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# Mathematical and numerical models for coupling surface and groundwater flows

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**Abstract**

In this paper we present some results about the coupling of Navier–Stokes and Shallow Water equations for surface flows and Darcy's equation for groundwater flows. We discuss suitable interface conditions and show the well-posedness of the coupled problem in the case of a linear Stokes problem. An iterative method to compute the solution is proposed. At each step this method requires the solution of one problem in the fluid part and one in the porous medium. Finally we introduce the Steklov–Poincaré equation associated to the coupled problem.


*Keywords:* Navier–Stokes equations; Darcy's law; Interface conditions; Domain decomposition; Steklov–Poincaré operators

## 1. Introduction

In this paper we consider the coupling between the surface and the subsurface motion of a fluid. In particular we consider two problems in which the lower part of the computational domain is formed by a porous medium while in the upper part we can have either a free-surface fluid or a confined fluid.

The behaviour of the fluid in the two regions is described by different partial differential equations: the Free Surface Navier–Stokes Equations or the Navier–Stokes system for the flow in the upper region, and Darcy's law in the lower one. This gives rise to coupled heterogeneous problems. To close the problems, in both cases we have to introduce suitable interface conditions relating the unknowns from the two subdomains across the interface.

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These coupled models have interesting applications. They can be used to simulate the effect of flooding in dry areas. When further coupled with transport-diffusion equations they can be used to study the propagation and diffusion of pollutants dispersed in water.

Moreover, similar models can be used to describe the behaviour of water in a basin due to the motion of a body (a ship, or a boat) beneath the free surface (see [4]). In fact such a problem can be studied by decomposing the computational domain into two parts. We have an upper region where the Navier–Stokes equations are used to describe the motion of water near the moving body; then we consider a deeper region where the effects due to the motion of the body can be omitted and simpler models, such as a Laplace equation for the velocity potential, can be adopted.

The above examples motivate our mathematical study of such coupled models and, in particular, our interest in finding effective numerical methods to solve them.

Having that in mind, we will first propose numerical approximations based on suitable finite element methods.

For the effective solution of the corresponding coupled finite-dimensional problem, we take inspiration from domain decomposition methods (see [17]). In fact we introduce a suitable Dirichlet–Neumann iterative method based on splitting interface conditions between upper and lower domains. In the linear case of the Stokes/Darcy coupling, we show that this iterative method converges with a rate independent of the grid size. This iterative method can actually be regarded as a preconditioned iterative procedure applied to the interface variables solely which are governed by a discrete Steklov–Poincaré operator.

The outline of the paper is as follows. In Section 2 we introduce the coupled models Shallow Water Equations/Darcy and Navier–Stokes/Darcy discussing in particular the interface conditions, referring to the existing literature. In Section 3 we present the idea at the basis of the iterative methods we use to compute the solutions of our problems. From Section 4 on, we consider the case of the Stokes/Darcy coupled problem. In Section 4 we write the weak form of the problem and prove its well-posedness, while in Section 5 we introduce a subdomain iterative method to solve it. The following two sections are dedicated to the Steklov–Poincaré operators associated to the problem. In particular in Section 6 we show that they are continuous and positive definite and in Section 7 we consider their discrete counterpart stemming from the finite element approximation of the coupled problem. Then we reformulate the Dirichlet–Neumann domain decomposition iteration as a preconditioned Richardson method, where the (optimal) preconditioner is obtained from a splitting of the local Steklov–Poincaré operators. Finally, we conclude by showing some numerical results which confirm the theoretical properties of the iterative method.

## 2. Modelling aspects

We consider a bounded domain $\Omega$ of $\mathbb{R}^3$ divided into two subdomains $\Omega_f$ and $\Omega_p$ such that $\overline{\Omega} = \overline{\Omega}_f \cup \overline{\Omega}_p$, $\Omega_f \cap \Omega_p = \emptyset$ and $\overline{\Omega}_f \cap \overline{\Omega}_p = \Gamma$. $\Gamma$ is the interface between the fluid and the porous medium. We denote by $\boldsymbol{n}_f$ the unit outward normal direction on $\partial\Omega_f$ and by $\boldsymbol{n}_p$ the normal direction on $\partial\Omega_p$, oriented outward. On the interface $\Gamma$ we have $\boldsymbol{n}_f = -\boldsymbol{n}_p$.

In the domain $\Omega_p$ we define the following quantity $\varphi$, called *piezometric head*:

$$\varphi := z + \frac{p_p}{\rho_f g}, \tag{1}$$

where $z$ is the elevation from a reference level, representing the potential energy per unit weight of fluid, $p_p$ is the pressure of the fluid, $\rho_f$ its density and $g$ is the gravity acceleration.

We describe the motion of the fluid in $\Omega_p$ by Darcy's law. It states that the fluid velocity $\boldsymbol{V}_p$ in $\Omega_p$ is proportional to the specific discharge vector $\boldsymbol{q} = -\boldsymbol{K}\nabla\varphi$, where $\boldsymbol{K}$ is the hydraulic conductivity tensor of the porous medium.

We have the following system of equations (see [1,19]): $\forall t > 0$,

$$\begin{cases} S_0 \dfrac{\partial \varphi}{\partial t} + \operatorname{div} \boldsymbol{q} = 0, & \forall \boldsymbol{x} \in \Omega_p, \\ \boldsymbol{q} = -\boldsymbol{K}\nabla\varphi, & \forall \boldsymbol{x} \in \Omega_p, \end{cases} \tag{2}$$

where $S_0$ is the specific mass storativity coefficient, and $\nabla$ denotes the gradient with respect to the 3D spatial coordinate $\boldsymbol{x} = (x, y, z)$.

The fluid motion in the water basin is described by the Navier–Stokes equations: $\forall t > 0$,

$$\begin{cases} \dfrac{\partial \boldsymbol{V}_f}{\partial t} + \mathbf{div}(\boldsymbol{V}_f \otimes \boldsymbol{V}_f) - \mathbf{div}\boldsymbol{T}(\boldsymbol{V}_f, p_f) = \boldsymbol{g}, & \forall \boldsymbol{x} \in \Omega_f, \\ \operatorname{div} \boldsymbol{V}_f = 0, & \forall \boldsymbol{x} \in \Omega_f. \end{cases} \tag{3}$$

$\boldsymbol{T}(\boldsymbol{V}_f, p_f)$ is the stress tensor defined as

$$\boldsymbol{T}(\boldsymbol{V}_f, p_f) = \nu\big(\nabla \boldsymbol{V}_f + \nabla^{\mathrm{T}} \boldsymbol{V}_f\big) - p_f \boldsymbol{I},$$

$\nu > 0$ is a given positive constant, $\boldsymbol{g}$ represents the external forces, while $\boldsymbol{V}_f$ and $p_f$ denote the unknown velocity and pressure.

