Proceedings of the 25th Chinese Control Conference
7–11 August, 2006, Harbin, Heilongjiang

Practical Swing-up Control System Design of Cart-type Double Inverted Pendulum

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Abstract: This paper describes an RTLinux-based swing-up control system design of a cart-type double inverted pendulum experimental setup. First, 4 steps control strategy for the inverted pendulum system is introduced. Second, for real-time application of the system, RTLinux-based control of first step and second step is mainly concerned, where the proposed control scheme swings up the first pendulum with controlling the motion of a cart and is robust to the second pendulum. Finally, real-time experiment is given to show the effectiveness of the control scheme.

Key Words: Cart-type double inverted pendulum, Swing-up control, Real-time experiment

1 INTRODUCTION

In recent years, to be able to guarantee sufficiently precise sampling times and to assure that the operation including computations is excuted in a short sampling time, RTLinux operating system is employed. Using RTLinux OS, this paper focuses on the description of the development of a practical swing-up control scheme with four steps for a serial double inverted pendulum (Deng *et al.*, 2006). The proposed control system combines some controllers in Inoue *et al.* (2004), Henmi *et al.* (2004), and Inoue *et al.* (2005) and the new control strategy shows a satisfied swing up control result. The 4 steps are described as follows, Step 1: to swing up the first pendulum by considering the serial second pendulum as parasitic dynamics, where the proposed control scheme swinging up the first pendulum with controlling the motion of a cart and being robust to the second pendulum. Step2: to stabilize the first pendulum and reduce swinging movement of the second pendulum. Step 3: to swing up the second pendulum while stabilizing the serial first pendulum at the upright position by sliding mode controller and energy controller, and Step 4: to stabilize the two pendulums around the unstable equilibrium state by sliding mode controller. In order to differentiate this contribution from the existing works, the main work in this paper is about real experimental system, where RTLinux OS is necessary so that the experimental system is able to obtain the exclusive utilization of processor within a restricted sampling time. In the experiment, first step and second step of the control strategy is mainly concerned.

The organization of this paper is as follows. In Section 2, problem setup is introduced. RTLInux-based inverted pendulum system is shown and RTLinux-based swing-up controller is designed in Section 3. Real-time experiment is given in Section 4.

IEEE Catalog Number: 06EX1310

This work is supported by Grant-in-Aid for Scientific Research from JSPS (Grant No. 16101005).

2 PROBLEM SETUP

In general, inverted pendulum system has several types, e.g., a single pendulum (Wiklund *et al.*, 1993), a double pendulum (Deng *et al.*, 2006), etc.. The use of the inverted pendulum is a traditional topic for verification and practice of various kinds of control theories, because the system is a nonlinear and underactuated mechanical system (Fantoni and Lozano, 2002) which it is hard to be controlled by real-time controller.

This paper considers a cart-type serial double pendulum. In the swing-up, the second pendulum is swinging at the top of the first pendulum and the effect of motion of the second pendulum works as strong disturbance to the control of the first pendulum. Hence the control scheme is required to be strongly robust to the disturbance, and this paper uses sliding mode control law as the controller, which is strongly robust to disturbance. In this paper, using RTLinux OS, the controller consists of two steps, Step 1: to swing up the first pendulum, Step 2: to stabilize the first pendulum at the upright position (Deng *et al.*, 2006).

In Section 3.1, RTLinux-based inverted pendulum is shown. Section 3.2 is for a controller of Step 1. A control scheme which swings up the first pendulum is given. For swinging up the first pendulum, the sliding mode control is applied. Section 3.3 is for a controller of Step 2. A control law to stabilize the first pendulum with sliding mode control is given. In Section 4, RTLinux-based real-time is given to show the effectiveness of the proposed scheme.

3 RTLINUX-BASED SWING-UP CONTROLLER DESIGN

3.1 RTLinux-based Inverted Pendulum System

RTLinux OS is used in the experiment in stead of Windows OS. In Windows OS, the control period has been controlled with 5[msec], and it is difficult to control a sampling period that is shorter than 5[msec]. In RTLinux, however, a shorter sampling period can be achieved. Then we will control an inverted pendulum system with small sampling period. RTLinux OS is real-time OS. The real-time OS is guaranteed that the processing of task turns without fail in the regulation time. RTLinux OS is what enhanced to use Linux hardware in real-time, and holds lose the function of Linux. In other words, RTLinux OS includes Linux OS function and a real-time OS function. However, it is not a real-time OS in strictness. It is only a scheduler and an interprocess communication that RTLinux OS offers. The computer, which is used to measure and to control, demands to process a real-time control calculation at high speed. RTLinux OS offers Linux OS as a virtual machine, and makes real-time processing coexist with processing as Linux OS by executing Linux task at the lowest priority level of real-time process. Real-time process means process to guarantee time limit between the beginning of processing timing and the end, which has a certain range. RTLinux OS is used for an inverted pendulum system control. In our lab, the composition of RTLinux system is shown in Fig. 1 (M: Motor; E: Encoder). RTLinux OS is roughly divided into the following two parts.

