCE OF WIRES

In this section we derive some fundamental results for the self partial inductance of wires having radii r_w. Again we assume that the current of the wire, I, is distributed uniformly over the wire cross section so that for the purpose of computing the $\mathbf{B}$ and $\mathbf{A}$ fields, we can concentrate the current I as a filament on the axis of the wire.

The fundamental problem for computing the self partial inductance of a wire is a wire of length l carrying a current I as shown in Fig. 5.9. We determine the self partial inductance of this segment of wire by integrating the magnetic flux density through the surface s between the wire surface, $y = r_\text{w}$,

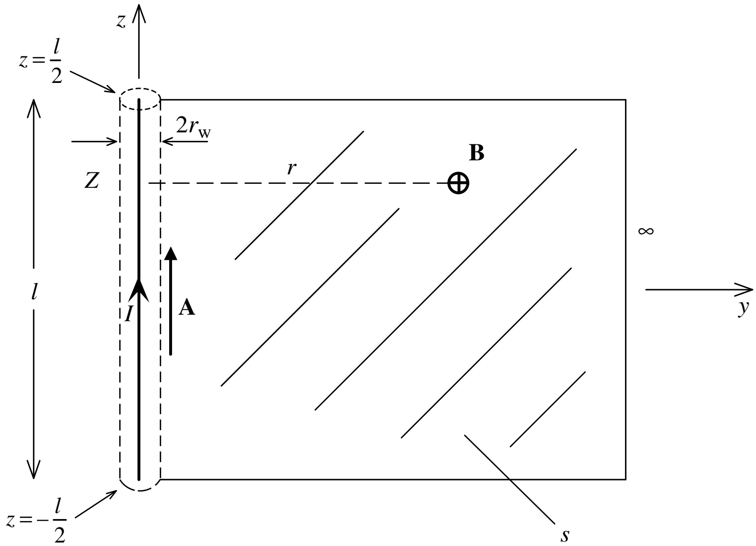


FIGURE 5.9. Determination of the self partial inductance of a wire.

and infinity, $y \to \infty$. The magnetic flux density was derived in Chapter 2 and given in (2.15):

$$\mathbf{B} = \frac{\mu_0 I}{4\pi r}\left[\frac{Z + l/2}{\sqrt{(Z + l/2)^2 + r^2}} - \frac{Z - l/2}{\sqrt{(Z - l/2)^2 + r^2}}\right]\mathbf{a}_\phi \tag{2.15}$$

The total flux through the surface s is

$$\begin{aligned}
\psi_\infty &= \int_{r=r_\mathrm{w}}^{\infty}\int_{Z=-l/2}^{l/2} B_\phi \, dZ\, dr \\
&= \frac{\mu_0 I}{4\pi}\int_{r=r_\mathrm{w}}^{\infty}\frac{1}{r}\int_{Z=-l/2}^{l/2}\left[\frac{Z + l/2}{\sqrt{(Z + l/2)^2 + r^2}}\right. \\
&\qquad \left.-\frac{Z - l/2}{\sqrt{(Z - l/2)^2 + r^2}}\right] dZ\, dr \\
&= 2\frac{\mu_0 I}{4\pi}\int_{r=r_\mathrm{w}}^{\infty}\frac{1}{r}\int_{\lambda=0}^{l}\frac{\lambda}{\sqrt{\lambda^2 + r^2}}\, d\lambda\, dr \\
&= \frac{\mu_0 I}{2\pi}\int_{r=r_\mathrm{w}}^{\infty}\frac{1}{r}\left[\sqrt{\lambda^2 + r^2}\right]_{\lambda=0}^{l} dr \\
&= \frac{\mu_0 I}{2\pi}\int_{r=r_\mathrm{w}}^{\infty}\frac{1}{r}\left(\sqrt{l^2 + r^2} - r\right) dr
\end{aligned} \tag{5.17a}$$

and we have used a change of variables, $\lambda = Z \pm l/2$, $d\lambda = dZ$, and integral 201.01 of Dwight [7]:

$$\int \frac{x}{\sqrt{x^2+a^2}}\,dx = \sqrt{x^2+a^2} \tag{D201.01}$$

Further integration with respect to r yields

$$\begin{aligned}
\psi_\infty &= \frac{\mu_0 I}{2\pi}\int_{r=r_\mathrm{w}}^{\infty}\left[\frac{\sqrt{l^2+r^2}}{r}-1\right]dr \\
&= \frac{\mu_0 I}{2\pi}\left[\sqrt{l^2+r^2}-l\,\ln\frac{l+\sqrt{l^2+r^2}}{r}-r\right]_{r=r_\mathrm{w}}^{r\to\infty} \\
&= -\frac{\mu_0 I}{2\pi}\,l\left[\ln\left(\frac{l}{r}+\sqrt{\left(\frac{l}{r}\right)^2+1}\right)-\sqrt{1+\left(\frac{r}{l}\right)^2}+\frac{r}{l}\right]_{r=r_\mathrm{w}}^{r\to\infty} \\
&= \frac{\mu_0 I}{2\pi}\,l\left[\ln\left(\frac{l}{r_\mathrm{w}}+\sqrt{\left(\frac{l}{r_\mathrm{w}}\right)^2+1}\right)-\sqrt{1+\left(\frac{r_\mathrm{w}}{l}\right)^2}+\frac{r_\mathrm{w}}{l}\right]
\end{aligned} \tag{5.17b}$$

and we have used integral 241.01 of Dwight [7]:

$$\int \frac{\sqrt{x^2+a^2}}{x}\,dx = \sqrt{x^2+a^2}-a\,\ln\frac{a+\sqrt{x^2+a^2}}{x} \tag{D241.01}$$

Hence, the self partial inductance is

$$\boxed{\begin{aligned}
L_p &= \frac{\psi_\infty}{I} \\
&= \frac{\mu_0}{2\pi}\,l\left[\ln\left(\frac{l}{r_\mathrm{w}}+\sqrt{\left(\frac{l}{r_\mathrm{w}}\right)^2+1}\right)-\sqrt{1+\left(\frac{r_\mathrm{w}}{l}\right)^2}+\frac{r_\mathrm{w}}{l}\right] \\
&= 2\times 10^{-7}\,l\left[\ln\left(\frac{l}{r_\mathrm{w}}+\sqrt{\left(\frac{l}{r_\mathrm{w}}\right)^2+1}\right)-\sqrt{1+\left(\frac{r_\mathrm{w}}{l}\right)^2}+\frac{r_\mathrm{w}}{l}\right]
\end{aligned}} \tag{5.18a}$$

and we have substituted $\mu_0/2\pi = 2\times 10^{-7}$. Using the inverse hyperbolic sine,

$$\begin{aligned}
\sinh^{-1}\frac{x}{a} &= \ln\left(\frac{x}{a}+\sqrt{\left(\frac{x}{a}\right)^2+1}\right) \\
&= -\sinh^{-1}\left(-\frac{x}{a}\right)
\end{aligned} \tag{D700.1}$$

gives an alternative form of the result:

$$\boxed{L_p = \frac{\mu_0}{2\pi}\, l\left[\sinh^{-1}\frac{l}{r_{\mathrm{w}}} - \sqrt{1+\left(\frac{r_{\mathrm{w}}}{l}\right)^2} + \frac{r_{\mathrm{w}}}{l}\right]} \tag{5.18b}$$

In a practical case, the length of the segment is usually much larger than the wire radius, $l \gg r_{\mathrm{w}}$, so we have the following approximations:

$$\ln\left[\frac{l}{r_{\mathrm{w}}} + \sqrt{\left(\frac{l}{r_{\mathrm{w}}}\right)^2 + 1}\right] = \ln\frac{2l}{r_{\mathrm{w}}} + \frac{1}{4}\left(\frac{r_{\mathrm{w}}}{l}\right)^2 - \frac{3}{32}\left(\frac{r_{\mathrm{w}}}{l}\right)^4 + \cdots \qquad \frac{l}{r_{\mathrm{w}}} \gg 1 \tag{D602.1}$$

and

$$\sqrt{1+\left(\frac{r_{\mathrm{w}}}{l}\right)^2} = 1 + \frac{1}{2}\left(\frac{r_{\mathrm{w}}}{l}\right)^2 - \frac{1}{8}\left(\frac{r_{\mathrm{w}}}{l}\right)^4 + \cdots \qquad \frac{r_{\mathrm{w}}}{l} \le 1 \tag{D5.3}$$

so that (5.18a) approximates to

$$\boxed{\begin{aligned} L_p &= \frac{\mu_0}{2\pi}\, l\left[\ln\frac{2l}{r_{\mathrm{w}}} - 1 + \frac{r_{\mathrm{w}}}{l} - \frac{1}{4}\left(\frac{r_{\mathrm{w}}}{l}\right)^2 + \cdots\right] \\ &\cong 2\times 10^{-7}\, l\left(\ln\frac{2\,l}{r_{\mathrm{w}}} - 1\right) \qquad l \gg r_{\mathrm{w}} \end{aligned}} \tag{5.18c}$$

Alternatively, we can determine the self partial inductance by integrating the vector magnetic potential along the wire surface also shown in Fig. 5.9. The vector magnetic potential **A** for this case was determined in Chapter 2 and given in (2.57):

$$A_z = \frac{\mu_0 I}{4\pi}\left(\sinh^{-1}\frac{Z+l/2}{r} - \sinh^{-1}\frac{Z-l/2}{r}\right) \tag{2.57}$$

Hence, we set up the integral

$$\begin{aligned} L_p &= \frac{\displaystyle\int_{Z=-l/2}^{l/2} A_z|_{r=r_{\mathrm{w}}}\, dZ}{I} \\ &= \frac{\mu_0}{4\pi}\int_{Z=-l/2}^{l/2}\left(\sinh^{-1}\frac{Z+l/2}{r_{\mathrm{w}}} - \sinh^{-1}\frac{Z-l/2}{r_{\mathrm{w}}}\right) dZ \\ &= 2\frac{\mu_0}{4\pi}\int_{\lambda=0}^{l}\left(\sinh^{-1}\frac{\lambda}{r_{\mathrm{w}}}\right) d\lambda \end{aligned}$$

$$
\begin{aligned}
&= 2\frac{\mu_0}{4\pi}\left[\lambda \sinh^{-1}\frac{\lambda}{r_w} - \sqrt{\lambda^2 + r_w^2}\right]_{\lambda=0}^{l} \\
&= \frac{\mu_0}{2\pi}\left(l \sinh^{-1}\frac{l}{r_w} - \sqrt{l^2 + r_w^2} + r_w\right) \\
&= \frac{\mu_0}{2\pi}l\left[\sinh^{-1}\frac{l}{r_w} - \sqrt{1 + \left(\frac{r_w}{l}\right)^2} + \frac{r_w}{l}\right] \\
&= \frac{\mu_0}{2\pi}l\left[\ln\left(\frac{l}{r_w} + \sqrt{\left(\frac{l}{r_w}\right)^2 + 1}\right) - \sqrt{1 + \left(\frac{r_w}{l}\right)^2} + \frac{r_w}{l}\right]
\end{aligned}
\tag{5.19}
$$

which is the same as (5.18a). We have used a change of variables, $\lambda = Z \pm l/2$, $d\lambda = dZ$, integral 730 of Dwight [7],

$$
\int \sinh^{-1}\frac{x}{a}\,dx = x\sinh^{-1}\frac{x}{a} - \sqrt{x^2 + a^2} \tag{D730}
$$

and the identity for inverse hyperbolic sine,

$$
\begin{aligned}
\sinh^{-1}\frac{x}{a} &= \ln\left(\frac{x}{a} + \sqrt{\left(\frac{x}{a}\right)^2 + 1}\right) \\
&= -\sinh^{-1}\left(-\frac{x}{a}\right)
\end{aligned}
\tag{D700.1}
$$

5.4 MUTUAL PARTIAL INDUCTANCE BETWEEN PARALLEL WIRES

Next, we determine another fundamental result: the mutual partial inductance between two parallel wires shown in Fig. 5.10. We first assume that both wires are of the same length and their endpoints are aligned. In the next section we derive the result for this situation but with the wires offset and their lengths different. The only difference between this computation and those for the self partial inductance of Section 5.3 is that here we integrate from $y = d + r_w$ to $y \to \infty$ rather than from the surface of the first wire. Hence, the integral in (5.17a) becomes

$$
\psi_\infty = \int_{r=d+r_w}^{\infty}\int_{Z=-l/2}^{l/2} B_\phi\, dZ\, dr \tag{5.20}
$$

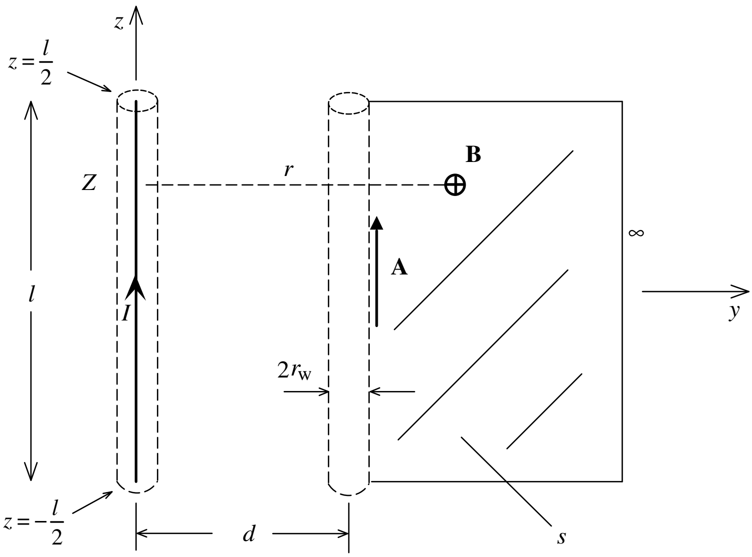


FIGURE 5.10. Determination of the mutual partial inductance between parallel wires.

