Project 1: Computational Physics - FYS3150

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1 Introduction

In this project we will investegate different approaches to solve the one-dimensional Poisson equation with Dirichlet boundary conditions given as follows:

$$u''(x) = f(x), \quad x \in (0,1), \quad u(0) = u(1) = 0$$

We will rewrite this as a set of linear equations, and solve it by a number of different computational approaches on either gaussian elimination or LU decomposition. We will solve the equation above with the function:

$$f(x) = 100e^{-10x}$$

Where the analytical solution then is given as:

$$u(x) = 1 - (1 - e^{-10})x - e^{-10x}$$

We will use the analytical solution to evaluate the precision of the numerical solutions for different steplength between the discretized gridpoints x_i.

2 Method

2.1 Rewritting the equation as a set of linear equations

In order to solve the Poisson equation numerically we discretize u as v_i with grid points $x_i = ih$ in the interval $x \in [x_0 = 0, x_1 = 1]$. We then have the step length $h = 1/(n+1)$. We use the following second derivative approximation

$$-u''(x_i) \approx -\frac{v_{i+1} + 2v_i - v_{i+1}}{h^2} = f(x_i)$$

$\iff$

$$-v_{i-1} + 2v_i - v_{i+1} = h^2 f(x_i)$$

We define the colum vector $\mathbf{v} = [v_1, v_2, \ldots, v_{n+1}]$ and try to setup the equation for every step i. As we do this we see a pattern appearing

$$\begin{bmatrix} 2 & -1 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} v_0 \\ \vdots \\ v_{n+1} \end{bmatrix} = h^2 f(x_0)$$

$$\begin{bmatrix} 2 & -1 & 0 & \cdots & \cdots & 0 \\ -1 & 2 & -1 & 0 & \cdots & \cdots \end{bmatrix} \begin{bmatrix} v_0 \\ \vdots \\ v_{n+1} \end{bmatrix} = h^2 \begin{bmatrix} f(x_0) \\ f(x_1) \end{bmatrix}$$

$$\vdots$$

$$\begin{bmatrix} 2 & -1 & 0 & \cdots & \cdots & 0 \\ -1 & 2 & -1 & 0 & \cdots & \cdots \\ 0 & -1 & 2 & -1 & 0 & \cdots \\ & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & & -1 & 2 & -1 \\ 0 & \cdots & & 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} v_0 \\ \vdots \\ v_{n+1} \end{bmatrix} = h^2 \begin{bmatrix} f(x_0) \\ f(x_1) \\ \vdots \\ f_{n+1} \end{bmatrix}$$

From this we see that we can write the problem as a linear set of equation:

$$\mathbf{A}\mathbf{v} = \tilde{\mathbf{g}}$$

With the following definitions:

$$\mathbf{A} = \begin{bmatrix} 2 & -1 & 0 & \cdots & \cdots & 0 \\ -1 & 2 & -1 & 0 & \cdots & \cdots \\ 0 & -1 & 2 & -1 & 0 & \cdots \\ & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & & -1 & 2 & -1 \\ 0 & \cdots & & 0 & -1 & 2 \end{bmatrix} \quad , \mathbf{v} = \begin{bmatrix} v_0 \\ v_1 \\ \vdots \\ v_{n+1} \end{bmatrix} \quad , \tilde{\mathbf{g}} = h^2 \begin{bmatrix} f(x_0) \\ f(x_1) \\ \vdots \\ f_{n+1} \end{bmatrix}$$

In this project we use $f(x) = 100e^{-10x}$. The solution for the Poisson equation in this case is given to be $u(x) = 1 - (1 - e^{-10})x - e^{-10x}$. We can ensure that this is true by inserting it and checking that the equarion holds remains true. First find the double derivative of u(x):

$$u'(x) = -(1 - e^{-10}) + 10e^{-10x}, \quad u''(x) = -100e^{-10x}$$

We now see that the solution satisfy the Poisson equation:

$$-u''(x) = 100e^{-10x} = f(x)$$

2.2 General solution using Gausian elimination

Forward / Backward sub FLOPS

2.3 Simplified problem specific solution

FLOPS

2.4 LU decomposition

FLOPS

2.5 Comparing precision and error

3 Implementation?

4 Results

5 Concluding remarks

6 Part a