In the case of a free surface fluid, we would like to replace Navier–Stokes equations by a simpler model based on the so-called Shallow Water equations. With this aim we characterize the domain $\Omega_f$ as follows. Let $\widehat{\Omega}$ be a bounded domain of $\mathbb{R}^2$ representing the undisturbed free surface of the fluid, while $z = h(x, y)$ and $z = \eta(x, y, t)$ are two functions describing respectively the bathymetry and the free surface with respect to a reference level $z = 0$. $\Omega_f$ is therefore the normal domain with respect to the $z$ axis defined as

$$\Omega_f = \big\{\boldsymbol{x} = (x, y, z) \mid (x, y) \in \widehat{\Omega},\ z \in (h, \eta)\big\}.$$

We describe the motion of the free surface fluid in $\Omega_f$ by the 3D non-hydrostatic Shallow Water equations (3D-NH-SWE) with constant density. The total pressure is the sum of a hydrostatic part and a hydrodynamic correction: $p_f = \rho g(\eta - z) + q$. We consider therefore the following model: $\forall t > 0$,

$$\begin{cases} \dfrac{D\boldsymbol{V}_f}{Dt} - \dfrac{\partial}{\partial z}\left(\nu_v \dfrac{\partial \boldsymbol{V}_f}{\partial z}\right) + \nabla q + \operatorname{diag}(g, g, 0) \cdot \nabla\eta = \boldsymbol{f}, & \forall \boldsymbol{x} \in \Omega_f, \\ \operatorname{div} \boldsymbol{V}_f = 0, & \forall \boldsymbol{x} \in \Omega_f, \\ \dfrac{\partial \eta}{\partial t} + \operatorname{div} \displaystyle\int_h^\eta \boldsymbol{u}_f \,\mathrm{d}z = \widetilde{Q}, & \forall (x, y) \in \widehat{\Omega}, \end{cases} \tag{4}$$

where $g$ is the gravity acceleration, $\boldsymbol{f} = (f_x, f_y, 0)^{\mathrm{T}}$ is the external force vector, $\boldsymbol{V}_f = (\boldsymbol{u}_f, w_f)^{\mathrm{T}}$ the velocity vector (hence $\boldsymbol{u}_f$ and $w_f$ represent its horizontal and vertical components respectively), $q$ is the hydrodynamic pressure and $\nu_v$ is the vertical viscosity coefficient. $\widetilde{Q}$ is equal to the normal component of the velocity $\boldsymbol{V}_f$ and is taken equal to zero when the bottom surface is impermeable. Finally $\nabla$ still denotes the gradient with respect to $\boldsymbol{x}$ while $D/Dt = \partial/\partial t + \boldsymbol{V}_f \cdot \nabla$ is the Lagrangian derivative.

### 2.1. Interface conditions

For the coupled model Free Surface Fluid/Darcy we propose the following interface conditions on $\Gamma$:

$$\begin{cases} \boldsymbol{V}_p \cdot \boldsymbol{n}_f = \boldsymbol{V}_f \cdot \boldsymbol{n}_f, \\ \dfrac{\partial \boldsymbol{u}_f}{\partial z} = \dfrac{\alpha_{BJ}\sqrt{3}}{\sqrt{\operatorname{tr}\boldsymbol{K}}}(\boldsymbol{u}_f - \boldsymbol{u}_p), \\ \rho_f g \varphi = \rho_f g H + p_p = p_f \end{cases} \tag{5}$$

where $H = \eta - h$ is the total height of the fluid in $\Omega_f$. We have denoted by $\boldsymbol{V}_p = \boldsymbol{q}/n$ the fluid velocity in $\Omega_p$, $n$ being the volumetric porosity. In particular, $\boldsymbol{u}_p$ indicates the vector of the two horizontal components of $\boldsymbol{V}_p$, while $w_p$ will denote its vertical component.

The parameter $\alpha_{BJ}$ is an empiric dimensionless coefficient whose value depends on the properties of the porous medium (see [15]).

We observe that condition $(5)_1$ imposes the continuity of the normal component of the velocity, however it allows a discontinuity of its tangential components.

In the case of Navier–Stokes fluid we consider the following set of interface conditions on $\Gamma$:

$$\begin{cases} \boldsymbol{V}_p \cdot \boldsymbol{n}_f = \boldsymbol{V}_f \cdot \boldsymbol{n}_f, \\ -\big[\big(\boldsymbol{T}(\boldsymbol{V}_f, p_f)\big) \cdot \boldsymbol{n}_f\big] \cdot \boldsymbol{\tau}_i = 0, \qquad i = 1, 2, \\ -\big[\big(\boldsymbol{T}(\boldsymbol{V}_f, p_f)\big) \cdot \boldsymbol{n}_f\big] \cdot \boldsymbol{n}_f = \rho_f g \varphi \end{cases} \tag{6}$$

where $\boldsymbol{\tau}_i$, $i = 1, 2$, are linear independent unit tangential vectors on the interface $\Gamma$. These conditions impose the continuity of the normal velocity on $\Gamma$, as well as that of the normal component of the normal stress.

We observe that conditions (6) generalize those proposed by Payne and Straughan in [16]. Indeed, they consider a flat interface $z = constant$ between the free fluid and the porous medium and therein require the following conditions:

$$\begin{cases} w_p = w_f, \\ \dfrac{\partial \boldsymbol{u}_f}{\partial z} + \nabla_{xy} w_f = \dfrac{\alpha_{BJ}\sqrt{3}}{\sqrt{\operatorname{tr}\boldsymbol{K}}}\boldsymbol{u}_f, \\ -2\nu \dfrac{\partial w_f}{\partial z} + p_f = p_p \end{cases} \tag{7}$$

where $\nabla_{xy}$ denotes the reduced gradient operator $(\partial/\partial x, \partial/\partial y)^{\mathrm{T}}$.

Now it can be easily seen that upon setting $\boldsymbol{n}_f = (0, 0, -1)^{\mathrm{T}}$ in (6), conditions $(6)_1$ and $(6)_3$ reduce to $(7)_1$ and $(7)_3$, respectively. Moreover, condition $(6)_2$ is similar to $(7)_2$ if one sets equal to zero the right term in $(7)_2$.

Finally let us consider the sets of interface conditions (5) and (7). First of all we notice that they both allow the pressure to be discontinuous across the interface $\Gamma$, as advocated also by Jäger and Mikelić (see [8,9]). Then we observe that conditions $(5)_2$ and $(7)_2$ are strictly linked. In fact condition $(7)_2$ was generalized by Jones (see [10]), according to the experimental results by Beavers and Joseph (see [2]), obtaining the following interface condition:

$$\frac{\partial \boldsymbol{u}_f}{\partial z} + \nabla_{xy} w_f = \frac{\alpha_{BJ}\sqrt{3}}{\sqrt{\operatorname{tr}\boldsymbol{K}}}(\boldsymbol{u}_f - \boldsymbol{u}_p). \tag{8}$$

Now we can see that condition (8) reduces to $(5)_2$ if we drop the term $\nabla_{xy} w_f$, which amounts to omitting the shear stress on the interface $\Gamma$.

### 2.2. Boundary conditions

For the Darcy equations we can consider the following boundary conditions. We define $\widetilde{\Gamma}_p = \partial\Omega_p \setminus \Gamma = \Gamma_p \cup \Lambda_p$ (see Fig. 2) and we assign the piezometric head $\varphi = \varphi_p$ on $\Gamma_p$ and the normal component of the velocity $\boldsymbol{V}_p \cdot \boldsymbol{n}_p$ on $\Lambda_p$.