1) Real-time process controlling the inverted pendulum system

2) User process (Non-real time process) controlling real-time process and monitoring the state of the inverted pendulum system

User process and real-time process are as follows.

1) Control PC runs main program, monitors the data of encoder count value etc. and stores the data in the file.

2) Main program sends RT-FIFO to the start signal and the stop signal.

3) When RT-FIFO receives start signal, it makes thread start or stop temporarily according to the sampling period arbitrarily decided by the designer. When RT-FIFO receives the stop signal, it makes thread stop.

4) Thread is a main program of the inverted pendulum system control and reads encoder values from the pendulum system. Main program calculates control input with converting the value into pendulum angles and outputs the calculated control input value to the servo motor. On the other hand, it sends to RT-FIFO the data, which is pendulum angles, control input, etc.

5) Input voltage is converted into rotating torque, and the torque is transmitted on the chain, and the cart moves straight line. Then, the pendulum system is controlled.

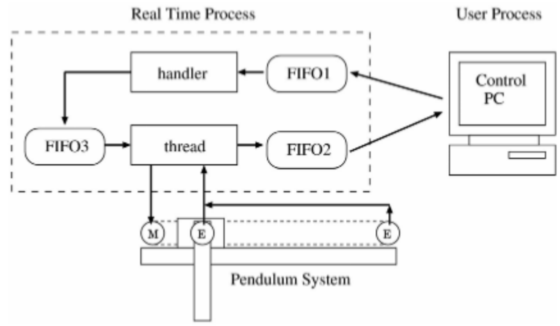

Figure 1: RTLinux System

6) The data that receives from thread is sent to main program and displayed on the screen in main program and stored in the file.

Where RT-FIFO is the part provided by RTLinux OS, and communicate between Real-time process and User process. RT-FIFO is fundamentally one direction communication.

3.2 Swing Up the First Pendulum with Considering the Second Pendulum (Step 1)

The dynamics equation for the serial double inverted pendulum system as depicted in Fig.3 is shown in Appendix. From the property of the inertia matrix, an important property that holds for the entire class of underactuated mechanical systems is the so-called collocated partial feedback linearization property. The collocated linearization refers to a control that linearizes the equations associated with the actuated degree of freedom q_2. Consider the equation (22),

$$M_{11}\ddot{q}_1 + M_{12}\ddot{q}_2 + C_1 + g_1 = 0 \qquad (1)$$

Also, from the uniform positive definiteness of the Matrix $M(q)$, the $l \times l$ matrix M_{11} with $l = n - m$ is invertible. Then, we have

$$\ddot{q}_1 = -M_{11}^{-1}(M_{12}\ddot{q}_2 + C_1 + g_1) \qquad (2)$$

Further, substituting (2) to equation (23), we obtain

$$\bar{M}_{22}\ddot{q}_2 + \bar{C}_2 + \bar{g}_2 = f \qquad (3)$$

where

$$\begin{aligned} \bar{M}_{22} &= M_{22} - M_{21}M_{11}^{-1}M_{12} \\ \bar{C}_2 &= C_2 - M_{21}M_{11}^{-1}C_1 \\ \bar{g}_2 &= g_2 - M_{21}M_{11}^{-1}g_1 \end{aligned}$$

and the $m \times m$ matrix $\bar{M}_{22}$ is symmetric and positive definite. As a result, a partial feedback linearizing controller can be obtained according to equation (3) as follows.

$$f = \bar{M}_{22} \cdot u + \bar{C}_2 + \bar{g}_2 \qquad (4)$$

where u was selected as new control input. The whole system can be rewritten as

$$\begin{aligned} M_{11}\ddot{q}_1 + C_1 + g_1 &= -M_{12}u \qquad (5) \\ \ddot{q}_2 &= u \qquad (6) \end{aligned}$$