It is easy to see that we only need to replace r_w with $d + r_w$ in the previous derivation for the self partial inductance in (5.17a)–(5.17b) and obtain

$$\boxed{\begin{aligned} M_p &= \frac{\psi_\infty}{I} \\ &= \frac{\mu_0}{2\pi} l \left[\ln\left(\frac{l}{d+r_w} + \sqrt{\left(\frac{l}{d+r_w}\right)^2 + 1} \right) \right. \\ &\qquad \left. - \sqrt{1 + \left(\frac{d+r_w}{l}\right)^2} + \frac{d+r_w}{l} \right] \\ &\cong 2 \times 10^{-7} l \left[\ln\left(\frac{l}{d} + \sqrt{\left(\frac{l}{d}\right)^2 + 1} \right) \right. \\ &\qquad \left. - \sqrt{1 + \left(\frac{d}{l}\right)^2} + \frac{d}{l} \right] \qquad d \gg r_w \end{aligned}} \tag{5.21a}$$

Using the inverse hyperbolic sine,

$$\begin{aligned} \sinh^{-1} \frac{x}{a} &= \ln\left(\frac{x}{a} + \sqrt{\left(\frac{x}{a}\right)^2 + 1} \right) \\ &= -\sinh^{-1}\left(-\frac{x}{a}\right) \end{aligned} \tag{D700.1}$$

gives an alternative form of the result:

$$\boxed{M_p = \frac{\mu_0}{2\pi} l \left[\sinh^{-1}\frac{l}{d} - \sqrt{1+\left(\frac{d}{l}\right)^2} + \frac{d}{l}\right] \qquad d \gg r_w} \tag{5.21b}$$

For wires that are very long compared to their separation, $l/d \gg 1$, or, equivalently, separations much smaller than their length, $d/l \ll 1$, the result in (5.21a) can be approximated by using

$$\ln\left[\frac{l}{d} + \sqrt{\left(\frac{l}{d}\right)^2 + 1}\right] = \ln\frac{2l}{d} + \frac{1}{4}\left(\frac{d}{l}\right)^2 - \frac{3}{32}\left(\frac{d}{l}\right)^4 + \cdots \qquad \frac{l}{d} > 1 \tag{D602.1}$$

$$\sqrt{1+\left(\frac{d}{l}\right)^2} = 1 + \frac{1}{2}\left(\frac{d}{l}\right)^2 - \frac{1}{8}\left(\frac{d}{l}\right)^4 + \cdots \qquad \frac{d}{l} \le 1 \tag{D5.3}$$

giving

$$\boxed{\begin{aligned} M_p &= \frac{\mu_0}{2\pi} l \left[\ln\frac{2l}{d} - 1 + \frac{d}{l} - \frac{1}{4}\left(\frac{d}{l}\right)^2 + \frac{1}{32}\left(\frac{d}{l}\right)^4 - \cdots\right] \\ &\cong \frac{\mu_0}{2\pi} l \left(\ln\frac{2l}{d} - 1\right) \qquad l \gg d \end{aligned}} \tag{5.21c}$$

For wires that are very short compared to their separation, $l/d \ll 1$, or, equivalently, separations much greater than their length, $d/l \gg 1$, the result in (5.21a) can be approximated by using

$$\ln\left[\frac{l}{d} + \sqrt{\left(\frac{l}{d}\right)^2 + 1}\right] = \frac{l}{d} - \frac{1}{6}\left(\frac{l}{d}\right)^3 + \frac{3}{40}\left(\frac{l}{d}\right)^5 - \cdots \qquad \frac{l}{d} < 1 \tag{D602.1}$$

$$\begin{aligned} \sqrt{1+\left(\frac{d}{l}\right)^2} &= \frac{d}{l}\sqrt{\left(\frac{l}{d}\right)^2 + 1} \\ &= \frac{d}{l} + \frac{1}{2}\left(\frac{l}{d}\right) - \frac{1}{8}\left(\frac{l}{d}\right)^3 + \frac{1}{16}\left(\frac{l}{d}\right)^5 - \cdots \qquad \frac{l}{d} \le 1 \end{aligned} \tag{D5.3}$$

giving

$$\boxed{M_p = \frac{\mu_0}{2\pi}\frac{l}{2d}\left[1 - \frac{1}{12}\left(\frac{l}{d}\right)^2 + \frac{1}{40}\left(\frac{l}{d}\right)^4 - \cdots\right] \qquad l \ll d} \tag{5.21d}$$

We can obtain the same result as in (5.21a) from the vector magnetic potential **A**:

$$M_p = \frac{\displaystyle\int_{Z=-l/2}^{l/2} A_z|_{r=d+r_\mathrm{w}}\, dZ}{I} \tag{5.22}$$

and evaluating A_z along the second wire at $y = d + r_\mathrm{w}$. Carrying through the same integration in (5.19) but with $r = r_\mathrm{w}$ replaced by $r = d + r_\mathrm{w}$ again gives (5.21a).

Finally, we show that the mutual partial inductance in (5.21a) can also be derived from the Neumann integral in (5.16):

$$\begin{aligned}
M_p &= \frac{\mu_0}{4\pi}\int_{c_1}\int_{c_2}\frac{d\mathbf{l}_1 \cdot d\mathbf{l}_2}{R_{12}} \\
&= \frac{\mu_0}{4\pi}\int_{z_2=-l/2}^{l/2} dz_2 \int_{z_1=-l/2}^{l/2} \frac{1}{\sqrt{(d+r_\mathrm{w})^2 + (z_1 - z_2)^2}}\, dz_1 \\
&= \frac{\mu_0}{4\pi}\int_{z_2=-l/2}^{l/2} dz_2 \int_{\lambda=-l/2-z_2}^{l/2-z_2} \frac{1}{\sqrt{(d+r_\mathrm{w})^2 + \lambda^2}}\, d\lambda \\
&= \frac{\mu_0}{4\pi}\int_{z_2=-l/2}^{l/2} \left[\ln\left(\lambda + \sqrt{(d+r_\mathrm{w})^2 + \lambda^2}\right)\right]_{\lambda=-l/2-z_2}^{l/2-z_2} dz_2 \\
&= \frac{\mu_0}{4\pi}\int_{z_2=-l/2}^{l/2} \left(\sinh^{-1}\frac{l/2 - z_2}{d+r_\mathrm{w}} + \sinh^{-1}\frac{l/2 + z_2}{d+r_\mathrm{w}}\right) dz_2 \\
&= \frac{\mu_0}{4\pi}\int_{\zeta=0}^{l} \left(2\sinh^{-1}\frac{\zeta}{d+r_\mathrm{w}}\right) d\zeta \\
&= 2\frac{\mu_0}{4\pi}\left[\zeta\sinh^{-1}\frac{\zeta}{d+r_\mathrm{w}} - \sqrt{\zeta^2 + (d+r_\mathrm{w})^2}\right]_{\zeta=0}^{l} \\
&= \frac{\mu_0}{2\pi}\left[l\sinh^{-1}\frac{l}{d+r_\mathrm{w}} - \sqrt{l^2 + (d+r_\mathrm{w})^2} + (d+r_\mathrm{w})\right] \\
&= \frac{\mu_0}{2\pi}\, l\left[\sinh^{-1}\frac{l}{d+r_\mathrm{w}} - \sqrt{1 + \left(\frac{d+r_\mathrm{w}}{l}\right)^2} + \frac{d+r_\mathrm{w}}{l}\right]
\end{aligned} \tag{5.23}$$

and the differential lengths $d\mathbf{l}_1$ and $d\mathbf{l}_2$ are parallel so that the dot product goes away: $d\mathbf{l}_1 \cdot d\mathbf{l}_2 = dl_1\, dl_2 = dz_1\, dz_2$. But (5.23) is the same as (5.21a). We have substituted a change of variables in the inner integral: $\lambda = z_1 - z_2$, $d\lambda = dz_1$, and a change of variables in the outer integral: $\zeta = l/2 \pm z_2$, $d\zeta = \pm dz_2$, and have again used the integrals

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln\left(x + \sqrt{x^2 + a^2}\right) \tag{D200.01}$$

and

$$\int \sinh^{-1} \frac{x}{a}\, dx = x \sinh^{-1} \frac{x}{a} - \sqrt{x^2 + a^2} \tag{D730}$$

We also used the important identity

$$\begin{aligned} \ln \frac{a + \sqrt{x^2 + a^2}}{-b + \sqrt{x^2 + b^2}} &= \sinh^{-1} \frac{a}{x} - \sinh^{-1}\left(-\frac{b}{x}\right) \\ &= \sinh^{-1} \frac{a}{x} + \sinh^{-1} \frac{b}{x} \end{aligned}$$

From these results we see that the self partial inductance L_p can be obtained from the mutual partial inductance simply by replacing $d + r_\mathrm{w}$ in M_p, with r_w, and vice versa. In other words,

$$L_p = M_p\big|_{d + r_\mathrm{w} \to r_\mathrm{w}} \tag{5.24}$$

Using M_p to get L_p in this way presupposes that both wires are of the same length and radii, and their endpoints are aligned.

5.5 MUTUAL PARTIAL INDUCTANCE BETWEEN PARALLEL WIRES THAT ARE OFFSET

Consider the case of two offset, parallel wires whose lengths are l and m shown in Fig. 5.11. The two wires are parallel to the z axis, have a center-to-center separation of d, and their endpoints are offset by a distance s. The radius of the second wire of length l is r_w. The radius of the first wire of length m carrying the current I which produces the magnetic field is immaterial since we assume that the current I is distributed uniformly over the cross section of that wire so that this current can be concentrated as a filament on the axis of the wire. The first wire carrying the current I has its lower end at the origin of the coordinate system, $z = 0$. The two ends of the other wire of length l are at positions $z = z_1$ and $z = z_2$. In all such problems of determining the mutual partial inductance between two parallel but offset wires using the result derived in this section, it is important to determine these wire lengths and positions, $z = 0$, z_1, and z_2, for each particular problem.

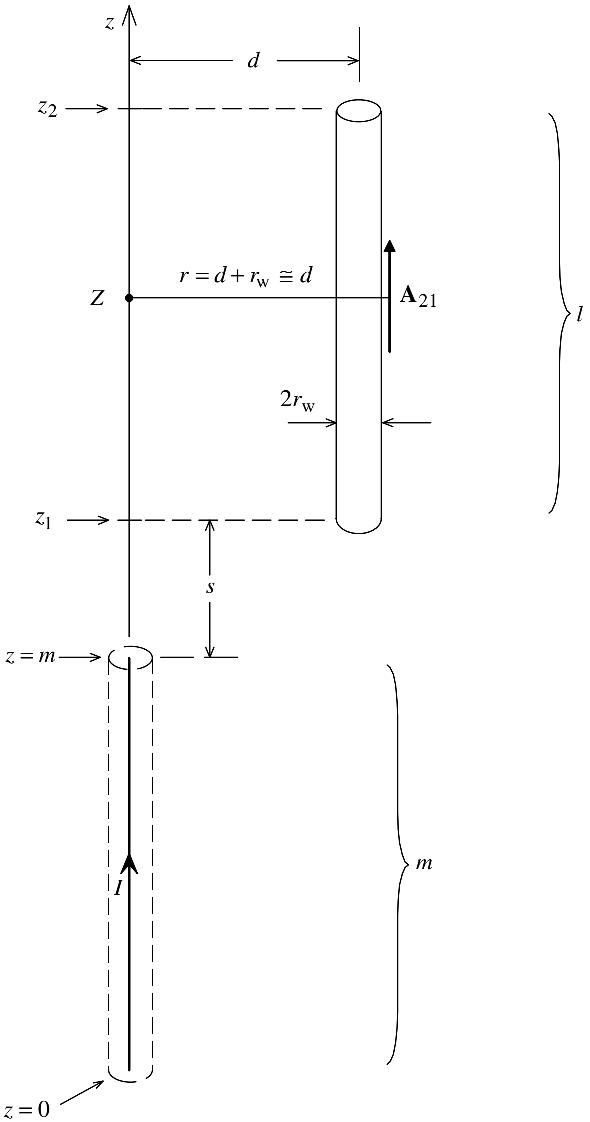


FIGURE 5.11. Mutual partial inductance between offset wires.