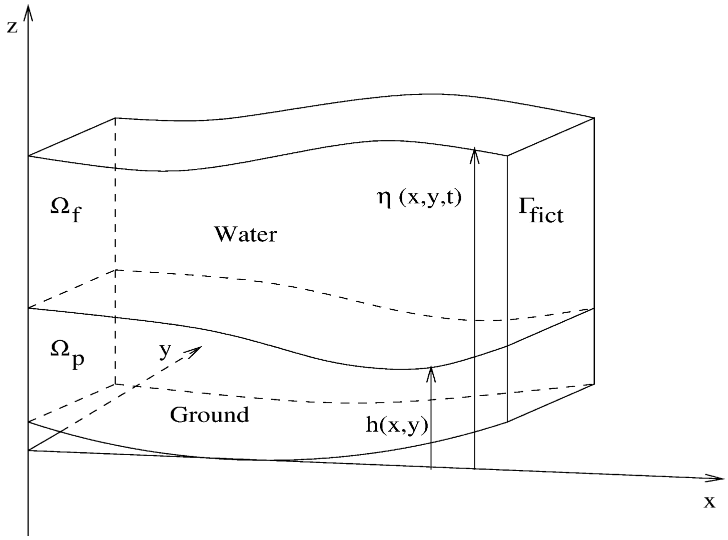


Fig. 1. Schematic representation of the domain of Free Surface/Darcy problem.

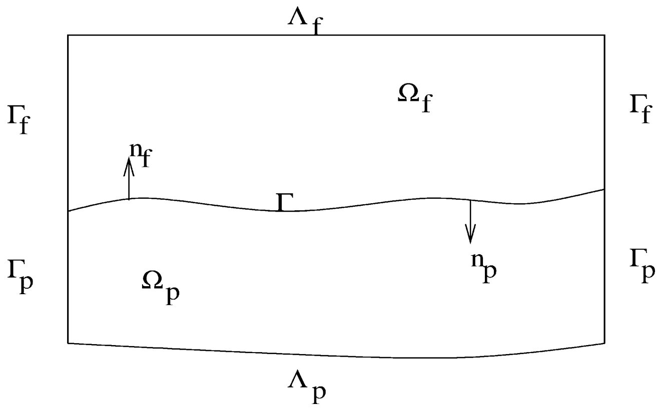


Fig. 2. Vertical section of the domain of Stokes/Darcy problem.

For the 3D-NH-SWE model (4) we consider on the free surface the kinematic boundary condition

$$\frac{\partial \eta}{\partial t} + \boldsymbol{u}_f \cdot \nabla_{xy}\eta = w_f, \quad \forall t > 0,\ \forall (x, y) \in \widehat{\Omega},\ z = \eta(x, y, t)$$

as well as the following condition which accounts for the wind effect:

$$\nu_v \frac{\partial \boldsymbol{u}_f}{\partial z} = c\|\boldsymbol{W}\|\boldsymbol{W}, \quad \forall t > 0,\ \forall (x, y) \in \widehat{\Omega},\ z = \eta(x, y, t)$$

where $c$ is a constant, $\boldsymbol{W}$ is the wind velocity vector and $\|\cdot\|$ denotes the Euclidean norm.

Finally on the lateral fictitious water-water boundary $\Gamma_{\text{fict}}$ (see Fig. 1) we assign the normal component of the velocity.

In the case of channeled fluid (problem (3)) we define $\widetilde{\Gamma}_f = \partial\Omega_f \setminus \Gamma = \Gamma_f \cup \Lambda_f$ (see Fig. 2) and we assign on $\Gamma_f$ the velocity $\boldsymbol{V}_f = 0$, while on $\Lambda_f$ we consider the natural boundary condition $\nu(\nabla \boldsymbol{V}_f + \nabla^{\mathrm{T}} \boldsymbol{V}_f) \cdot \boldsymbol{n}_f - p_f \boldsymbol{n}_f = \boldsymbol{l}$, $\boldsymbol{l} = (l_1, l_2)$ being a given vector function.

## 3. Subdomain iterative methods for the coupled problems

Either one of the coupled problems previously considered can be reformulated in a compact form as follows: we have a problem $P_f(\xi_f) = 0$ in the domain $\Omega_f$, where $\xi_f$ indicates all the unknowns therein (velocity, pressure and free surface location in the case of free surface flow), and a problem $P_p(\xi_p) = 0$ in $\Omega_p$, where $\xi_p$ represents the unknowns of Darcy's problem. Moreover we have three interface conditions involving pressure and velocity on $\Gamma$ that we write concisely as:

$$\boldsymbol{V}_p \cdot \boldsymbol{n}_f = \boldsymbol{V}_f \cdot \boldsymbol{n}_f, \tag{9}$$

$$\Psi_f(\boldsymbol{u}_f, p_f) = \Psi_p(\boldsymbol{u}_p), \tag{10}$$

$$\Phi_f(\boldsymbol{u}_f, p_f) = \Phi_p(p_p). \tag{11}$$

Our aim is to solve the coupled problems by an appropriate numerical scheme based on domain decomposition methods and, in particular, inspired by the Dirichlet–Neumann method in heterogeneous domain decomposition theory (see [17]).

More precisely, the iterative substructuring method that we advocate for the solution of our problem can be set in the following abstract form: at any fixed time level $t > 0$,

(1) solve $P_p(\xi_p) = 0$ in $\Omega_p$ using (9) as boundary condition on $\Gamma$ (this yields a Neumann boundary condition on $\varphi$);
(2) solve $P_f(\xi_f) = 0$ in $\Omega_f$ using (10) and (11) as boundary conditions on $\Gamma$;
(3) finally use a suitable relaxation depending on a positive parameter $\theta$ to enforce the continuity of the normal velocity $\boldsymbol{V}_p \cdot \boldsymbol{n}_f = \boldsymbol{V}_f \cdot \boldsymbol{n}_f$ at the following iterative step;
(4) iterate this procedure till convergence, according to a suitable convergence test.

It can be easily seen that by this method we compute the solution of the coupled problem through the independent solution of Darcy's equation in $\Omega_p$ and of the free fluid problem in $\Omega_f$.

We observe that the way the interface conditions (9)–(11) are enforced depends on the weak formulation adopted for each problem and, consequently, on the specific numerical approximation used.

In fact for the coupling of the 3D Shallow Water System with the Darcy equation a different approach is used, namely:

(1) solve the problem in the porous medium using $(5)_3$ as boundary condition on $\Gamma$ (this yields a Dirichlet boundary condition on $\varphi$);
(2) solve the 3D Shallow Water using $(5)_1$ and $(5)_2$ as boundary conditions on $\Gamma$;
(3) finally use a suitable relaxation depending on a positive parameter $\theta$ to enforce that $\rho_f g\varphi = \rho_f g H + p_p = p_f$ at the following iterative step;
(4) iterate this procedure till convergence, according to a suitable convergence test.

The numerical results presented in Section 8 are obtained using this latter approach whereas the theoretical analysis is carried out using the first one.

Finally, we point out that the above iterative method can be improved by a suitable re-interpretation. In fact, as we will show in Section 6 for the coupled problem Navier–Stokes/Darcy, one step of the iterative procedure can be regarded as a preconditioned Richardson iteration for a suitable Steklov–Poincaré equation on $\Gamma$. This allows to use more effective iterative solvers (e.g., GMRES or Krylov) with the same preconditioner, which is optimal in the sense that it yields a convergence rate independent of the grid-size of our numerical approximation.

## 4. Weak formulation and Finite Element Approximation

We consider only the coupled problem Stokes/Darcy, whereas for the problem Free Surface Fluid/Darcy we refer to previous works, in particular [7,12,14,13]. We remember that the Stokes problem is the linear counterpart of (3). It is obtained by dropping the second nonlinear term in (3), which is justified if the fluid velocity is small enough.