Using the collocated linearization method, the original system (22), (23) is feedback equivalent to the system (5), (6). We consider a swing-up controller for (5) and (6) as follows.

$$f = -T_1 \text{sign}(-T_2 \text{sign}(d_1\theta + \dot{\theta}) + d_2\phi + \dot{\phi} + d_3 z + \dot{z}) \quad (7)$$
$$s_1 = d_1\theta + \dot{\theta} \quad (8)$$
$$s_2 = -T_2 \text{sign}(d_1\theta + \dot{\theta}) + d_2\phi + \dot{\phi} + d_3 z + \dot{z} \quad (9)$$

where, original controller f makes nonlinear system (5) and linear system (6) be stable. Then the values s_2 and $\dot{s_2}$ have different signs. Hence the plane $s_2 = 0$ is reached within a finite time interval. For the motion $d_2\phi + \dot{\phi} + d_3 z + \dot{z} = T_2 \text{sign}(d_1\theta + \dot{\theta})$, namely $s_2 = 0$, the sliding mode is satisfied by the condition of the linear system $\dot{z} + d_3 z + d_2\phi + \dot{\phi} = \tilde{\tau}$ being controllable by sliding mode control. The main reason is as follows. From the relation between $(\phi, \dot{\phi}, z, \dot{z})$ and $(\theta, \dot{\theta})$ described by system dynamics, instead of $(\phi, \dot{\phi}, z, \dot{z})$, equation $\dot{z} + d_3 z + d_2\phi + \dot{\phi} = \tilde{\tau}$ can be rewritten by $(\theta, \dot{\theta})$ related equation. Then we consider sliding mode controller $\tau = T_2 \text{sign}(d_1\theta + \dot{\theta})$ such that the system is stable. The detailed proof of stability is omitted.

According to the above explanation, after a finite time interval the state will reach the intersection of the planes $s_2 = 0$ and $d_1\theta + \dot{\theta} = 0$. Since $s_1 = 0$, the following first equation is obtained.

$$\dot{\theta} = -d_1\theta \quad (10)$$

The two-dimensional sliding mode is asymptotically stable, its order is two less than the order of the original system and the motion does not depend on the disturbances (Utkin *et al.*, 1999).

It is worthy to say that when the pendulum swung up in the neighborhood $\theta = 0$ by controller (7), the controller switched to the controller of Step 2. That is, $-\theta_0 \le \theta \le \theta_0$, where θ_0 is a small positive constant. Concerning the reason, besides the reason of stabilizing the first pendulum, the another reason is to avoid the time derivative of $\text{sign}(d_1\theta + \dot{\theta})$ at $d_1\theta + \dot{\theta} = 0$.

3.3 Stabilizing the First Pendulum and Controlling the Second Pendulum (Step 2)

In this section, a control method which stabilizes the pendulum is proposed.

For the stabilizing controller of the pendulum at the unstable equilibrium point, a state feedback stabilization controller is derived by using sliding mode controller (Deng *et al.*, 2006; Henmi *et al.*, 2004).

Due to the robustness of the sliding mode controller, the overall system with the parasitic effect can be stabilized by choosing the design parameter by a priori trial and error. In the following, the control law which stabilizes the pendulum is considered. By neglecting ϕ in (17), the dynamics of θ becomes

$$I_1\ddot{\theta} + (c_1 + c_2)\dot{\theta} - b_3 g \sin\theta + b_3 \cos\theta \ddot{z} = 0 \quad (11)$$

The control input f is defined as

$$f = \ddot{z} \quad (12)$$

The state variables are chosen as

$$\begin{aligned} x_2 &= [x_{21},\ x_{22},\ x_{23},\ x_{24}]^T \\ &= [\theta,\ \dot{\theta},\ z,\ \dot{z}]^T \end{aligned} \quad (13)$$

With (12) and (13), the linearized state-space equation of (11) around the unstable equilibrium point of the first pendulum ($(\theta, \dot{\theta}) = (0, 0)$) is

$$\begin{aligned} \dot{x}_2 &= A_2 x_2 + B_2 f \\ &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{b_3 g}{b_1} & -\frac{c_1+c_2}{b_1} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} x_2 + \begin{bmatrix} 0 \\ -\frac{b_3}{b_1} \\ 0 \\ 1 \end{bmatrix} f \end{aligned} \quad (14)$$