Again we have three methods for calculating the mutual partial inductance between the two wire segments: the magnetic flux linkage method using **B**, the vector magnetic potential method using **A**, and the Neumann integral. For this problem we choose to use the vector magnetic potential method using **A**. We must integrate the vector magnetic potential due to the current I of the first wire of length m along the surface of the second wire of length l with a line integral:

$$M_p = \frac{\int_l \mathbf{A}_{21}|_{r=d+r_w\cong d} \cdot d\mathbf{l}}{I} = \frac{\int_{Z=z_1}^{z_2} A_{21}(Z,r)|_{r=d}\, dZ}{I} \tag{5.25}$$

where $\mathbf{A}_{21}$ is the vector magnetic potential along the surface of the second wire that is produced by the current I of the first wire. Hence, we need the result for the vector magnetic potential from a wire of length m carrying a current I. This was derived in Chapter 2 from Fig. 2.24 and given in (2.57). Note in Fig. 2.24 that the origin of the coordinate system at $z = 0$ was located at the midpoint of the wire. We must modify that result to fit Fig. 5.11 by rederiving the result for the case where the lower end of the wire is at $z = 0$. Carrying through the development that led to (2.57) yields for this case

$$\begin{aligned} A_{21}(Z,r) &= \frac{\mu_0 I}{4\pi} \left\{ \ln\left(Z + \sqrt{Z^2 + r^2}\right) \right. \\ &\quad \left. - \ln\left[(Z-m) + \sqrt{(Z-m)^2 + r^2}\right] \right\} \\ &= \frac{\mu_0 I}{4\pi} \left(\sinh^{-1}\frac{Z}{r} - \sinh^{-1}\frac{Z-m}{r} \right) \end{aligned} \tag{5.26}$$

and we have again used the identity

$$\begin{aligned} \sinh^{-1}\frac{x}{a} &= -\sinh^{-1}\left(-\frac{x}{a}\right) \\ &= \ln\left[\frac{x}{a} + \sqrt{\left(\frac{x}{a}\right)^2 + 1}\right] \\ &= \ln\left(x + \sqrt{x^2 + a^2}\right) - \ln a \end{aligned} \tag{D700.1}$$

Hence, (5.25) becomes

$$\begin{aligned} M_p &= \frac{\int_{Z=z_1}^{z_2} A_{21}|_{r=d}\, dZ}{I} \\ &= \frac{\mu_0}{4\pi} \int_{Z=z_1}^{z_2} \left(\sinh^{-1}\frac{Z}{d} - \sinh^{-1}\frac{Z-m}{d} \right) dZ \end{aligned} \tag{5.27}$$

Carrying through with the integration of (5.27) gives

$$\boxed{\begin{aligned} M_p &= \frac{\mu_0}{4\pi} \int_{Z=z_1}^{z_2} \left(\sinh^{-1} \frac{Z}{d} - \sinh^{-1} \frac{Z-m}{d} \right) dZ \\ &= \frac{\mu_0}{4\pi} \left(\int_{Z=z_1}^{z_2} \sinh^{-1} \frac{Z}{d}\, dZ - \int_{\lambda=z_1-m}^{z_2-m} \sinh^{-1} \frac{\lambda}{d}\, d\lambda \right) \\ &= \frac{\mu_0}{4\pi} \left[z_2 \sinh^{-1} \frac{z_2}{d} - z_1 \sinh^{-1} \frac{z_1}{d} - (z_2-m) \sinh^{-1} \frac{z_2-m}{d} \right. \\ &\qquad + (z_1-m) \sinh^{-1} \frac{z_1-m}{d} - \sqrt{z_2^2+d^2} + \sqrt{z_1^2+d^2} \\ &\qquad \left. + \sqrt{(z_2-m)^2+d^2} - \sqrt{(z_1-m)^2+d^2} \right] \end{aligned}} \tag{5.28}$$

where we have used a change of variables, $\lambda = Z - m, d\lambda = dZ$, in the second integral and have used integral 730 of Dwight [7]:

$$\int \sinh^{-1} \frac{x}{a}\, dx = x \sinh^{-1} \frac{x}{a} - \sqrt{x^2+a^2} \tag{D730}$$

In the case where the two wires lie on the z axis, $d = 0$, as shown in Fig. 5.12, we could reintegrate (5.27) for $r = r_w$ or simply substitute $d = r_w$ into (5.28) to give

$$\boxed{\begin{aligned} M_{p(d=r_w)} = \frac{\mu_0}{4\pi} & \left[z_2 \sinh^{-1} \frac{z_2}{r_w} - z_1 \sinh^{-1} \frac{z_1}{r_w} - (z_2-m) \sinh^{-1} \frac{z_2-m}{r_w} \right. \\ & + (z_1-m) \sinh^{-1} \left(\frac{z_1-m}{r_w} \right) - \sqrt{z_2^2+r_w^2} + \sqrt{z_1^2+r_w^2} \\ & \left. + \sqrt{(z_2-m)^2+r_w^2} - \sqrt{(z_1-m)^2+r_w^2} \right] \end{aligned}} \tag{5.29a}$$

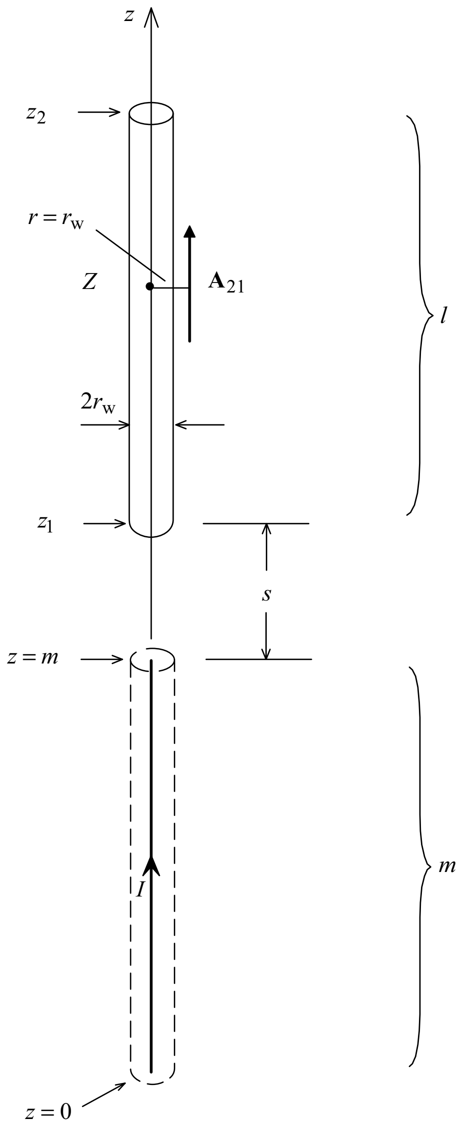


FIGURE 5.12. Aligned but offset wires.

Substituting the dimensions gives

$$\boxed{\begin{aligned} M_{p\,(d=r_{\mathrm{w}})} &= \frac{\mu_0}{4\pi}\left[(l+s+m)\,\sinh^{-1}\frac{l+s+m}{r_{\mathrm{w}}} - (m+s)\,\sinh^{-1}\frac{m+s}{r_{\mathrm{w}}}\right. \\ &\quad - (l+s)\,\sinh^{-1}\frac{l+s}{r_{\mathrm{w}}} + s\sinh^{-1}\frac{s}{r_{\mathrm{w}}} - \sqrt{(l+s+m)^2+r_{\mathrm{w}}^2} \\ &\quad \left. + \sqrt{(m+s)^2+r_{\mathrm{w}}^2} + \sqrt{(l+s)^2+r_{\mathrm{w}}^2} - \sqrt{s^2+r_{\mathrm{w}}^2}\right] \\ &\cong \frac{\mu_0}{4\pi}\left\{(l+s+m)\left[\ln\left(\frac{2\,(l+s+m)}{r_{\mathrm{w}}}\right)-1\right]\right. \\ &\quad - (m+s)\left[\ln\left(\frac{2\,(m+s)}{r_{\mathrm{w}}}\right)-1\right] \\ &\quad \left. - (l+s)\left[\ln\left(\frac{2\,(l+s)}{r_{\mathrm{w}}}\right)-1\right] + s\left[\ln\left(\frac{2s}{r_{\mathrm{w}}}\right)-1\right]\right\} \end{aligned}}$$

(5.29b)

In terms of the self partial inductances of a wire of radius r_{w} and length l obtained in (5.18b),

$$L_l = \frac{\mu_0}{2\pi}\left(l\sinh^{-1}\frac{l}{r_{\mathrm{w}}} - \sqrt{l^2+r_{\mathrm{w}}^2} + r_{\mathrm{w}}\right) \tag{5.18b}$$

the result for aligned but offset wires in (5.29a,b) can be written as

$$\boxed{\begin{aligned} 2M_{p(d=r_{\mathrm{w}})} &= \left(L_{z_2} + L_{z_1-m}\right) - \left(L_{z_2-m} + L_{z_1}\right) \\ &= \left(L_{l+s+m} + L_s\right) - \left(L_{l+s} + L_{m+s}\right) \end{aligned}} \tag{5.29c}$$

Notice that (5.29c) gives $2M_p$ since the self partial inductance L_l in (5.18b) is multiplied by $\mu_0/2\pi$, whereas the result for M_p in (5.29a,b) is multiplied by $\mu_0/4\pi$.

There is a simple explanation for why the result for the mutual partial inductance between two aligned but offset wires can be written in terms of the self partial inductances of wires of various lengths obtained previously, as in (5.29c). Recall that the self partial inductance of a wire is the ratio of the magnetic flux between that wire and infinity, ψ_l, and the current of that wire:

$$L_l = \frac{\psi_l}{I}$$

Figure 5.13 shows that a current on each wire segment produces not only flux between that segment and infinity but also between each of the other segments and infinity. For example, observe from Fig. 5.13 that superimposing the fluxes

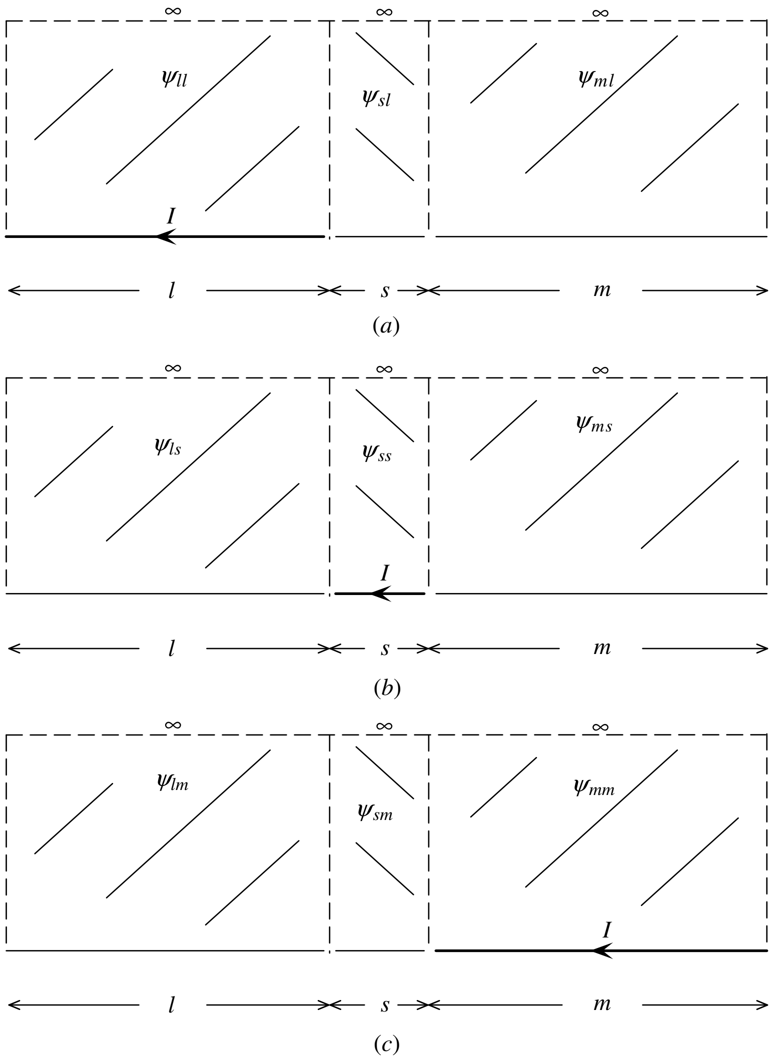


FIGURE 5.13. Mutual partial inductance for aligned but offset wires in terms of fluxes to infinity.

opposite each segment that are due to currents on the other three segments gives

$$\psi_l = \psi_{ll} + \psi_{ls} + \psi_{lm} \tag{5.30a}$$

$$\psi_s = \psi_{sl} + \psi_{ss} + \psi_{sm} \tag{5.30b}$$

$$\psi_m = \psi_{ml} + \psi_{ms} + \psi_{mm} \tag{5.30c}$$

where the notation ψ_{ij} denotes the flux to infinity opposite segment i due to a current I *only* on segment j. Keep in mind that these mutual inductances are reciprocal (i.e., $M_{ij} = M_{ji}$). But if all three segments have current I on them, they produce the total flux ψ_{l+s+m}. Hence, the self inductance of a wire of total length $l + s + m$ can be written as

$$\begin{aligned}
L_{l+s+m} &= \frac{\psi_{l+s+m}}{I} \\
&= \frac{\psi_l}{I} + \frac{\psi_s}{I} + \frac{\psi_m}{I} \\
&= \frac{\psi_{ll}}{I} + \frac{\psi_{ls}}{I} + \frac{\psi_{lm}}{I} \\
&\quad + \frac{\psi_{sl}}{I} + \frac{\psi_{ss}}{I} + \frac{\psi_{sm}}{I} \\
&\quad + \frac{\psi_{ml}}{I} + \frac{\psi_{ms}}{I} + \frac{\psi_{mm}}{I}
\end{aligned} \tag{5.31}$$