Moreover, for simplicity we limit our analysis to the stationary case. To this end, let us introduce the following functional spaces:

$$H^1_{\Gamma_f} := \{v \in H^1(\Omega_f) \mid v = 0 \text{ on } \Gamma_f\}, \tag{12}$$

$$H_f := \left(H^1_{\Gamma_f}\right)^3, \tag{13}$$

$$H_p := \{\psi \in H^1(\Omega_p) \mid \psi = 0 \text{ on } \Gamma_p\}, \tag{14}$$

$$Q := L^2(\Omega_f), \tag{15}$$

$$W := H_f \times H_p. \tag{16}$$

Notice that $W$ is a Hilbert space with respect to the norm

$$\|\underline{w}\|_W := \frac{1}{\sqrt{2}}\left(\|\vec{\boldsymbol{w}}\|^2_{H^1(\Omega_f)} + \|\psi\|^2_{H^1(\Omega_b)}\right)^{1/2}, \quad \forall \underline{w} = (\vec{\boldsymbol{w}}, \psi) \in W.$$

Finally we consider on $\Gamma$ the trace space (see [11])

$$\Lambda := H^{1/2}_{00}(\Gamma). \tag{17}$$

We define the following bilinear forms:

$$
\begin{aligned}
&\tilde{a}: W \times W \to \mathbb{R}\\
&\tilde{a}(\underline{v}, \underline{w}) := \int_{\Omega_f} n\nu\big(\nabla \boldsymbol{v} + \nabla^{\mathrm{T}} \boldsymbol{v}\big) \cdot \nabla \boldsymbol{w} \,\mathrm{d}\Omega_f + \int_{\Omega_p} \rho_f g \nabla\psi \cdot \boldsymbol{K} \nabla\varphi \,\mathrm{d}\Omega_p \\
&\qquad\qquad + \int_{\Gamma} n\rho_f g \varphi \boldsymbol{w} \cdot \boldsymbol{n}_f \,\mathrm{d}\Gamma - \int_{\Gamma} n\rho_f g \psi \boldsymbol{v} \cdot \boldsymbol{n}_f \,\mathrm{d}\Gamma
\end{aligned}
\tag{18}
$$

for all $\underline{v} = (\boldsymbol{v}, \varphi)$ and $\underline{w} = (\boldsymbol{w}, \psi)$ in $W$;

$$
\begin{aligned}
&\tilde{b}: W \times Q \to \mathbb{R}\\
&\tilde{b}(\underline{w}, q) := -\int_{\Omega_f} nq \operatorname{div} \boldsymbol{w} \,\mathrm{d}\Omega_f, \quad \forall \underline{w} = (\boldsymbol{w}, \psi) \in W,\ q \in Q.
\end{aligned}
\tag{19}
$$

Finally we indicate by $E_p : H^{1/2}(\Gamma_p) \to H^1(\Omega_p)$ a continuous extension operator such that $E_p\mu = \mu$ on $\Gamma_p$, for all $\mu \in H^{1/2}(\Gamma_p)$. So we can define the function $\varphi_0 \in H_p$ as $\varphi_0 := \varphi - E_p\varphi_p$, where $\varphi_p$ is the boundary datum introduced for Darcy's equation.

Then we can introduce the following linear functional

$$
\begin{aligned}
&F : W \to \mathbb{R}\\
&\langle F, \underline{w} \rangle := \int_{\Omega_f} n\boldsymbol{g} \cdot \boldsymbol{w} \,\mathrm{d}\Omega_f + \int_{\Lambda_f} n\boldsymbol{l} \cdot \boldsymbol{w} \,\mathrm{d}\Lambda_f - \int_{\Omega_p} \rho_f g \nabla\psi \cdot \boldsymbol{K} \nabla E_p\varphi_p \,\mathrm{d}\Omega_p \\
&\qquad\qquad - \int_{\Gamma} n\rho_f g E_p\varphi_p \boldsymbol{w} \cdot \boldsymbol{n}_f \,\mathrm{d}\Gamma, \quad \forall \underline{w} = (\boldsymbol{w}, \psi) \in W.
\end{aligned}
\tag{20}
$$

Using the above definitions, the stationary coupled problem Stokes/Darcy can be written in the following weak form: find $\underline{u} = (\boldsymbol{V}_f, \varphi_0) \in W$, $p \in Q$:

$$
\begin{cases}
\tilde{a}(\underline{u}, \underline{v}) + \tilde{b}(\underline{v}, p) = \langle F, \underline{v} \rangle, & \forall \underline{v} \in W,\\
\tilde{b}(\underline{u}, q) = 0, & \forall q \in Q.
\end{cases}
\tag{21}
$$

We remark that the interface conditions (6) are easily incorporated in the above weak model: in fact, as one can see by integrating by parts Stokes and Darcy's equations, they are natural conditions on $\Gamma$.

Using the theory developed by Brezzi (see [3]) for saddle point problems, the following result can be proved:

**Theorem 1.** *Problem* (21) *is well posed, i.e., there exists a unique solution* $(\underline{u}, p) \in W \times Q$ *to* (21), *depending continuously on the data of the problem.*

For the proof see [5].

We introduce now a Galerkin finite element approximation to problem (21).

Taken $h > 0$, we consider a regular triangulation $\mathcal{T}_h$ of the domain $\overline{\Omega}_f \cup \overline{\Omega}_p$ made up of tetrahedra such that the triangulations $\mathcal{T}_{f,h}$ and $\mathcal{T}_{p,h}$ induced on $\Omega_f$ and $\Omega_p$ are compatible on $\Gamma$, that is they share the same faces on $\Gamma$. Finally we suppose the triangulation $\mathcal{M}_h$ induced on $\Gamma$ to be quasi-uniform (e.g., [18]).

Denoting by $\mathbb{P}_r$ the space of polynomials of degree less or equal to $r$, $r$ being a fixed non negative integer, we introduce the following discrete spaces:

$$H_{p,h} := \{\psi_h \in C^0(\overline{\Omega}_p) \mid \psi_h = 0 \text{ on } \Gamma_p \text{ and } \psi_{h|K} \in \mathbb{P}_1(K),\ \forall K \in \mathcal{T}_{p,h}\} \tag{22}$$

$$Q_h := \{q_h \in L^2(\Omega_f) \mid q_{h|K} \in \mathbb{P}_0(K),\ \forall K \in \mathcal{T}_{f,h}\} \tag{23}$$

$$H_{f,h} := (X^2_{f,h})^3 \tag{24}$$

where

$$X^2_{f,h} := \{v_h \in C^0(\overline{\Omega}_f) \mid v_h = 0 \text{ on } \Gamma_f \text{ and } v_{h|K} \in \mathbb{P}_2(K),\ \forall K \in \mathcal{T}_{f,h}\}.$$

Other choices of finite element spaces can be made as well.

Let us denote by $W_h$ the family of finite-dimensional spaces $W_h := H_{f,h} \times H_{p,h}$. Finally, we consider the space

$$\Lambda_h := \{v_{h|\Gamma} \mid v_h \in X^2_{f,h}\} \tag{25}$$

to approximate the trace space $\Lambda$ on $\Gamma$.

The Galerkin approximation to problem (21) reads: find $\underline{u}_h = (\boldsymbol{V}_{fh}, \varphi_{0h}) \in W_h$, $p_h \in Q_h$:

$$\begin{cases} \tilde{a}(\underline{u}_h, \underline{v}_h) + \tilde{b}(\underline{v}_h, p_h) = \langle F, \underline{v}_h\rangle, & \forall \underline{v}_h \in W_h, \\ \tilde{b}(\underline{u}_h, q_h) = 0, & \forall q_h \in Q_h. \end{cases} \tag{26}$$

We remark that in the finite-dimensional case we have to consider a suitable discrete extension of the boundary data $\varphi_p$, thus possibly modifying the functional $F$.