A sliding mode controller is used to keep robustness for stabilization of the first pendulum. The control input f which stabilizes (14) is

$$\begin{aligned} f &= -(S_2 B_2)^{-1}(S_2 A_2 x_2 + R_2 \text{sign}(\sigma_2) + K_2 \sigma_2) \\ \sigma_2 &= S_2 x_2 \end{aligned} \quad (15)$$

where $R_2 > 0, K_2 > 0$ and S_2 is the solution of the following Riccati equation with $\epsilon_2 > 0$

$$P_2(A_2 + \epsilon_2 I) + (A_2 + \epsilon_2 I)^T P_2 - P_2 B_2 B_2^T P_2 + Q_2 = 0$$
$$S_2 = B_2^T P_2$$

4 REAL-TIME EXPERIMENT

In order to show the performance of the proposed scheme in this paper, an experiment of the swinging up control of the considered system is conducted. The parameters in (17), (18) and (19) are selected to be those of an experimental system of a serial double inverted pendulums in our laboratory. These parameters and the parameters of the control law are given in Table 1.

The experiment is conducted by using the serial double pendulum system in order to confirm whether the first pendulum can be swung up and can be stabilized by the proposed scheme in this report. The experimental result is shown in Fig. 2. Fig. 2 illustrates the responses of $\theta,\ \dot{\theta},\ \phi,\ \dot{\phi},\ z,\ \dot{z},\ u$. The parameters of the control rule is shown in Table 3. The initial states of the system are given by $(\theta(0), \dot{\theta}(0), \phi(0), \dot{\phi}(0), z(0), \dot{z}(0)) = (\pi, 0, \pi, 0, 0, 0)$ and the time when controllers are switched from Step 1 to Step 2 is at the condition of $'\cos\theta > 0.9'$ being satisfied. The gain is decided through some experiments. The sampling period is 5[msec], because a large noise runs into the velocity element when the experiment is conducted in 1[msec] and we consider that it influences the control. From the experimental result, the first pendulum was swung up and stabilized at the unstable equilibrium point. Hence, the experiments were given to show the effectiveness of the proposed controller.

5 CONCLUSION

In this paper, an RTLinux-based swing-up control for a cart-type serial double inverted pendulum experimental system is proposed. Real-time experiment is given to show the effectiveness of the proposed scheme.

Table 1: Value of system parameters and control parameters

m_1	$0.18[kg]$	m_2	$0.10[kg]$
n_1	$0.078[kg]$	n_2	$0.05[kg]$
l_1	$0.105[m]$	l_2	$0.115[m]$
J_{n1}	$2.8 \times 10^{-5}[kgm^2]$	J_{n2}	$2.0 \times 10^{-6}[kgm^2]$
I_1	$0.0089[kgm^2]$	I_2	$0.0018[kgm^2]$
c_1	$0.0001[kgm^2/s]$	c_2	$0.002[kgm^2/s]$
L	$0.38[m]$	g	$9.8[m/s^2]$
M	$0.44[kg]$	T_1	70.0
T_2	40.0	d_1	30.0
d_2	30.0	d_3	70.0
K_2	1.0	R_2	5.0