The key to simplifying this and writing it in the form of (5.29c) is to *write the result in terms of the self partial inductances of segments of a single length so that we can use the result derived in* (*5.18a,b*) *without having to rederive a new result* [which we have already done in (5.29)]. To do this, note that the total fluxes given by (5.31) can be written as

$$\begin{aligned}
L_{l+s+m} = &\underbrace{\frac{\psi_{ll}}{I} + \frac{\psi_{ls}}{I} + \frac{\psi_{sl}}{I} + \frac{\psi_{ss}}{I}}_{L_{l+s}} + \underbrace{\frac{\psi_{mm}}{I} + \frac{\psi_{ms}}{I} + \frac{\psi_{sm}}{I} + \frac{\psi_{ss}}{I}}_{L_{m+s}} \\
&- \underbrace{\frac{\psi_{ss}}{I}}_{L_s} + \underbrace{\frac{\psi_{lm}}{I} + \frac{\psi_{ml}}{I}}_{2M_p}
\end{aligned} \tag{5.32a}$$

Solving this gives the result in (5.29c) since

$$\begin{aligned}
2M_p &= \frac{M_{lm} + M_{ml}}{I} \\
&= (L_{l+s+m} + L_s) - (L_{l+s} + L_{m+s})
\end{aligned} \tag{5.32b}$$

This gives a very basic principle for adding inductors in series where the inductors have not only their self inductance but also mutual inductances between each other:

$$\boxed{\begin{aligned}
L_{1+2+3} &= L_1 + M_{12} + M_{13} + L_2 + M_{12} + M_{23} + L_3 + M_{13} + M_{23} \\
&= L_1 + L_2 + L_3 + 2M_{12} + 2M_{13} + 2M_{23}
\end{aligned}} \tag{5.33}$$

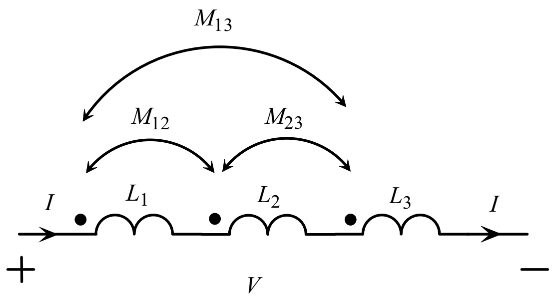


FIGURE 5.14. Adding inductors in series.

This can be verified from the electric circuit diagram in Fig. 5.14 by determining the total voltage across the series combination using the dot convention [1,2].

The basic result in (5.28) for parallel, offset wires with $d \neq 0$ can be written similarly in terms of the result in Section 5.4 for the mutual partial inductance between two identical wires of lengths l and separation d whose endpoints coincide as shown in Fig. 5.10 and given in (5.21b):

$$M_l = \frac{\mu_0}{2\pi}\left(l \sinh^{-1}\frac{l}{d} - \sqrt{l^2 + d^2} + d\right) \qquad d \gg r_{\mathrm{w}} \tag{5.21b}$$

Hence, the result in (5.28) can be written in terms of (5.21b) as

$$\boxed{\begin{aligned} 2M_p &= \left(M_{z_2} + M_{z_1 - m}\right) - \left(M_{z_1} + M_{z_2 - m}\right) \\ &= \left(M_{l+s+m} + M_s\right) - \left(M_{m+s} + M_{l+s}\right) \end{aligned}} \tag{5.34}$$

Notice again that (5.34) gives $2M_p$ since M_l in (5.21b) is multiplied by $\mu_0/2\pi$, whereas the result for M_p in (5.28) is multiplied by $\mu_0/4\pi$. Note from (5.21b) that

$$M_0 = 0 \tag{5.35a}$$

and

$$M_{-l} = M_l \tag{5.35b}$$

with (5.35b) resulting from the identity $\sinh^{-1}(-x) = -\sinh^{-1} x$. If the wires overlap, replace s with $-s$ in (5.34).

We can easily determine the mutual partial inductance between the various offset structures shown in Fig. 5.15 by using the basic result in (5.34) *and* comparing each of these structures to Fig. 5.11, from which (5.34) was derived in order to (1) determine the location point of $z = 0$ on those structures, and (2) hence to determine the values of z_1 and z_2 in (5.34). For example,

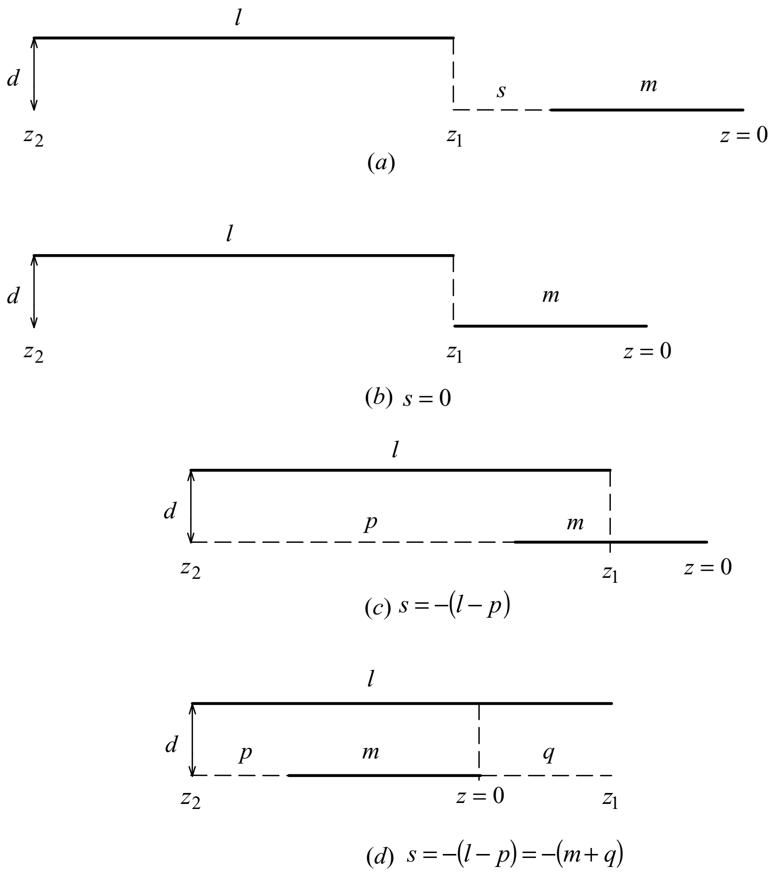


FIGURE 5.15. Using the basic relation in (5.34) to determine the mutual inductance for other offset structures.

in Fig. 5.15(a) we identify $z_2 = l + s + m$ and $z_1 = m + s$. Hence, for the structure in Fig. 5.15(a) we obtain

$$2M_p = (M_{l+s+m} + M_s) - (M_{m+s} + M_{s+l})$$

Similarly, for the case in Fig. 5.15(b) we identify $z_2 = l + m$ and $z_1 = m$ or, equivalently, $s = 0$. Hence, for the structure in Fig. 5.15(b) we obtain

$$\begin{aligned} 2M_p &= (M_{l+m} + M_0) - (M_m + M_l) \\ &= M_{l+m} - M_m - M_l \end{aligned}$$

For the case in Fig. 5.15(c) we identify $z_2 = p + m$ and $z_1 = p + m - l$ or, equivalently, $s = -(l - p)$. Hence, for the structure in Fig. 5.15(c) we obtain

$$\begin{aligned} 2M_p &= \left(M_{p+m} + M_{p-l}\right) - \left(M_{p+m-l} + M_p\right) \\ &= \left(M_{p+m} + M_{l-p}\right) - \left(M_{p+m-l} + M_p\right) \end{aligned}$$

For the case in Fig. 5.15(d) we identify $z_2 = p + m$ and $z_1 = -q = p + m - l$ or, equivalently, $s = -(l - p) = -(m + q)$. Hence, for the structure in Fig. 5.15(d) we obtain

$$\begin{aligned} 2M_p &= \left(M_{p+m} + M_{-q-m}\right) - \left(M_{-q} + M_p\right) \\ &= \left(M_{p+m} + M_{q+m}\right) - \left(M_q + M_p\right) \end{aligned}$$

Figure 5.16 shows how we could have easily obtained the basic result in (5.34) by using lumped-circuit analysis principles and the dot convention

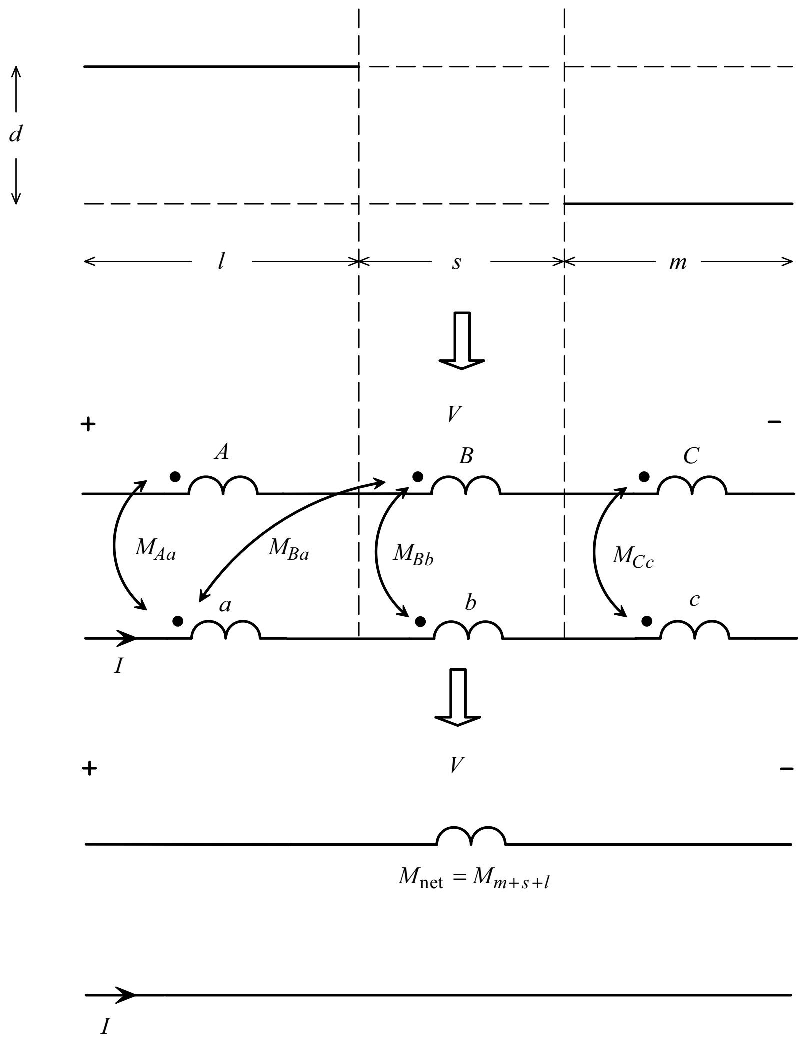


FIGURE 5.16. Combining mutual partial inductances that are in series.

[1,2]. We have shown an equivalent circuit for two parallel conductors of total length $m + s + l$ along with the mutual inductances between the three segments of lengths m, s, and l. Denoting the voltage between the endpoints of the ends of the top conductor as V, and passing a current I through the lower conductor, the total contribution to V due to the mutual inductances between all segments is

$$V = M_{\text{net}} \frac{dI}{dt}$$

Using the dot convention [1,2] and analyzing this circuit for the total voltage contributed to V by the mutual inductances between the segments, the net mutual inductance between the entire lengths is

$$M_{\text{net}} = M_{Aa} + M_{Ab} + M_{Ac} + M_{Ba} + M_{Bb} + M_{Bc} + M_{Ca} + M_{Cb} + M_{Cc}$$

But

$$\begin{aligned} M_{\text{net}} &= M_{m+s+l} \\ M_{s+l} &= M_{Aa} + M_{Ab} + M_{Ba} + M_{Bb} \\ M_{m+s} &= M_{Bb} + M_{Bc} + M_{Cb} + M_{Cc} \\ M_s &= M_{Bb} \end{aligned}$$

Hence, we can write

$$M_{m+s+l} = M_{s+l} + M_{m+s} - M_s + M_{Ac} + M_{Ca}$$

However, the mutual inductance we desire between the two conductors of lengths l and m is

$$2M_p = M_{Ac} + M_{Ca}$$

Solving the last two relations gives the basic relation in (5.34), which we derived through a lengthy integration!