## 5. Iterative procedure for the coupled problem Stokes/Darcy

We shall now introduce an iterative method which allows to solve the coupled stationary problem Stokes/Darcy through the independent solution of the Stokes system in $\Omega_f$ and Darcy's equation in $\Omega_p$, following the ideas presented in Section 3.

In particular we will assign condition $(6)_1$ to Darcy's equation in $\Omega_p$, while conditions $(6)_2$ and $(6)_3$ are imposed on the Stokes system specifying the normal and tangential components of normal stress.

More precisely the iterative scheme reads as follows:

let $\lambda_h^0 \in \Lambda_h$ be an initial guess; solve for $k \geqslant 0$:

$$\begin{cases} \text{find } \varphi_h^{k+1} \in H_{p,h}: \\ \displaystyle\int_{\Omega_p} \nabla\psi_h \cdot \boldsymbol{K}\nabla\varphi_h^{k+1}\, \mathrm{d}\Omega_p - \int_\Gamma n\psi_h\lambda_h^k\, \mathrm{d}\Gamma = 0, \quad \forall \psi_h \in H_{p,h}, \end{cases} \tag{27}$$

$$
\begin{cases}
\text{find } \boldsymbol{V}_{fh}^{k+1} \in H_{f,h}, \quad p_h^{k+1} \in Q_h: \\
\displaystyle\int_{\Omega_f} \nu\big(\nabla \boldsymbol{V}_{fh}^{k+1} + \nabla^{\mathrm{T}} \boldsymbol{V}_{fh}^{k+1}\big) \cdot \nabla \boldsymbol{W}_h \, \mathrm{d}\Omega_f - \int_{\Omega_f} p_h^{k+1} \operatorname{div} \boldsymbol{W}_h \, \mathrm{d}\Omega_f \\
\quad \displaystyle + \int_{\Gamma} \rho_f g \varphi_h^{k+1} \boldsymbol{W}_h \cdot \boldsymbol{n}_f \, \mathrm{d}\Gamma = \int_{\Omega_f} \boldsymbol{g} \cdot \boldsymbol{W}_h \, \mathrm{d}\Omega_f + \int_{\Lambda_f} \boldsymbol{l} \cdot \boldsymbol{W}_h \, \mathrm{d}\Lambda_f, \\
\displaystyle\int_{\Omega_f} q_h \operatorname{div} \boldsymbol{V}_{fh}^{k+1} \, \mathrm{d}\Omega_f = 0, \\
\forall \boldsymbol{W}_h \in H_{f,h}, \quad \forall q_h \in Q_h
\end{cases}
\tag{28}
$$

with

$$
\lambda_h^{k+1} := \theta\big(\boldsymbol{V}_{fh}^{k+1} \cdot \boldsymbol{n}_f\big)_{|\Gamma} + (1-\theta)\lambda_h^k, \tag{29}
$$

$\theta$ being a positive relaxation parameter.

The iterative procedure proposed consists then of three steps: an initial guess $\lambda_h^0$ being fixed for the interface condition $(6)_1$ on $\Gamma$,

(1) solve problem (27);
(2) use the value $\varphi_h^{k+1}$ to solve problem (28);
(3) compute $\lambda_h^{k+1}$ using (29) and restart from step (1) iterating till convergence, using as convergence test:

$$
\frac{\|\lambda_h^{k+1} - \lambda_h^k\|_{L^2(\Gamma)}}{\|\lambda_h^{k+1}\|_{L^2(\Gamma)}} \leqslant \varepsilon,
$$

where $\varepsilon$ is an assigned tolerance.

The above iterative scheme (27)–(29) can be proven to converge by reinterpreting it through the Steklov–Poincaré operators associated to the coupled problem Stokes/Darcy.

The study of these operators in order to prove the convergence of the iterative method will be the subject of a future paper (see [6]).

## 6. Steklov–Poincaré Equation associated to the coupled problem Stokes/Darcy

Our aim is now to reformulate the coupled problem Stokes/Darcy through an autonomous interface equation on $\Gamma$ and a family of subproblems of reduced size upon the subdomains $\Omega_f$ and $\Omega_p$. For simplicity we again limit our analysis to the stationary case.

We consider the interface condition $\boldsymbol{V}_p \cdot \boldsymbol{n}_f = \boldsymbol{V}_f \cdot \boldsymbol{n}_f$ on $\Gamma$ and we choose as governing variable on $\Gamma$

$$
\lambda \equiv \boldsymbol{V}_f \cdot \boldsymbol{n}_f = -\frac{1}{n} \boldsymbol{K} \nabla \varphi \cdot \boldsymbol{n}_f
$$

(we recall that $\boldsymbol{V}_p = -\frac{1}{n} \boldsymbol{K} \nabla \varphi$ in $\Omega_p$).

We can then solve both Stokes and Darcy's problems by lifting $\lambda$ on $\Gamma$. Formally speaking,

$$\begin{cases} \text{Stokes}(\breve{\boldsymbol{\omega}}, \breve{\pi}) & \text{in } \Omega_f, \\ \breve{\boldsymbol{\omega}} \cdot \boldsymbol{n}_f = \lambda & \text{on } \Gamma, \\ -\big[\big(\boldsymbol{T}(\breve{\boldsymbol{\omega}}, \breve{\pi})\big) \cdot \boldsymbol{n}_f\big] \cdot \boldsymbol{\tau}_i = 0 & (i = 1, 2) \text{ on } \Gamma, \\ + \text{boundary conditions on } \widetilde{\Gamma}_f, \end{cases} \tag{30}$$

$$\begin{cases} \text{Darcy}(\breve{\varphi}) & \text{in } \Omega_p, \\ -\frac{1}{n}\boldsymbol{K}\nabla\breve{\varphi} \cdot \boldsymbol{n}_f = \lambda & \text{on } \Gamma, \\ + \text{boundary conditions} & \text{on } \widetilde{\Gamma}_p. \end{cases} \tag{31}$$

Thanks to the linearity of Stokes' and Darcy's problems we can decompose the variables $\breve{\omega}$, $\breve{\pi}$ and $\breve{\varphi}$ as sums of two components: the first one depending only on $\lambda$, the second one on all the remaining data of the problems. With this aim we consider the following splitting:

$$\breve{\boldsymbol{\omega}} = \boldsymbol{\omega}(\lambda) + \boldsymbol{\omega}^*, \qquad \breve{\pi} = \pi(\lambda) + \pi^* \quad \text{and} \quad \breve{\varphi} = \tilde{\varphi}(\lambda) + \varphi^*. \tag{32}$$

It can be easily seen that the two problems (30) and (31) are equivalent to the coupled problem Stokes/Darcy if and only if the following condition on $\Gamma$ is satisfied:

$$-\big[\big(\boldsymbol{T}(\breve{\boldsymbol{\omega}}, \breve{\pi})\big) \cdot \boldsymbol{n}_f\big] \cdot \boldsymbol{n}_f = \rho_f g \breve{\varphi}. \tag{33}$$

Using (32), Eq. (33) can be written as:

$$\big[\boldsymbol{T}\big(\boldsymbol{\omega}(\lambda), \pi(\lambda)\big) \cdot \boldsymbol{n}_f\big] \cdot \boldsymbol{n}_f + \tilde{\varphi}(\lambda) = -\big[\boldsymbol{T}(\boldsymbol{\omega}^*, \pi^*) \cdot \boldsymbol{n}_f\big] \cdot \boldsymbol{n}_f - \varphi^* \quad \text{on } \Gamma.$$

Thus problems (30) and (31) are equivalent to the coupled problem Stokes/Darcy if and only if $\lambda$ is the solution to the *Steklov–Poincaré interface equation*

$$S\lambda = \chi \quad \text{on } \Gamma.$$

$S$ is a pseudo-differential operator defined from the trace space $H_{00}^{1/2}(\Gamma)$ and its dual space $H^{-1/2}(\Gamma)$ (see [11] for the definition of these spaces), while $\chi$ is a continuous linear functional from $H_{00}^{1/2}(\Gamma)$ in $\mathbb{R}$.