Table 2: Value of Control Parameter

T_1	7.0	T_2	5.0
K_2	6.0	R_2	6.0
d_1	8.0	d_2	8.0
d_3	45.0		

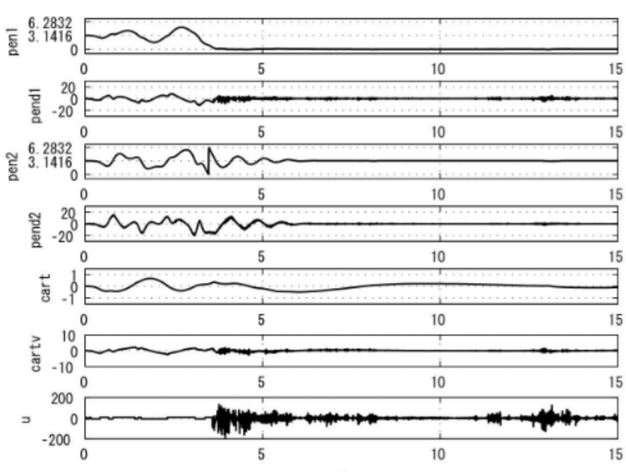

Figure 2: Results of the Experiment

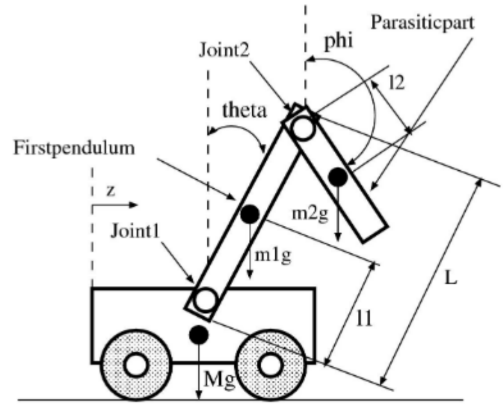

Figure 3: Illustration of the inverted pendulum system

Acknowledgment

The authors would like to thank technician Mr. N. Ueki and former student Mr. M. Kosugi at Okayama University for their contribution to this work.

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APPENDIX

The controlled system is a serial double pendulum as depicted in Fig. 3. The dynamics of the considered system is modelled by Lagrange method. The parameter of kinetic

energy (A_1, A_2), potential energy (B_1, B_2), and loss energy (C_1, C_2) of the first pendulum and the parasitic part, the kinetic energy (A_3) of the cart and the kinetic energy (A_4, A_5) of joint of the pendulums are shown as follow.
First Pendulum:

$$\begin{aligned} A_1 &= \frac{1}{2}J_1\dot{\theta}^2 + \frac{1}{2}m_1\left\{\frac{d}{dt}(z + l_1 \sin\theta)\right\}^2 \\ &\quad + \frac{1}{2}m_1\left\{\frac{d}{dt}(l_1 \cos\theta)\right\}^2 \\ B_1 &= m_1 g l_1 \cos\theta \\ C_1 &= \frac{1}{2}c_1\dot{\theta}^2 \end{aligned}$$

Parasitic Part:

$$\begin{aligned} A_2 &= \frac{1}{2}J_2\dot{\phi}^2 + \frac{1}{2}m_2\left\{\frac{d}{dt}(z + L\sin\theta + l_2\sin\phi)\right\}^2 \\ &\quad + \frac{1}{2}m_2\left\{\frac{d}{dt}(L\cos\theta + l_2\cos\phi)\right\}^2 \\ B_2 &= m_2 g(L\cos\theta + l_2\cos\phi) \\ C_2 &= \frac{1}{2}c_2(\dot{\theta} - \dot{\phi})^2 \end{aligned}$$

Cart:

$$\begin{aligned} A_3 &= \frac{1}{2}M\dot{z}^2 \\ B_3 &= 0 \end{aligned}$$

Joint between first pendulum and cart:

$$\begin{aligned} A_4 &= \frac{1}{2}I_{n1}\dot{\theta}^2 + \frac{1}{2}n_1\dot{z}^2 \\ B_4 &= 0 \end{aligned}$$

Joint of both pendulums:

$$\begin{aligned} A_5 &= \frac{1}{2}I_{n2}\dot{\phi}^2 + \frac{1}{2}n_1\left\{\frac{d}{dt}(z + L\sin\theta)\right\}^2 \\ &\quad + \frac{1}{2}n_1\left\{\frac{d}{dt}(L\cos\theta)\right\}^2 \\ B_5 &= n_2 g L\cos\theta \end{aligned}$$

Using these equations, the kinetic energy (A), potential energy (B), and loss energy (C) of the serial double inverted pendulum system are given by

$$A = \sum_i A_i, \quad B = \sum_i B_i, \quad C = \sum_i C_i$$

Therefore, the Lagrange equation of this system is shown by

$$\frac{d}{dt}\frac{\partial A}{\partial \dot{x}} - \frac{\partial A}{\partial x} + \frac{\partial B}{\partial x} + \frac{\partial C}{\partial \dot{x}} = 0 \tag{16}$$

where x is variable of (θ, ϕ, z).

The dynamics of considered system is given as

$$\begin{aligned} & b_1\ddot{\theta} + b_2\cos(\theta-\phi)\ddot{\phi} + b_2\sin(\theta-\phi)\dot{\phi}^2 \\ &\quad + (c_1+c_2)\dot{\theta} - c_2\dot{\phi} - b_3 g\sin\theta + b_3\cos\theta\ddot{z} = 0 \quad (17) \\ & b_2\cos(\theta-\phi)\ddot{\theta} + b_4\ddot{\phi} - b_2\sin(\theta-\phi)\dot{\theta}^2 \\ &\quad - c_2\dot{\theta} + c_2\dot{\phi} - b_5 g\sin\phi + b_5\cos\phi\ddot{z} = 0 \quad (18) \\ & b_3\cos\theta\ddot{\theta} + b_5\cos\phi\ddot{\phi} - b_5\sin\phi\dot{\phi}^2 \\ &\quad - b_3\sin\theta\dot{\theta}^2 + b_6\ddot{z} = f \quad (19) \end{aligned}$$