5.6 MUTUAL PARTIAL INDUCTANCE BETWEEN WIRES AT AN ANGLE TO EACH OTHER

We first consider a special case of two straight wires of lengths l and m that are inclined with respect to each other at an angle θ and joined at one end (or at least infinitesimally close) as shown in Fig. 5.17. The solution for the mutual partial inductance for this special case can be adapted to give the solution for a large class of similar problems, as we will see. This will be very similar

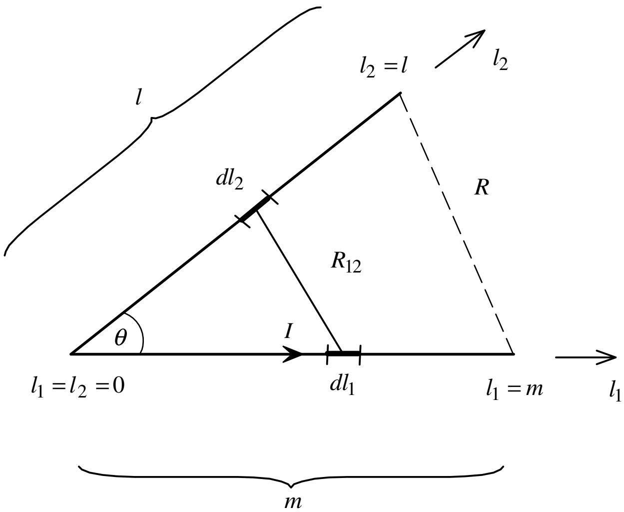


FIGURE 5.17. Wires inclined at an angle to each other.

to our recognizing that the mutual partial inductance for the case for two parallel but offset wires shown in Fig. 5.11 could be obtained in terms of the solution for two equal-length wires whose endpoints are aligned and shown in Fig. 5.10. This adaptation is given in (5.34) in terms of the mutual partial inductance of two equal-length parallel wires whose endpoints are aligned given in (5.21a,b).

We obtain the mutual partial inductance for the configuration in Fig. 5.17 using the Neumann integral:

$$\begin{aligned} M_p &= \frac{\mu_0}{4\pi} \int_{l_2} \int_{l_1} \frac{d\mathbf{l}_1 \cdot d\mathbf{l}_2}{R_{12}} \\ &= \frac{\mu_0}{4\pi} \cos\theta \int_{l_2} \int_{l_1} \frac{1}{R_{12}} \, dl_1 \, dl_2 \end{aligned} \tag{5.36a}$$

where l_1 and l_2 are the contours along the axes of the two wires, and R_{12} is the distance between the differential segments dl_1 and dl_2 given by

$$R_{12} = \sqrt{l_1^2 + l_2^2 - 2 l_1 l_2 \cos\theta} \tag{5.36b}$$

and we have used the law of cosines. The dot product of the vector differential segments becomes $d\mathbf{l}_1 \cdot d\mathbf{l}_2 = \cos\theta \, dl_1 \, dl_2$.

We can place this integral in an integrable form using the following technique [17]. We can show that (5.36a) can be written as

$$\begin{aligned} M_p &= \frac{\mu_0}{4\pi} \cos\theta \int_{l_2} \int_{l_1} \frac{1}{R_{12}}\, dl_1\, dl_2 \\ &= \frac{\mu_0}{4\pi} \cos\theta \int_{l_2} \int_{l_1} \left[\frac{d}{dl_1}\left(\frac{l_1}{R_{12}}\right) + \frac{d}{dl_2}\left(\frac{l_2}{R_{12}}\right)\right] dl_1\, dl_2 \\ &= \frac{\mu_0}{4\pi} \cos\theta \left[l_1 \int_{l_2} \frac{1}{R_{12}}\, dl_2 + l_2 \int_{l_1} \frac{1}{R_{12}}\, dl_1\right] \end{aligned} \tag{5.37}$$

This first equivalence in (5.37) can be shown, using R_{12} from (5.36b), to give

$$\begin{aligned} \frac{d}{dl_1}\left(\frac{l_1}{R_{12}}\right) &= \frac{R_{12} - l_1\, R_{12}^{-1}\, (l_1 - l_2 \cos\theta)}{R_{12}^2} \\ &= \frac{1}{R_{12}} - \frac{l_1\, (l_1 - l_2 \cos\theta)}{R_{12}^3} \end{aligned} \tag{5.38a}$$

$$\begin{aligned} \frac{d}{dl_2}\left(\frac{l_2}{R_{12}}\right) &= \frac{R_{12} - l_2\, R_{12}^{-1}\, (l_2 - l_1 \cos\theta)}{R_{12}^2} \\ &= \frac{1}{R_{12}} - \frac{l_2\, (l_2 - l_1 \cos\theta)}{R_{12}^3} \end{aligned} \tag{5.38b}$$

and we have used

$$\frac{d\left(\frac{u}{v}\right)}{dx} = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2} \tag{D65}$$

Hence,

$$\frac{d}{dl_1}\left(\frac{l_1}{R_{12}}\right) + \frac{d}{dl_2}\left(\frac{l_2}{R_{12}}\right) = \frac{1}{R_{12}} \tag{5.39}$$

The second equivalence in (5.37) can easily be shown from

$$\begin{aligned} \int_{l_2}\int_{l_1}\left[\frac{d}{dl_1}\left(\frac{l_1}{R_{12}}\right)\right] dl_1\, dl_2 &= \int_{l_2}\left[\int_{l_1}\frac{d}{dl_1}\left(\frac{l_1}{R_{12}}\right) dl_1\right] dl_2 \\ &= \int_{l_2} \frac{l_1}{R_{12}}\, dl_2 \\ &= l_1 \int_{l_2} \frac{1}{R_{12}}\, dl_2 \end{aligned} \tag{5.40a}$$

$$\int_{l_1}\int_{l_2}\left[\frac{d}{dl_2}\left(\frac{l_2}{R_{12}}\right)\right]dl_2\,dl_1 = \int_{l_1}\left[\int_{l_2}\frac{d}{dl_2}\left(\frac{l_2}{R_{12}}\right)dl_2\right]dl_1$$

$$= \int_{l_1}\frac{l_2}{R_{12}}\,dl_1$$

$$= l_2\int_{l_1}\frac{1}{R_{12}}\,dl_1 \tag{5.40b}$$

and we have obtained the equivalence in (5.37). But the last result in (5.37) can easily be integrated using integral 380.001 from Dwight [7]:

$$\int\frac{dx}{\sqrt{x^2+bx+c}} = \ln\left(2\sqrt{x^2+bx+c}+2x+b\right) \tag{D380.001}$$

and the equation for R_{12} in (5.36b) to give

$$\int_{l_i}\frac{1}{R_{12}}\,dl_i = \int_{l_i}\frac{1}{\sqrt{l_i^2+l_j^2-2l_il_j\cos\theta}}\,dl_i$$

$$= \ln\left(2\sqrt{l_i^2+l_j^2-2l_il_j\cos\theta}+2l_i-2l_j\cos\theta\right)$$

$$= \ln\left(\sqrt{l_i^2+l_j^2-2l_il_j\cos\theta}+l_i-l_j\cos\theta\right) \tag{5.41}$$

The factor of 2 cancels out when we evaluate at the upper and lower limits of the integral.

Now we apply this result to the problem of Fig. 5.17. For economy of notation we denote the mutual partial inductance between the two segments as

$$M_p = \frac{\mu_0}{4\pi}N \tag{5.42}$$

and N becomes, in terms of the limits of the integrals,

$$N = \int_{l_2=A}^{B}\int_{l_1=a}^{b}\frac{1}{R_{12}}\,dl_1dl_2$$

$$= l_1\int_{l_2}\frac{1}{R_{12}}\,dl_2 + l_2\int_{l_1}\frac{1}{R_{12}}\,dl_1$$

$$= \left[l_1\int_{l_2=A}^{B}\frac{1}{R_{12}}dl_2\right]_{l_1=a}^{b} + \left[l_2\int_{l_1=a}^{b}\frac{1}{R_{12}}dl_1\right]_{l_2=A}^{B}$$

$$= b\left\{\ln\left[\sqrt{b^2+B^2-2bB\cos\theta}+B-b\cos\theta\right]\right.$$

$$
\begin{aligned}
& - \ln \left[\sqrt{b^2 + A^2 - 2bA\cos\theta} + A - b\cos\theta\right]\Big\} \\
& -a \Big\{ \ln\left[\sqrt{a^2 + B^2 - 2aB\cos\theta} + B - a\cos\theta\right] \\
& - \ln \left[\sqrt{a^2 + A^2 - 2aA\cos\theta} + A - a\cos\theta\right]\Big\} \\
& +B \Big\{ \ln\left[\sqrt{b^2 + B^2 - 2bB\cos\theta} + b - B\cos\theta\right] \\
& - \ln \left[\sqrt{a^2 + B^2 - 2aB\cos\theta} + a - B\cos\theta\right]\Big\} \\
& -A \Big\{ \ln\left[\sqrt{b^2 + A^2 - 2bA\cos\theta} + b - A\cos\theta\right] \\
& - \ln \left[\sqrt{a^2 + A^2 - 2aA\cos\theta} + a - A\cos\theta\right]\Big\} \\
= {} & b\ln\frac{R_{bB} + B - b\cos\theta}{R_{bA} + A - b\cos\theta} - a\ln\frac{R_{aB} + B - a\cos\theta}{R_{aA} + A - a\cos\theta} \\
& + B\ln\frac{R_{bB} + b - B\cos\theta}{R_{aB} + a - B\cos\theta} - A\ln\frac{R_{Ab} + b - A\cos\theta}{R_{aA} + a - A\cos\theta}
\end{aligned}
\tag{5.43}
$$

The beginning and ending coordinates of the two lines are denoted as a,b for l_1 and A,B for l_2. The distances R_{ij} are the distances between the endpoints of the segments. For the problem in Fig. 5.17, we obtain

$$
\begin{aligned}
N &= \left[l_1 \int_{l_2} \frac{1}{R_{12}}\, dl_2 + l_2 \int_{l_1} \frac{1}{R_{12}}\, dl_1\right] \\
&= \left\{ \left[l_1 \int_{l_2=0}^{l} \frac{1}{R_{12}} dl_2\right]_{l_1=0}^{m} + \left[l_2 \int_{l_1=0}^{m} \frac{1}{R_{12}} dl_1\right]_{l_2=0}^{l} \right\} \\
&= m\ln\frac{R_{ml} + l - m\cos\theta}{R_{m0} + 0 - m\cos\theta} - 0\ln\frac{R_{0l} + l - 0\cos\theta}{R_{00} + 0 - 0\cos\theta} \\
&\quad + l\ln\frac{R_{ml} + m - l\cos\theta}{R_{0l} + 0 - l\cos\theta} - 0\ln\frac{R_{0m} + m - 0\cos\theta}{R_{00} + 0 - 0\cos\theta} \\
&= l\ln\frac{R + m - l\cos\theta}{l - l\cos\theta} + m\ln\frac{R + l - m\cos\theta}{m - m\cos\theta}
\end{aligned}
\tag{5.44}
$$

and, by using l’Hôpital’s rule,

$$
\underbrace{\lim}_{x\to 0} x\ln(x) = 0 \tag{D605}
$$

The distance between the endpoints is denoted as

$$\begin{aligned} R &= R_{ml} \\ &= \sqrt{l^2 + m^2 - 2\,l\,m\,\cos\theta} \end{aligned} \tag{5.45}$$

and $R_{m0} = m$ and $R_{0l} = l$. Substituting the result in (5.44) into (5.42) gives the mutual partial inductance between the two segments in Fig. 5.17:

$$\boxed{M_p = \frac{\mu_0}{4\pi}\cos\theta\left(l\ \ln\frac{R + m - l\cos\theta}{l - l\cos\theta} + m\ \ln\frac{R + l - m\cos\theta}{m - m\cos\theta}\right)} \tag{5.46a}$$

But this result can be put into an equivalent form as [14]

$$\boxed{M_p = \frac{\mu_0}{4\pi}\cos\theta\left(l\ \ln\frac{R + m + l}{R + l - m} + m\ \ln\frac{R + l + m}{R + m - l}\right)} \tag{5.46b}$$

To demonstrate the equivalence between the two forms of the result in (5.46) we need to show that

$$\frac{R + m - l\cos\theta}{l - l\cos\theta} = \frac{R + m + l}{R + l - m} \tag{5.47}$$

This can be shown directly by multiplying it out to give

$$\begin{aligned} &R^2 + R\,(l - l\cos\theta) + (m - l\cos\theta)\,(l - m) \\ &\quad \stackrel{?}{=} R\,(l - l\cos\theta) + (m + l)\,(l - l\cos\theta) \end{aligned}$$

or

$$\begin{aligned} R^2 &\stackrel{?}{=} (m + l)(l - l\cos\theta) - (m - l\cos\theta)(l - m) \\ &= l^2 + m^2 - 2ml\cos\theta \end{aligned}$$

which is satisfied.