The above approach can be replicated on the Finite Element Galerkin approximation of the coupled problem to yield a discrete equation on $\Gamma$: find $\lambda_h \in \Lambda_h$ such that

$$S_h\lambda_h = \chi_h, \tag{34}$$

where $S_h$ and $\chi_h$ are Galerkin approximations to $S$ and $\chi$ respectively.

In its turn, $S_h$ can be split as sum of two operators $S_h^f$ and $S_h^p$, associated to the subproblems (30) and (31) respectively.

The following theorem holds (see [6]):

**Theorem 2.** *The operator $S_h^f$ is continuous and coercive on $\Lambda_h$ while $S_h^p$ is continuous and positive definite on $\Lambda_h$, both uniformly with respect to $h$. Therefore, as $S_h = S_h^f + S_h^p$, $S_h$ and $S_h^f$ are spectrally equivalent, i.e., there exist two constants $C_1$ and $C_2$, independent of $h$, such that*

$$C_1 \leqslant \frac{\nu_{\max}((S_h^f)^{-1}S_h)}{\nu_{\min}((S_h^f)^{-1}S_h)} \leqslant C_2,$$

$\nu_{\max}$, $\nu_{\min}$ *being the maximum and minimum eigenvalues of* $(S_h^f)^{-1}S_h$.

**Remark 3.** As a consequence of Theorem 2, $S_h^f$ is an optimal preconditioner to $S_h$ for the solution of the discrete Steklov–Poincaré problem (34).

As already anticipated in Section 3, the iterative scheme (27)–(29) can be interpreted as a preconditioned Richardson procedure for the Steklov–Poincaré problem (34) on $\Gamma$. In particular the following equivalence result holds:

**Proposition 4.** *The iterative scheme* (27)–(29) *is equivalent to the preconditioned Richardson method with preconditioner* $S_h^f$

$$S_h^f\left(\lambda_h^{k+1}-\lambda_h^k\right)=\theta\left(\chi_h-S_h\lambda_h^k\right)$$

*for the discrete Steklov–Poincaré equation* $S_h\lambda_h=\chi_h$ *on* $\Gamma$.

Thus the Dirichlet–Neumann type method (27)–(29) embodies in itself an optimal preconditioner for the system induced on the interface unknowns. We expect, therefore, that the number of iterations needed by the iterative method (27)–(29) to reach convergence is independent of the parameter $h$ chosen for the mesh of the computational domain.

## 7. Algebraic formulation of the coupled problem Stokes/Darcy and Schur complement matrix

We shall now introduce the algebraic formulation of the coupled problem Stokes/Darcy in the stationary case and we will interpret the iterative scheme (27)–(29) in matrix form.

We denote by $\boldsymbol{u}_0$ the vector of the values of the unknown $\boldsymbol{V}_f$ at the nodes of $\Omega_f\setminus\Gamma$ and moreover those of $\boldsymbol{V}_f\cdot\boldsymbol{\tau}_i$ at the nodes lying on the interface $\Gamma$.

Then let $\boldsymbol{u}_\Gamma$ indicate the vector of the values of $\boldsymbol{V}_f\cdot\boldsymbol{n}_f$ at the nodes of $\Gamma$, and $\boldsymbol{p}$ the vector of the values of the unknown pressure $p$ at the nodes of $\Omega_f$.

Finally $\boldsymbol{\Psi}_{\text{int}}$ indicates the vector of the values of the piezometric head $\varphi_{0h}$ at the nodes on $\Omega_p\setminus\Gamma$ and $\boldsymbol{\Psi}_\Gamma$ those at the nodes on $\Gamma$.

Using this notation the coupled problem Stokes/Darcy (21) can be written in the following matrix form:

$$\begin{pmatrix} A & B^{\mathrm{T}} & A_\Gamma & 0 & 0\\ B_1 & 0 & B_\Gamma & 0 & 0\\ A_{f\Gamma} & B_\Gamma^{\mathrm{T}} & A_{\Gamma\Gamma} & M_\Gamma & 0\\ 0 & 0 & -M_\Gamma^{\mathrm{T}} & \widetilde{A}_\Gamma & A_{p\Gamma}^{\mathrm{T}}\\ 0 & 0 & 0 & A_{p\Gamma} & A_p \end{pmatrix}\begin{pmatrix}\boldsymbol{u}_0\\ \boldsymbol{p}\\ \boldsymbol{u}_\Gamma\\ \boldsymbol{\Psi}_\Gamma\\ \boldsymbol{\Psi}_{\text{int}}\end{pmatrix}=\begin{pmatrix}\boldsymbol{f}_f\\ 0\\ \boldsymbol{f}_{f\Gamma}\\ \boldsymbol{f}_{p\Gamma}\\ \boldsymbol{f}_p\end{pmatrix}. \tag{35}$$

### 7.1. Interpretation in matrix form of the substructuring iterative method

Let us indicate by $N_\Gamma$ the number of nodes lying on $\Gamma$ and by $\underline{\lambda}^k\in\mathbb{R}^{N_\Gamma}$ the vector of the values of $\lambda_h$ at the $k$th step at the nodes of $\Gamma$. Then scheme (27)–(29) corresponds to the following steps.

Let $\underline{\lambda}^k$ be given; solve the following algebraic system:

$$\begin{pmatrix}\widetilde{A}_\Gamma & A_{p\Gamma}^{\mathrm{T}}\\ A_{p\Gamma} & A_p\end{pmatrix}\begin{pmatrix}\boldsymbol{\Psi}_\Gamma^{k+1}\\ \boldsymbol{\Psi}_{\text{int}}^{k+1}\end{pmatrix}=\begin{pmatrix}\boldsymbol{f}_{p\Gamma}+M_\Gamma^{\mathrm{T}}\underline{\lambda}^k\\ \boldsymbol{f}_p\end{pmatrix} \tag{36}$$

(which corresponds to (27)).

In particular, eliminating $\boldsymbol{\Psi}_{\text{int}}^{k+1}$ from (36), we obtain

$$\big(\widetilde{A}_\Gamma - A_{p\Gamma}^{\mathrm{T}} A_p^{-1} A_{p\Gamma}\big)\boldsymbol{\Psi}_\Gamma^{k+1} = \boldsymbol{f}_{p\Gamma} - A_{p\Gamma}^{\mathrm{T}} A_p^{-1} \boldsymbol{f}_p + M_\Gamma^{\mathrm{T}} \underline{\lambda}^k. \tag{37}$$

Now use $\boldsymbol{\Psi}_\Gamma^{k+1}$ to compute the unknown vector $\boldsymbol{u}_\Gamma^{k+1}$ by solving the following system:

$$\begin{pmatrix} A & B^{\mathrm{T}} & A_\Gamma \\ B_1 & 0 & B_\Gamma \\ A_{f\Gamma} & B_\Gamma^{\mathrm{T}} & A_{\Gamma\Gamma} \end{pmatrix} \begin{pmatrix} \boldsymbol{u}_0^{k+1} \\ \boldsymbol{p}^{k+1} \\ \boldsymbol{u}_\Gamma^{k+1} \end{pmatrix} = \begin{pmatrix} \boldsymbol{f}_f \\ 0 \\ \boldsymbol{f}_{f\Gamma} - M_\Gamma \boldsymbol{\Psi}_\Gamma^{k+1} \end{pmatrix} \tag{38}$$

(which corresponds to solve the Stokes problem (28)).