where $b_1, b_2, b_3, b_4, b_5, b_6$ are expressed as

$$\begin{aligned} b_1 &= I_1 + m_1 l_1^2 + m_2 L^2 + n_2 L^2 + I_{n1} \\ b_2 &= m_2 l_2 L \\ b_3 &= m_1 L_1 + m_2 L + n_2 L \\ b_4 &= I_2 + m_2 l_2^2 + I_{n2} \\ b_5 &= m_2 l_2 \\ b_6 &= M + m_1 + m_2 + n_1 + n_2 \end{aligned}$$

and the following notations are used through the paper (i=1, 2) :

- z : position of the cart
- θ : angular position of the first pendulum from the vertical line
- ϕ : angular position of the second pendulum from the vertical line
- m_i : mass of the pendulum or the parasitic part
- n_i : mass of the joint 1 and 2
- l_i : length from the i th joint to the center of mass of the pendulum or the parasitic part
- I_{ni} : inertia of mass of the i th joint around the center of gravity
- I_i : inertia of the pendulum or the parasitic part around the joint
- c_i : viscosity of each joint
- L : length of the first pendulum
- g : gravity acceleration
- M : mass of the cart

The equation for the displacement of the cart z[m] is approximated as

$$f = \ddot{z} \tag{20}$$

where f is the acceleration command input to the amplifier of the servo motor, the ratio of the attained acceleration [m/s^2] to the acceleration command input is 1.
The dynamics of the considering system (17), (18), (19) are arranged as follows.

$$M(q)\ddot{q} + C(q, \dot{q}) + g(q) = F \tag{21}$$

where

$$
\begin{aligned}
M(q) &= \begin{bmatrix} b_1 & b_2\cos(\theta-\phi) & b_3\cos\theta \\ b_2\cos(\theta-\phi) & b_4 & b_5\cos\phi \\ b_3\cos\theta & b_5\cos\phi & b_6 \end{bmatrix} \\
C(q,\dot{q}) &= \\
& \begin{bmatrix} c_1+c_2 & -c_2+b_2\sin(\theta-\phi) & 0 \\ -c_2-b_2\sin(\theta-\phi) & c_2 & 0 \\ -b_3\sin\theta\dot{\theta} & -b_5\sin\phi\dot{\phi} & 0 \end{bmatrix} \\
q &= \begin{bmatrix} \theta \\ \phi \\ z \end{bmatrix}, g(q) = \begin{bmatrix} -b_3 g\sin\theta \\ -b_5 g\sin\phi \\ 0 \end{bmatrix}, F = \begin{bmatrix} 0 \\ 0 \\ f \end{bmatrix}
\end{aligned}
$$

In the following, the so-called collocated partial feedback linearization method (Spong, 1996) is summarized. By partitioning the vector q, we get $q^T = [q_1^T, q_2^T]$, with q_1 corresponding to the passive and q_2 corresponding to the actuated variables. The Lagrange equations of the dynamics of an n -degree of freedom mechanical system with q_1 passive coordinates and q_2 actuated coordinates can be re-described in the following form:

$$M_{11}\ddot{q}_1 + M_{12}\ddot{q}_2 + C_1(q,\dot{q}) + g_1(q) = 0 \tag{22}$$

$$M_{21}\ddot{q}_1 + M_{22}\ddot{q}_2 + C_2(q,\dot{q}) + g_2(q) = f \tag{23}$$

where

$$
\begin{aligned}
M(q) &= \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}, C(q,\dot{q})\dot{q} = \begin{bmatrix} C_1(q,\dot{q}) \\ C_2(q,\dot{q}) \end{bmatrix} \\
q &= \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}, g(q) = \begin{bmatrix} g_1(q) \\ g_2(q) \end{bmatrix}
\end{aligned}
$$

and $M(q)$ is the symmetric, positive definite system inertia matrix. C is the vector of Coriolis and centripetal torques, and g is the vector derived from the potential energy, such as gravitational and elastic generalized forces. The vector f represents the input of the generalized forces produced by the m actuators at Yu *et al*(1994).