In fact, a more general result can be proven which will be useful for other situations. Consider the triangles shown in Fig. 5.18. Each triangle is composed of two sides labeled R and R' with included angles θ and θ' with respect to the horizontal axes. These sides R and R' make projections on the horizontal axes of P and P', respectively, where $P = R\cos\theta$ and $P' = R'\cos\theta'$. The total length on the horizontal axis between the intersections of each line with the horizontal axis is denoted as T. We can prove the following important

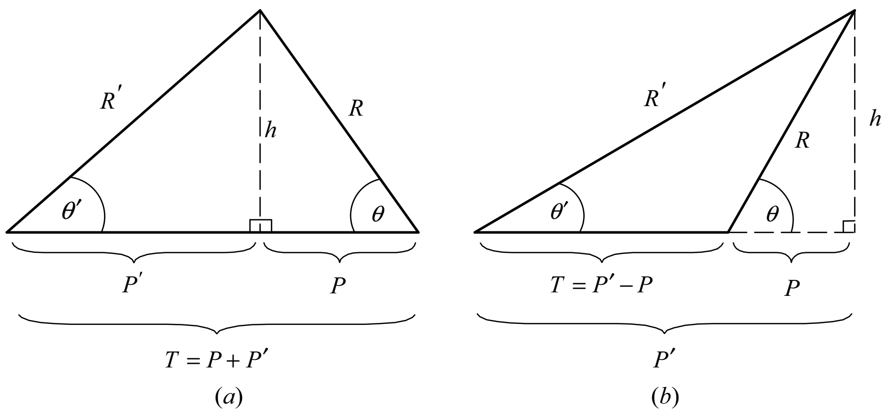


FIGURE 5.18. Important theorem.

equivalences. For the left triangle in Fig. 5.18(a) we have

$$\begin{aligned}\ln\frac{R+P}{R'-P'} &= \ln\frac{R'+P'}{R-P}\\ &= \ln\frac{R+R'+T}{R+R'-T}\\ &= 2\tanh^{-1}\frac{T}{R+R'}\end{aligned}\tag{5.48a}$$

and for the right triangle in Fig. 5.18(b) we have

$$\begin{aligned}\ln\frac{R-P}{R'-P'} &= \ln\frac{R'+P'}{R+P}\\ &= \ln\frac{R+R'+T}{R+R'-T}\\ &= 2\tanh^{-1}\frac{T}{R+R'}\end{aligned}\tag{5.48b}$$

The conversion of (5.48a) to (5.48b) is accomplished simply by replacing P in (5.48a) with $-P$. This is somewhat evident since P in Fig. 5.18(a) adds to P' to give the total length between the endpoints of R and R', which is denoted as $T = P' + P$, whereas in Fig. 5.18(b) P subtracts from P' to give $T = P' - P$. The identity for Fig. 5.17 in (5.47) follows from the identity in (5.48a).

The proofs of (5.48) are fairly simple by comparing the arguments of the log functions. For example, (5.48a) gives

$$\frac{R+P}{R'-P'} \stackrel{?}{=} \frac{R'+P'}{R-P} \stackrel{?}{=} \frac{R+R'+T}{R+R'-T}$$

Multiplying these out gives

$$(R+P)(R-P) \stackrel{?}{=} (R'+P')(R'-P')$$

But $R^2 = P^2 + h^2$ and $R'^2 = P'^2 + h^2$. Substituting $T = P + P'$, we need to show that

$$\frac{R+P}{R'-P'} \stackrel{?}{=} \frac{R+R'+(P+P')}{R+R'-(P+P')}$$

Multiplying this out and canceling common terms gives

$$R^2 - P^2 \stackrel{?}{=} R'^2 - P'^2$$

which is satisfied. The results in (5.48b) can be verified similarly. The last results in (5.48) are verified using the identity for the inverse hyperbolic tangent:

$$\tanh^{-1} x = \frac{1}{2} \ln \frac{1+x}{1-x} \qquad x^2 < 1 \tag{D702}$$

Hence, a further equivalent form for the result in (5.46) for Fig. 5.17 can be obtained in terms of the inverse hyperbolic tangent as

$$\boxed{\begin{aligned} M_p &= \frac{\mu_0}{4\pi} \cos\theta \left(l \ln \frac{R+m+l}{R+l-m} + m \ln \frac{R+l+m}{R+m-l} \right) \\ &= \frac{\mu_0}{2\pi} \cos\theta \left(l \tanh^{-1} \frac{m}{R+l} + m \tanh^{-1} \frac{l}{R+m} \right) \end{aligned}} \tag{5.46c}$$

This solution process for the configuration of Fig. 5.17 can readily be adapted to obtain the mutual partial inductance between two segments that do not physically join at a common point but are inclined at an angle θ to each other as shown in Fig. 5.19. Extend the segments of lengths l and m to a point where they join, thereby generating the extension lengths α and β. Adapting

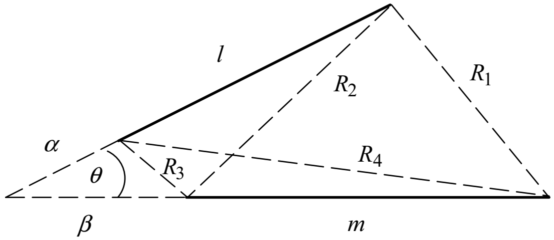


FIGURE 5.19. More general case of Fig. 5.17.

the result in (5.43) gives the result for Fig. 5.19 as

$$
\begin{aligned}
N &= l_1 \int_{l_2} \frac{1}{R_{12}}\, dl_2 + l_2 \int_{l_1} \frac{1}{R_{12}}\, dl_1 \\
&= \left\{ \left[l_1 \int_{l_2=\alpha}^{\alpha+l} \frac{1}{R_{12}} dl_2 \right]_{l_1=\beta}^{\beta+m} + \left[l_2 \int_{l_1=\beta}^{\beta+m} \frac{1}{R_{12}} dl_1 \right]_{l_2=\alpha}^{\alpha+l} \right\} \\
&= (\beta + m)\ \ln \frac{R_{(\beta+m)(\alpha+l)} + (\alpha + l) - (\beta + m)\ \cos\theta}{R_{(\beta+m)\alpha} + \alpha - (\beta + m)\ \cos\theta} \\
&\quad - \beta \ln \frac{R_{\beta(\alpha+l)} + (\alpha + l) - \beta \cos\theta}{R_{\beta\alpha} + \alpha - \beta \cos\theta} \\
&\quad + (\alpha + l)\ \ln \frac{R_{(\beta+m)(\alpha+l)} + (\beta + m) - (\alpha + l)\ \cos\theta}{R_{\beta(\alpha+l)} + \beta - (\alpha + l)\ \cos\theta} \\
&\quad - \alpha \ln \frac{R_{\alpha(\beta+m)} + (\beta + m) - \alpha \cos\theta}{R_{\beta\alpha} + \beta - \alpha \cos\theta}
\end{aligned}
\tag{5.49}
$$

Denoting the distances between the endpoints of the lines as shown in Fig. 5.19 gives

$$
\begin{aligned}
R_1 &= R_{(\alpha+l)(\beta+m)} \\
&= \sqrt{(\alpha + l)^2 + (\beta + m)^2 - 2\,(\alpha + l)\,(\beta + m)\ \cos\theta}
\end{aligned}
\tag{5.50a}
$$

$$
\begin{aligned}
R_2 &= R_{(\alpha+l)\beta} \\
&= \sqrt{(\alpha + l)^2 + \beta^2 - 2\,(\alpha + l)\,\beta \cos\theta}
\end{aligned}
\tag{5.50b}
$$

$$
\begin{aligned}
R_3 &= R_{\alpha\beta} \\
&= \sqrt{\alpha^2 + \beta^2 - 2\alpha\beta \cos\theta}
\end{aligned}
\tag{5.50c}
$$

$$
\begin{aligned}
R_4 &= R_{\alpha(\beta+m)} \\
&= \sqrt{\alpha^2 + (\beta + m)^2 - 2\alpha\,(\beta + m)\ \cos\theta}
\end{aligned}
\tag{5.50d}
$$

Hence, the mutual partial inductance between the two segments of Fig. 5.19 becomes

$$\boxed{\begin{aligned} M_p &= \frac{\mu_0}{4\pi} N \\ &= \frac{\mu_0}{4\pi} \left[(\beta + m) \ln \frac{R_1 + (\alpha + l) - (\beta + m)\cos\theta}{R_4 + \alpha - (\beta + m)\cos\theta} \right. \\ &\quad - \beta \ln \frac{R_2 + (\alpha + l) - \beta\cos\theta}{R_3 + \alpha - \beta\cos\theta} \\ &\quad + (\alpha + l) \ln \frac{R_1 + (\beta + m) - (\alpha + l)\cos\theta}{R_2 + \beta - (\alpha + l)\cos\theta} \\ &\quad \left. - \alpha \ln \frac{R_4 + (\beta + m) - \alpha\cos\theta}{R_3 + \beta - \alpha\cos\theta} \right] \end{aligned}} \tag{5.51a}$$

Using the identities in (5.48) and the inverse hyperbolic tangent identity in (D702) gives equivalent forms as

$$\boxed{\begin{aligned} M_p &= \frac{\mu_0}{4\pi} N \\ &= \frac{\mu_0}{4\pi} \left[(\beta + m) \ln \frac{R_1 + R_4 + l}{R_1 + R_4 - l} - \beta \ln \frac{R_2 + R_3 + l}{R_2 + R_3 - l} \right. \\ &\quad \left. + (\alpha + l) \ln \frac{R_1 + R_2 + m}{R_1 + R_2 - m} - \alpha \ln \frac{R_4 + R_3 + m}{R_4 + R_3 - m} \right] \\ &= \frac{\mu_0}{2\pi} \left[(\beta + m) \tanh^{-1} \frac{l}{R_1 + R_4} - \beta \tanh^{-1} \frac{l}{R_2 + R_3} \right. \\ &\quad \left. + (\alpha + l) \tanh^{-1} \frac{m}{R_1 + R_2} - \alpha \tanh^{-1} \frac{m}{R_4 + R_3} \right] \end{aligned}} \tag{5.51b}$$

We can then use the previous result for the mutual partial inductance between two segments of lengths x and y that are joined at a common point that was derived for Fig. 5.17 and given in (5.46c) to obtain the mutual partial inductance for Fig. 5.19 indirectly. Denote the result for Fig. 5.17 as

$$\begin{aligned} M_{x,y} &= \frac{\mu_0}{4\pi} \cos\theta \left(x \ln \frac{R + x + y}{R + x - y} + y \ln \frac{R + y + x}{R + y - x} \right) \\ &= \frac{\mu_0}{2\pi} \cos\theta \left(x \tanh^{-1} \frac{y}{R + x} + y \tanh^{-1} \frac{x}{R + y} \right) \end{aligned} \tag{5.52a}$$

where θ is the included angle where they are joined and

$$R = \sqrt{x^2 + y^2 - 2xy\cos\theta} \tag{5.52b}$$

Visualize the structure of Fig. 5.19 as consisting of four such structures, each consisting of the following lengths, with each pair being joined at a common point: (1) $x = \alpha + l$, $y = \beta + m$, (2) $x = \alpha$, $y = \beta$, (3) $x = \alpha + l$, $y = \beta$, and (4) $x = \alpha$, and $y = \beta + m$. We can then obtain the total mutual partial inductance for the structure in Fig. 5.19 of overall lengths $\alpha + l$ and $\beta + m$ to give, in a fashion similar to that of Fig. 5.16,

$$M_{\alpha+l,\beta+m} = M_{\alpha,\beta} + M_{\alpha,m} + M_{l,\beta} + M_{l,m}$$

The desired result is $M_{l,m}$ giving

$$M_{l,m} = M_{\alpha+l,\beta+m} - M_{\alpha,\beta} - M_{\alpha,m} - M_{l,\beta}$$

But the result for Fig. 5.17 does not apply to generating $M_{\alpha,m}$ or $M_{l,\beta}$ since the two lengths in each of these are not joined at a common point. So we write this as

$$M_{l,m} = M_{\alpha+l,\beta+m} - M_{\alpha,\beta} - \underbrace{\left(M_{\alpha,m} + M_{\alpha,\beta}\right)}_{M_{\alpha,\beta+m}} - \underbrace{\left(M_{l,\beta} + M_{\alpha,\beta}\right)}_{M_{\alpha+l,\beta}} + 2M_{\alpha,\beta}$$

giving

$$\boxed{M_p = \left(M_{\alpha+l,\beta+m} + M_{\alpha\beta}\right) - \left(M_{\alpha+l,\beta} + M_{\beta+m,\alpha}\right)} \tag{5.53}$$

Using the result for two segments joined at one end in (5.52) gives the result as

$$\begin{aligned}
M_p = \frac{\mu_0}{4\pi}\cos\theta \Bigg[&(\alpha+l)\ \ln\frac{R_1 + (\alpha+l) + (\beta+m)}{R_1 + (\alpha+l) - (\beta+m)} \\
&+ (\beta+m)\ \ln\frac{R_1 + (\beta+m) + (\alpha+l)}{R_1 + (\beta+m) - (\alpha+l)} \\
&+\alpha\ \ln\frac{R_3 + \alpha + \beta}{R_3 + \alpha - \beta} + \beta\ \ln\frac{R_3 + \beta + \alpha}{R_3 + \beta - \alpha} \\
&- (\alpha+l)\ \ln\frac{R_2 + (\alpha+l) + \beta}{R_2 + (\alpha+l) - \beta} - \beta\ \ln\frac{R_2 + \beta + (\alpha+l)}{R_2 + \beta - (\alpha+l)} \\
&- (\beta+m)\ \ln\frac{R_4 + (\beta+m) + \alpha}{R_4 + (\beta+m) - \alpha} - \alpha\ \ln\frac{R_4 + \alpha + (\beta+m)}{R_4 + \alpha - (\beta+m)}\Bigg]
\end{aligned} \tag{5.54a}$$

This result can be simplified to

$$\boxed{\begin{aligned} M_p = \frac{\mu_0}{4\pi} \cos\theta \Bigg[&(\alpha + l) \ln \frac{R_1 + R_2 + m}{R_1 + R_2 - m} + (\beta + m) \ln \frac{R_1 + R_4 + l}{R_1 + R_4 - l} \\ &-\alpha \ln \frac{R_3 + R_4 + m}{R_3 + R_4 - m} - \beta \ln \frac{R_2 + R_3 + l}{R_2 + R_3 - l} \Bigg] \end{aligned}} \tag{5.54b}$$

which agrees with (5.51b).