Finally, following (29), we set

$$\underline{\lambda}^{k+1} := \theta \boldsymbol{u}_\Gamma^{k+1} + (1-\theta)\underline{\lambda}^k \tag{39}$$

and we iterate restarting from (36) till the convergence test

$$\frac{\|\underline{\lambda}^{k+1} - \underline{\lambda}^k\|_{\mathbb{R}^{N_\Gamma}}}{\|\underline{\lambda}^{k+1}\|_{\mathbb{R}^{N_\Gamma}}} \leqslant \varepsilon$$

is satisfied for a suitable tolerance $\varepsilon$ ($\|\cdot\|_{\mathbb{R}^{N_\Gamma}}$ indicates the Euclidean norm in $\mathbb{R}^{N_\Gamma}$).

### *7.2. Algebraic formulation of the discrete Steklov–Poincaré operator $S_h$: The Schur complement matrix*

At the algebraic level the discrete Steklov–Poincaré operator $S_h$ introduced in (34) is the Schur complement matrix $\Sigma_h$ of the matrix associated to the coupled problem Stokes/Darcy with respect to the unknown vector $\boldsymbol{u}_\Gamma$.

After elimination of the unknowns $\boldsymbol{u}_0$, $\boldsymbol{p}$, $\boldsymbol{\Psi}_\Gamma$ and $\boldsymbol{\Psi}_{\text{int}}$ in (35), we obtain the reduced system:

$$\Sigma_h \boldsymbol{u}_\Gamma = \chi_h \tag{40}$$

where

$$\begin{aligned} \Sigma_h :=\ & \big[\big(B_\Gamma^{\mathrm{T}} - A_{f\Gamma} A^{-1} B^{\mathrm{T}}\big)\big(B_1 A^{-1} B^{\mathrm{T}}\big)^{-1}\big(B_\Gamma - B_1 A^{-1} A_\Gamma\big) + \big(A_{\Gamma\Gamma} - A_{f\Gamma} A^{-1} A_\Gamma\big)\big] \\ & + \big[M_\Gamma\big(\widetilde{A}_\Gamma - A_{p\Gamma}^{\mathrm{T}} A_p^{-1} A_{p\Gamma}\big)^{-1} M_\Gamma^{\mathrm{T}}\big] \end{aligned} \tag{41}$$

and

$$\begin{aligned} \chi_h :=\ & \boldsymbol{f}_{f\Gamma} - A_{f\Gamma} A^{-1} \boldsymbol{f}_f - \big(B_\Gamma^{\mathrm{T}} - A_{f\Gamma} A^{-1} B^{\mathrm{T}}\big)\big(B_1 A^{-1} B^{\mathrm{T}}\big)^{-1}\big(B_1 A^{-1} \boldsymbol{f}_f\big) \\ & - M_\Gamma\big(\widetilde{A}_\Gamma - A_{p\Gamma}^{\mathrm{T}} A_p^{-1} A_{p\Gamma}\big)^{-1}\big(\boldsymbol{f}_{p\Gamma} - A_{p\Gamma}^{\mathrm{T}} A_p^{-1} \boldsymbol{f}_p\big). \end{aligned}$$

We observe that in (41) the first term in square brackets arises from the domain $\Omega_f$, whereas the second one from $\Omega_p$.

Therefore we can split matrix $\Sigma_h$ as sum of two submatrices $\Sigma_h^f$ and $\Sigma_h^p$ defined in the following way:

$$\Sigma_h^f := \big[\big(B_\Gamma^{\mathrm{T}} - A_{f\Gamma} A^{-1} B^{\mathrm{T}}\big)\big(B_1 A^{-1} B^{\mathrm{T}}\big)^{-1}\big(B_\Gamma - B_1 A^{-1} A_\Gamma\big) + \big(A_{\Gamma\Gamma} - A_{f\Gamma} A^{-1} A_\Gamma\big)\big], \tag{42}$$

$$\Sigma_h^p := M_\Gamma\big(\widetilde{A}_\Gamma - A_{p\Gamma}^{\mathrm{T}} A_p^{-1} A_{p\Gamma}\big)^{-1} M_\Gamma^{\mathrm{T}}. \tag{43}$$

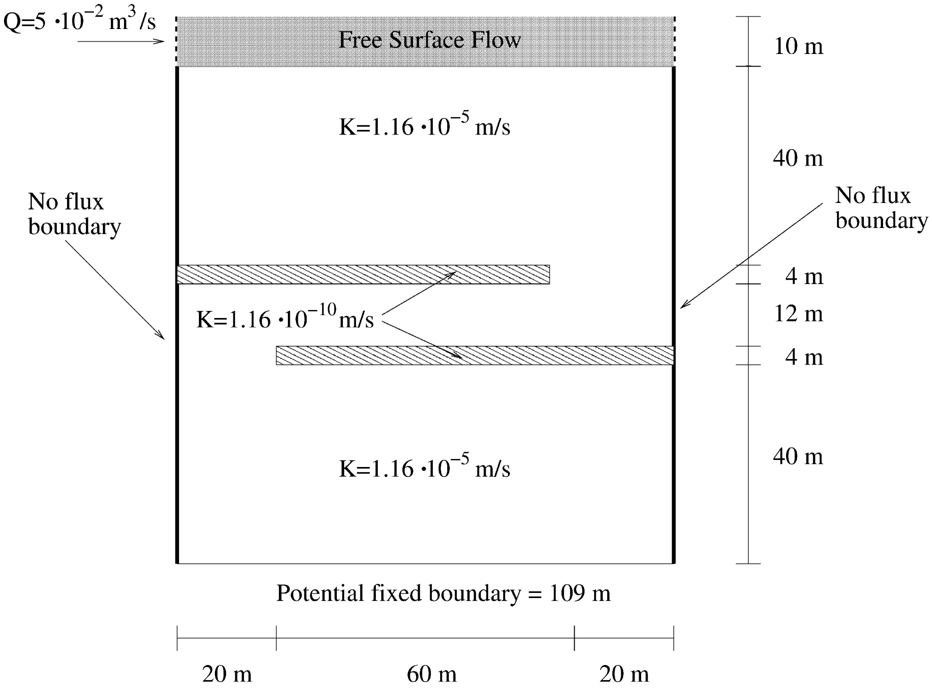


Fig. 3. Setting for the test case.

We observe that (42) and (43) are the algebraic counterpart of the operators $S_h^f$ and $S_h^p$ which decompose the global Steklov–Poincaré discrete operator $S_h$ as $S_h = S_h^f + S_h^p$: in particular $\Sigma_h^f$ is the counterpart of $S_h^f$ and $\Sigma_h^p$ of $S_h^p$.

Finally we underline that, according to Proposition 4, the matrix $\Sigma_h^f$ defined in (42) provides an optimal preconditioner for the Steklov–Poincaré interface problem (40) on $\Gamma$. The same preconditioner can be used with other iterative methods to solve the interface problem such as, e.g., GMRES or, more generally, Krylov methods.

## 8. Numerical results

In this section we present some numerical results concerning the coupling of the 3D Shallow Water system with the Darcy equation.