The equivalence of (5.54a) and (5.54b) can be shown with the following important identity for triangles. Consider the three triangles shown in Fig. 5.20. Triangle T_1 has sides of a, b, and R_1. Triangle T_2 has sides of a, c, and R_2. Triangle T_3 has sides of R_1, $b - c$, and R_2 and is formed from triangles T_1 and T_2 as $T_3 = T_1 - T_2$. It is a simple matter to prove the following

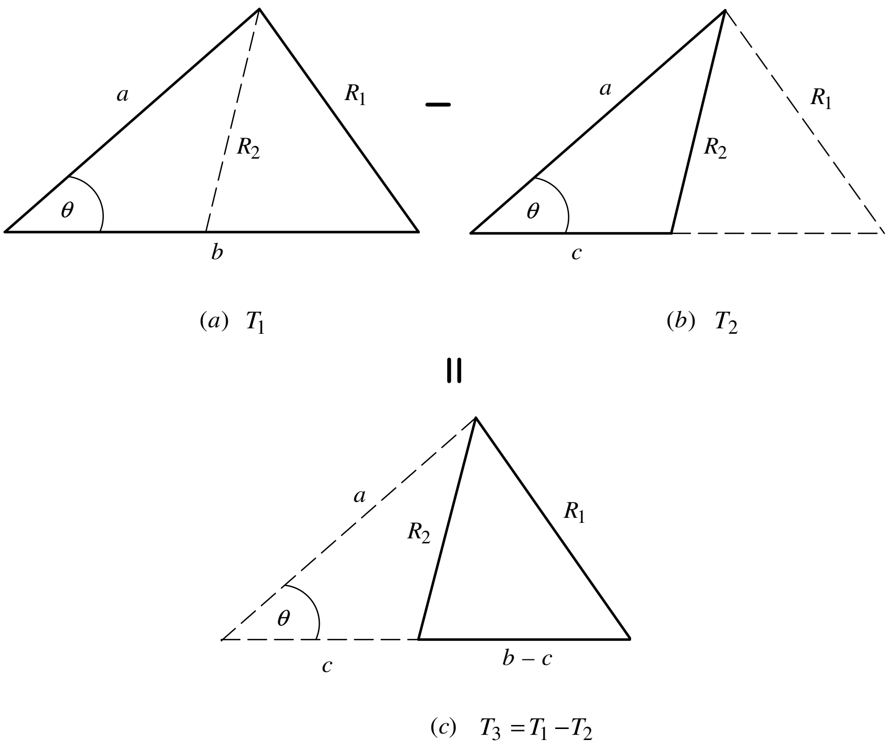


FIGURE 5.20. Important identity for triangles.

identity for these three related triangles:

$$\ln \frac{R_1 + a + b}{R_1 + a - b} - \ln \frac{R_2 + a + c}{R_2 + a - c} = \ln \frac{R_1 + R_2 + (b - c)}{R_1 + R_2 - (b - c)} \tag{5.55a}$$

and R_1 and R_2 are given by the law of cosines:

$$R_1^2 = a^2 + b^2 - 2ab\cos\theta \tag{5.55b}$$

$$R_2^2 = a^2 + c^2 - 2ac\cos\theta \tag{5.55c}$$

The important identity in (5.55a) can easily be verified by multiplying out the arguments of the logarithms as $\ln A - \ln B = \ln C \Rightarrow A/B = C$. Applying (5.55a) to (5.54a) gives the equivalence to (5.54b).

Figure 5.21 shows the general configuration for skewed and displaced conductors. The general result for this was derived by G.A. Campbell in 1915 [17]. This figure is modeled after that of Grover [14], pp. 56, who clearly explained the general result obtained by Campbell. The first conductor is of length l and its endpoints are denoted as A and B. It is shown as lying in a plane. The second conductor is of length m and its endpoints are denoted as a and b. It is shown as lying in another plane. These two planes containing the two conductors are parallel and separated by distance d between the two

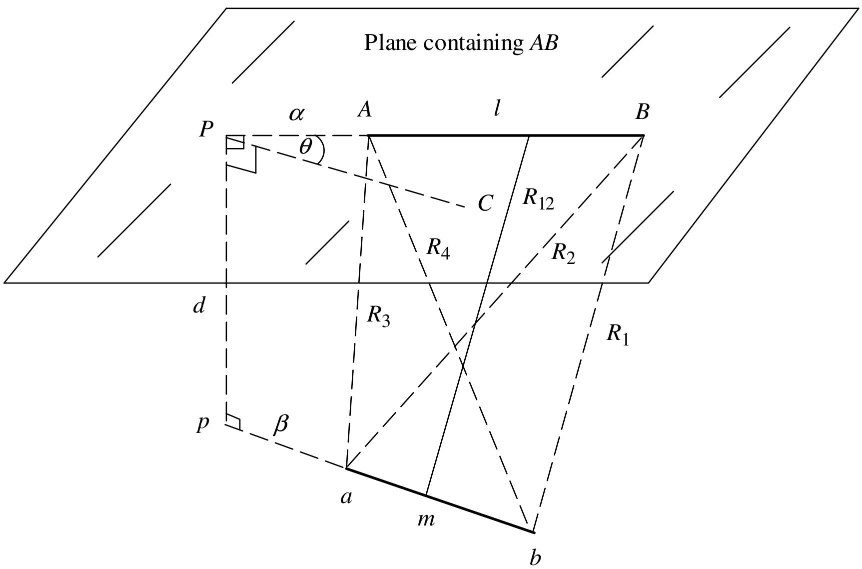


FIGURE 5.21. General configuration for skewed and displaced conductors.

planes. The line Pp between the two planes is of length d and is mutually perpendicular to the two planes containing the two conductors. Hence, Pp is said to be the *common perpendicular* to the two conductors. The endpoints of the conductors, A and a, are displaced from points P and p by distances α and β, respectively. The line PC lying in the plane containing AB is parallel to the line ab representing the second conductor and is at an angle θ to the first conductor AB. This is what is meant by the two conductors AB and ab having an angle of inclination of θ with respect to each other. If the displacement between the planes, d, is zero, $d = 0$, then the angle θ between the two conductors is the same as in the previous results.

The Neumann integral in (5.36) remains the same for this case:

$$M_p = \frac{\mu_0}{4\pi} \cos\theta \int_{l_1} \int_{l_1} \frac{1}{R_{12}}\, dl_1\, dl_2 \tag{5.56a}$$

where l_1 and l_2 again denote the contours along the two conductors of lengths m and l, respectively, and

$$R_{12} = \sqrt{d^2 + l_1^2 + l_2^2 - 2l_1 l_2 \cos\theta} \tag{5.56b}$$

Carrying through with a similar development as before gives

$$\begin{aligned}
M_p &= \frac{\mu_0}{4\pi} \cos\theta \int_{l_2} \int_{l_1} \frac{1}{R_{12}}\, dl_1\, dl_2 \\
&= \frac{\mu_0}{4\pi} \cos\theta \int_{l_2} \int_{l_1} \left[\frac{d}{dl_1} \left(\frac{l_1}{R_{12}} \right) + \frac{d}{dl_2} \left(\frac{l_2}{R_{12}} \right) - \frac{d^2}{R_{12}^3} \right] dl_1\, dl_2 \\
&= \frac{\mu_0}{4\pi} \cos\theta \left(l_1 \int_{l_2} \frac{1}{R_{12}}\, dl_2 + l_2 \int_{l_1} \frac{1}{R_{12}}\, dl_1 \right. \\
&\qquad \left. - \frac{d}{\sin\theta} \int_{l_2} \int_{l_1} \frac{d\ \sin\theta}{R_{12}^3}\, dl_1\, dl_2 \right)
\end{aligned} \tag{5.57a}$$

and one can similarly show using R_{12} in (5.56b), as was done previously for $d = 0$, that

$$\frac{d}{dl_1} \left(\frac{l_1}{R_{12}} \right) + \frac{d}{dl_2} \left(\frac{l_2}{R_{12}} \right) - \frac{d^2}{R_{12}^3} = \frac{1}{R_{12}} \tag{5.57b}$$

Again, the last result in (5.57a) can be integrated, using (D380.001), to yield

$$
\begin{aligned}
M_p &= \frac{\mu_0}{4\pi}\cos\theta\left(l_1\int_{l_2}\frac{1}{R_{12}}\,dl_2 + l_2\int_{l_1}\frac{1}{R_{12}}\,dl_1\right.\\
&\quad\left.-\frac{d}{\sin\theta}\int_{l_2}\int_{l_1}\frac{d\,\sin\theta}{R_{12}^3}\,dl_1\,dl_2\right)\\
&= \frac{\mu_0}{4\pi}\cos\theta\left[\left[l_1\ln(R_{12}+l_2-l_1\cos\theta)+l_2\ln(R_{12}+l_1-l_2\cos\theta)\right.\right.\\
&\quad\left.\left.-\frac{\Omega d}{\sin\theta}\right]_{l_2=PA}^{l_2=PB}\right]_{l_1=pa}^{l_1=pb}\\
&= \frac{\mu_0}{2\pi}\left(pB'\tanh^{-1}\frac{ab}{aB+Bb} - pA'\tanh^{-1}\frac{ab}{aA+Ab}\right.\\
&\quad\left.+Pb'\tanh^{-1}\frac{AB}{Ab+bB} - Pa'\tanh^{-1}\frac{AB}{Aa+aB} - \frac{\Omega d}{\tan\theta}\right)
\end{aligned}
\tag{5.58a}
$$

where the solid angle Ω is

$$
\begin{aligned}
\Omega &= \tan^{-1}\left(\frac{Pp}{Bb}\cot\theta + \frac{PB}{Pp}\frac{pb}{Bb}\sin\theta\right) - \tan^{-1}\left(\frac{Pp}{Ba}\cot\theta + \frac{PB}{Pp}\frac{pa}{Ba}\sin\theta\right)\\
&\quad - \tan^{-1}\left(\frac{Pp}{Ab}\cot\theta + \frac{PA}{Pp}\frac{pb}{Ab}\sin\theta\right)\\
&\quad + \tan^{-1}\left(\frac{Pp}{Aa}\cot\theta + \frac{PA}{Pp}\frac{pa}{Aa}\sin\theta\right)
\end{aligned}
\tag{5.58b}
$$

In (5.58a) primes denote the projection of the point on one conductor perpendicular to and onto the other conductor, and the inverse hyperbolic tangent is again defined in terms of the natural logarithm as

$$\tanh^{-1}x = \frac{1}{2}\ln\frac{1+x}{1-x} \qquad x^2 < 1 \tag{D702}$$

Grover [14] simplified this in terms of the quantities in Fig. 5.21 and the result becomes

$$
\boxed{
\begin{aligned}
M_p = \frac{\mu_0}{2\pi}\cos\theta&\left[(\alpha+l)\tanh^{-1}\frac{m}{R_1+R_2} + (\beta+m)\tanh^{-1}\frac{l}{R_1+R_4}\right.\\
&\left.-\alpha\tanh^{-1}\frac{m}{R_3+R_4} - \beta\tanh^{-1}\frac{l}{R_2+R_3}\right] - \frac{\mu_0}{4\pi}\frac{\Omega d}{\tan\theta}
\end{aligned}
}
\tag{5.59a}
$$

where the solid angle Ω is

$$\boxed{\begin{aligned}\Omega = \tan^{-1} &\frac{d^2 \cos\theta + (\alpha + l)(\beta + m)\sin^2\theta}{dR_1 \sin\theta} \\ - \tan^{-1} &\frac{d^2 \cos\theta + (\alpha + l)\,\beta \sin^2\theta}{dR_2 \sin\theta} \\ + \tan^{-1} &\frac{d^2 \cos\theta + \alpha\,\beta \sin^2\theta}{dR_3 \sin\theta} \\ - \tan^{-1} &\frac{d^2 \cos\theta + \alpha\,(\beta + m)\sin^2\theta}{dR_4 \sin\theta}\end{aligned}} \tag{5.59b}$$

The distances between the ends of the two conductors are shown in Fig. 5.21 and are $R_1 = Bb$, $R_2 = Ba$, $R_3 = Aa$, and $R_4 = Ab$. Using the law of cosines, these distances are

$$R_1^2 = d^2 + (\alpha + l)^2 + (\beta + m)^2 - 2\,(\alpha + l)\,(\beta + m)\,\cos\theta \tag{5.60a}$$

$$R_2^2 = d^2 + (\alpha + l)^2 + \beta^2 - 2\,\beta\,(\alpha + l)\,\cos\theta \tag{5.60b}$$

$$R_3^2 = d^2 + \alpha^2 + \beta^2 - 2\,\alpha\,\beta\,\cos\theta \tag{5.60c}$$

$$R_4^2 = d^2 + \alpha^2 + (\beta + m)^2 - 2\,\alpha\,(\beta + m)\,\cos\theta \tag{5.60d}$$

The only difference between this result and the result for Fig. 5.19 given in (5.51b) is the solid angle Ω, which goes away for $d = 0$.

5.7 NUMERICAL VALUES OF PARTIAL INDUCTANCES AND SIGNIFICANCE OF INTERNAL INDUCTANCE

It is helpful to obtain some representative values of the self and mutual partial inductances for typical configurations. The self partial inductance of a wire of radius r_w and length l is obtained as

$$\begin{aligned}L_p = 2 \times 10^{-7}\, l\, &\left[\ln\left(\frac{l}{r_w} + \sqrt{\left(\frac{l}{r_w}\right)^2 + 1}\right)\right. \\ &\left. - \sqrt{1 + \left(\frac{r_w}{l}\right)^2} + \frac{r_w}{l}\right]\end{aligned} \tag{5.18a}$$

Observe that this depends on the ratio of the wire length and the wire radius: l/r_w. It is typical to specify wire radii r_w in mils (1000 mils=1 in. and 1 in.=2.54 cm). Also observe that the length of the wire, l, also appears outside the equation. Hence, it is not possible to speak of an absolute per-unit-length inductance as is the case for a two-wire transmission line of infinite length. Nevertheless, we can divide both sides of (5.18a) by the wire length and obtain a universal plot of the ratio of self partial inductance per unit length, L_p/l, versus the ratio l/r_w as

$$\frac{L_p}{l} = 5.08 \left[\ln\left(\frac{l}{r_w} + \sqrt{\left(\frac{l}{r_w}\right)^2 + 1} \right) - \sqrt{1 + \left(\frac{r_w}{l}\right)^2} + \frac{r_w}{l} \right] \quad \text{nH/in.} \tag{5.61}$$

This is shown in Fig. 5.22 for ratios of $10 \le l/r_w \le 500$. For example, a No. 20 gauge (AWG) wire is a common wire size and has a radius of 16 mils. Hence, the last ratio of 500 plotted represents a wire length of 8 in. for a No. 20 gauge wire (30.02 nH/in.), and a ratio of 10 represents a length of 0.16 in., or about 3/16 in. (10.63 nH/in.). Because the wire length l appears outside the result, it is not possible to state a single per-unit-length value of the self partial inductance. But the plot in Fig. 5.22 indicates that a reasonable rule of thumb for practical wire sizes and wire lengths is a value of between 15 and 30 nH/in.

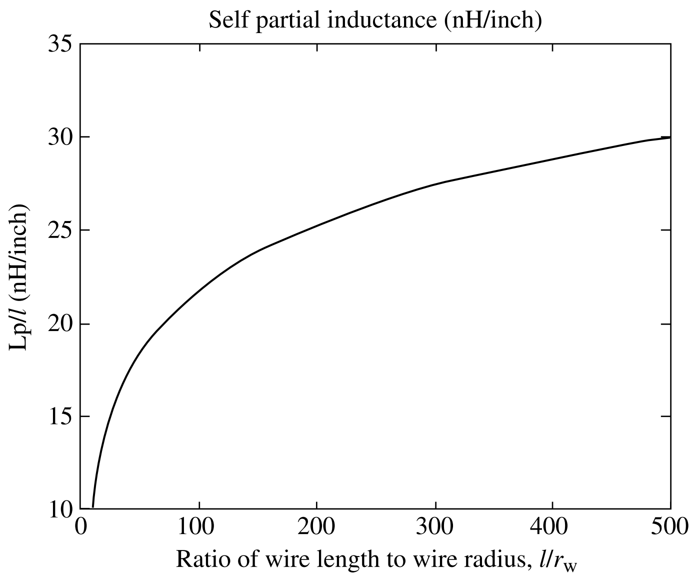


FIGURE 5.22. Plot of L_p/l (nH/in.) vs. the ratio of wire length to wire radius, l/r_w.

We obtained the dc per-unit-length value of the internal inductance of a wire (which is independent of the wire radius) as

$$\begin{aligned} l_{\text{internal}} &= \frac{\mu_0}{8\pi} \\ &= 0.5 \times 10^{-7}\,\text{H/m} \\ &= 1.27\,\text{nH/in.} \end{aligned} \tag{5.62}$$

Technically, this should be multiplied by the wire length and added to the external self partial inductance in (5.18a) to give the total self partial inductance:

$$L_{p,\text{total}} = L_{p,(5.18\text{a})} + l_{\text{internal}} \times l \tag{5.63}$$

But we see from Fig. 5.22 that for practical situations the internal inductance of the wire can generally be neglected. Furthermore, the value for the internal inductance in (5.62) is its value at dc. As frequency is increased from zero, the current tends to move toward the surface of the wire, and hence the internal inductance goes to zero. This gives further support to the observation that the internal inductance can generally be neglected.

The mutual partial inductance between two wires of common length l and separation d is obtained as

$$\begin{aligned} M_p = 2 \times 10^{-7}\, l \Bigg[&\ln\left(\frac{l}{d} + \sqrt{\left(\frac{l}{d}\right)^2 + 1}\right) \\ &- \sqrt{1 + \left(\frac{d}{l}\right)^2} + \frac{d}{l} \Bigg] \end{aligned} \tag{5.21a}$$

As was the case for self partial inductance, notice that this depends on the ratio of wire length to wire separation, l/d. But the wire length, l, also appears outside the result, so it is not possible to speak of an absolute value of per-unit-length mutual inductance, as is the case for a transmission line of infinite length. Nevertheless, we can divide both sides of (5.21a) by the wire length and obtain a universal plot of the ratio of the per-unit-length mutual partial inductance, M_p/l, versus the ratio l/d as

$$\frac{M_p}{l} = 5.08 \left[\ln\left(\frac{l}{d} + \sqrt{\left(\frac{l}{d}\right)^2 + 1}\right) - \sqrt{1 + \left(\frac{d}{l}\right)^2} + \frac{d}{l} \right] \qquad \text{nH/in.} \tag{5.64}$$

This is plotted in Fig. 5.23 for ratios of $1 \leq l/d \leq 100$. For example, a ratio of 80 would apply to two wires of length 5 in. and a separation between them of 0.0625 in., or 1/16 in. (20.77 nH/in.), and a ratio of 10 would apply to

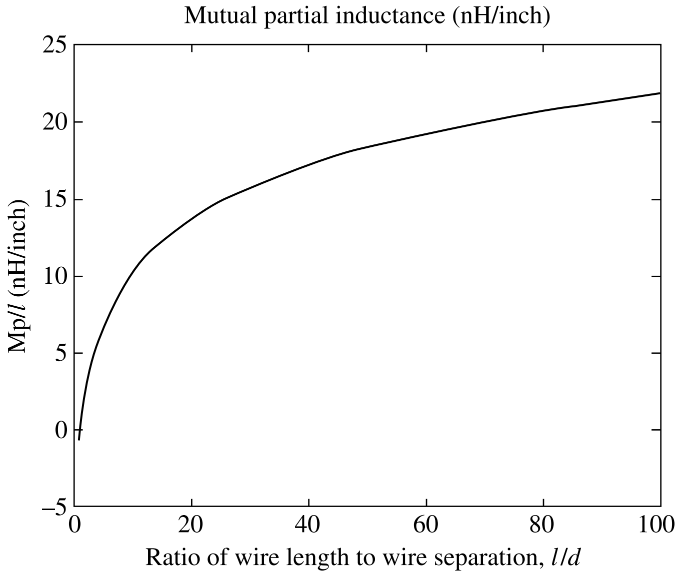


FIGURE 5.23. Plot of M_p/l in (nH/in.) vs. the ratio of wire length to wire separation l/d.

two wires of length 5 in. and a separation between them of 1/2 in. (10.63 nH/in.). Observe that as the wire separation increases without bound (i.e., the ratio goes to zero), the mutual partial inductance goes to zero: an expected result. Similarly, as the wire separation goes to zero (approaches the radii of the wires) (i.e., the ratio increases), the mutual partial inductance approaches the self partial inductance shown in Fig. 5.22: again, an expected result.

5.8 CONSTRUCTING LUMPED EQUIVALENT CIRCUITS WITH PARTIAL INDUCTANCES

Unlike the case of loop inductances, for current loops whose borders are bounded by piecewise-linear segments of wires, there are no further partial inductances to be derived. We simply "put together" the partial inductances (self and mutual) derived previously in this chapter and "turn the crank." We construct an equivalent lumped-circuit model that can be solved with, for example, the SPICE circuit analysis computer program [2]. To do so, we finally need to discuss the allocation of the dots in an inductor equivalent circuit of the segments. The key to doing so is to be able to determine correctly the total voltage developed across the segment, magnitude and polarity, by using the dot convention that replicates the derivation of that inductance in this chapter. The self partial inductance of a segment determines the voltage across

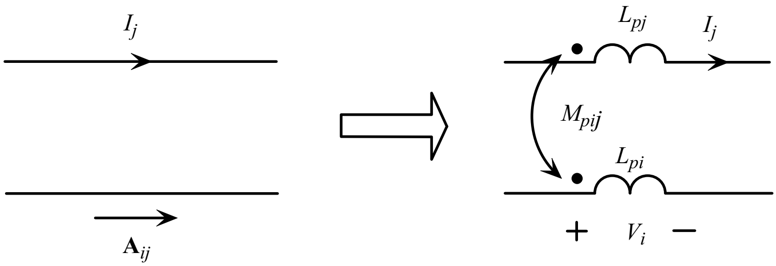


FIGURE 5.24. Mutual partial inductance between pairs of segments.

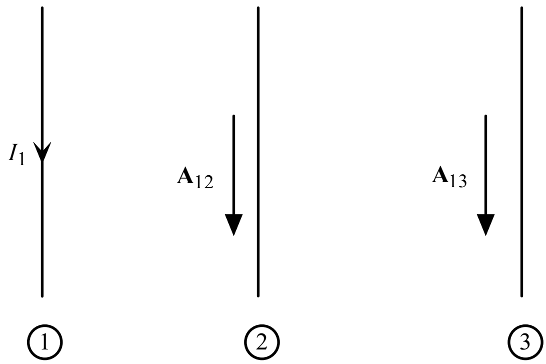
it that is due to the current through the element according to the passive sign convention: Current entering one end of the inductance produces a voltage across the element that is positive at that end. The dots do not have anything to do with this self voltage.

Let us first address the situation for a pair of segments shown in Fig. 5.24. The dots are placed on the ends of two elements so that the magnitude and polarity of the contribution to the voltage across one of the elements that is due to the current through the other element via the mutual partial inductance between the associated segments will be determined correctly. *The key to doing so is to replicate the situation for which the mutual partial inductance between two segments was as derived in this chapter.* Note that if a current I on one segment enters the dotted end of that segment, a voltage $M_p dI/dt$ will be developed along the other segment that is positive at the dotted end of that segment:

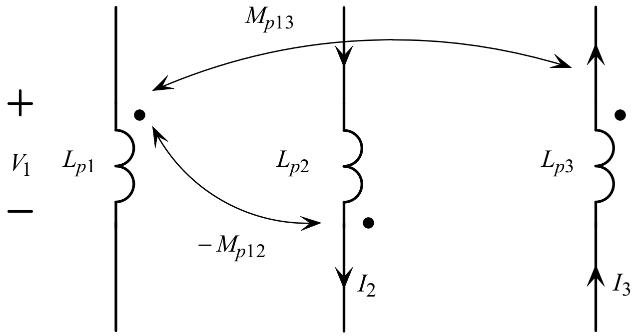
$$V_i = M_{pij}\frac{dI_j}{dt}$$

The key to getting the dots placed correctly *on a pair of segments* is observed to be in the relation between the current in one segment and the direction of the vector magnetic potential $\mathbf{A}$ along the other segment, which was used in the derivations of the mutual partial inductance. The vector magnetic potential $\mathbf{A}$ is everywhere parallel to the current that produced it. Hence, the positive terminal of the induced voltage is on the end of the segment that $\mathbf{A}$ enters, as shown in Fig. 5.24. In other words, $\mathbf{A}$ points *from* the positive terminal of the induced voltage *to* the negative terminal of the induced voltage.

This can be done easily for a pair of elements. For more than two coupled segments we must *arbitrarily* place the dots on the ends of the inductor symbols for each segment. But some of the mutual partial inductances may turn out to be *negative* for that placement. A good example of this is the rectangular loop shown in Fig. 5.1. The inductive equivalent circuit is shown in Fig. 5.3 and the dots are assigned arbitrarily. Observe that a current directed



(*a*)



(*b*)

FIGURE 5.25. Assigning dots to the segments.

down through the right segment of the loop produces a vector magnetic potential along the left side of the loop that is also directed downward. But this is opposite the assigned dotted terminal of the left inductor. Hence, M_{13} here is negative.

Figure. 5.25 shows an example of this. Assigning the dots *arbitrarily* gives the inductive equivalent circuit in Fig. 5.25(b). The voltage across the first inductor is assigned the polarity of positive at its dotted end. Directing a current I_1 through the first inductor that enters the assigned dotted end of it as shown in Fig. 5.25(a) generates vector magnetic potentials along the other segments that enters the dotted end of the third segment (assigned arbitrarily) so that the mutual inductance between the first and third inductors is positive as M_{p13}. But current I_1 that enters the dotted terminal of the first inductor (arbitrarily assigned) generates a vector magnetic potential that enters the undotted end of the second segment (assigned arbitrarily) so that the mutual partial inductance between that pair is $-M_{p12}$, where M_{p12} here has a positive value. Hence, using the dot convention, the voltage generated across the first inductor that is due only to the mutual inductances is

$$\begin{aligned} V_1 &= -\left(-M_{p12}\right)\frac{dI_2}{dt} - M_{p13}\frac{dI_3}{dt} \\ &= +M_{p12}\frac{dI_2}{dt} - M_{p13}\frac{dI_3}{dt} \end{aligned}$$

Computer-aided circuit analysis programs such as SPICE require that all self inductances such as L_p here must be positive. However, there are no restrictions on the signs of any of the mutual inductances: Some may have negative values.