We have considered a situation in which a water channel 5 m wide and 100 m long with a water depth of 10 m covers a porous media whose height is 100 m. At the inlet of the channel a discharge of $5 \times 10^{-2}$ $\mathrm{m}^3/\mathrm{s}$ is prescribed while at the outlet a non-reflective boundary condition is enforced. As for the porous media we have considered an isotropic heterogeneous domain with hydraulic conductivity of $1.16 \times 10^{-5}$ m/s intersected by two less previous barriers of $1.16 \times 10^{-10}$ m/s; the specific storativity $S_0$ is equal to 0.02 $\mathrm{m}^{-1}$, while the porosity is $n = 0.01$. On the vertical boundary of the porous domain a no-flux condition is imposed. In the basin the free surface is initially flat and the velocity is zero, in the

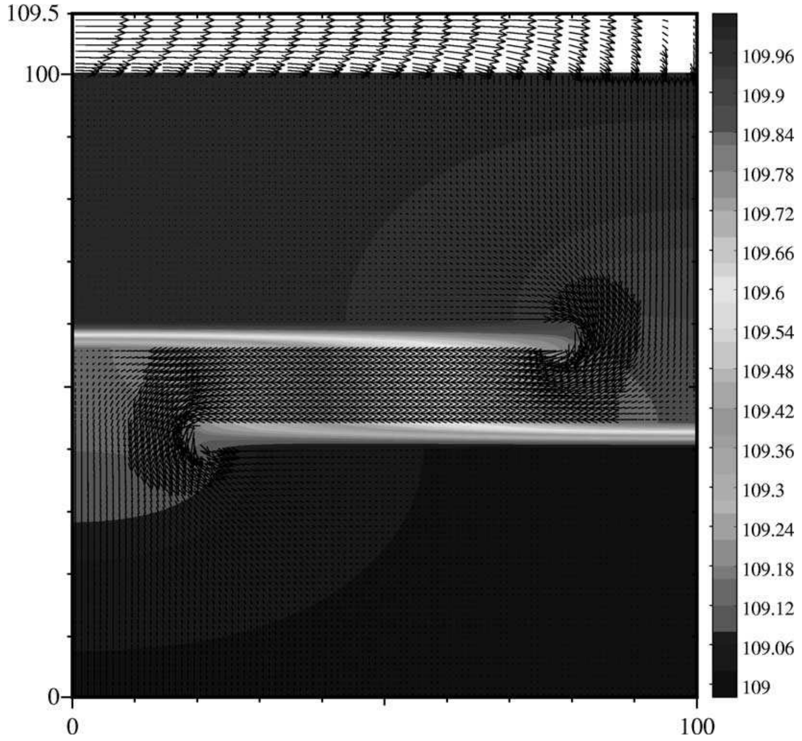


Fig. 4. Velocity field in the whole domain and piezometric head in the porous medium.

Table 1
Iterations for different values of the discretization parameter $h$

| $h$ | Triangles | Nodes | Iterations |
|---|---|---|---|
| 1 | 1398 | 702 | 15 |
| 0.5 | 4687 | 2348 | 11 |
| 0.25 | 17933 | 8976 | 10 |
| 0.125 | 70961 | 35500 | 10 |

porous media the initial potential head is 0 (see Fig. 3). Finally $\alpha_{BJ} = 1$, the time step is set to $\Delta t = 120$ s and the viscosity of the water is equal to $10^{-2}$ $\mathrm{m}^2/\mathrm{s}$.

The 3D Shallow Water System is solved using the multi-layer approach described in [7] using 13 layers whereas for the porous media a mixed finite element approach is adopted. The spatial discretization for both the free surface and the porous media is obtained extruding a 2D unstructured triangular mesh (characterized by a discretization parameter $h$) along the vertical direction giving rise to a 3D mesh of prisms.

The tolerance for the stopping criterium of the iterative method is $tol = 10^{-5}$ and the relaxation parameter $\theta = 0.5$.

In Table 1 it is shown that the number of iterations is bounded independently of the discretization parameter $h$.

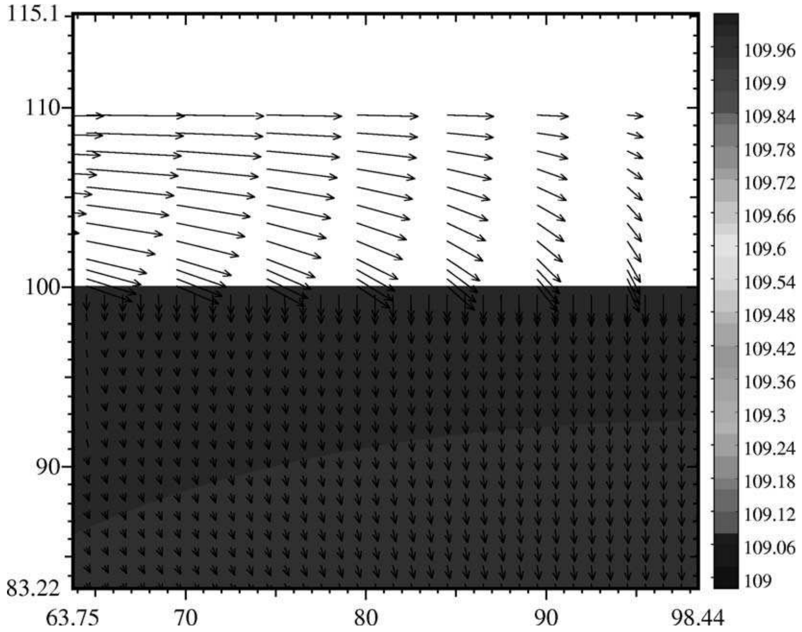


Fig. 5. Zoom of the velocity field across the interface near the outlet.

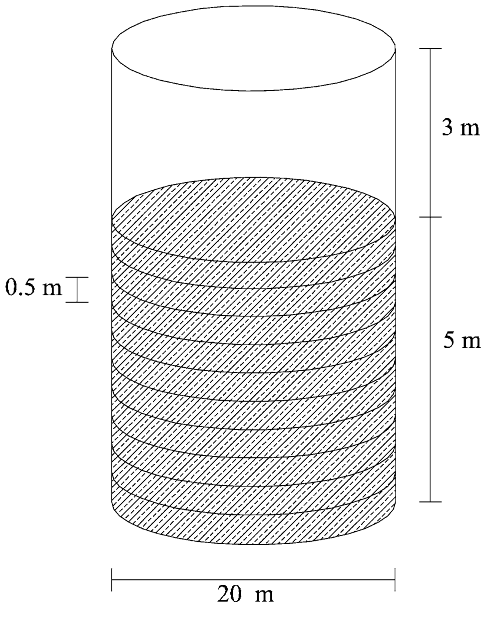


Fig. 6. Schematic representation of the computational domain.

Other numerical results have been obtained considering a cylindrical domain with a circular base with a diameter of 10 m and whose height is 8 m. It is filled in its lower part up to 5 m by a porous medium

Table 2
Iterations for different values of the discretization parameter $h$ in the case of cylindrical domain

| $h$ | Triangles | Nodes | Iterations |
|---|---|---|---|
| 1 | 514 | 288 | 8 |
| 0.5 | 2084 | 1087 | 8 |
| 0.25 | 8168 | 4209 | 8 |
| 0.125 | 32836 | 16669 | 7 |

(see Fig. 6). The computational mesh is obtained as in the above example, extruding an unstructured triangular mesh of the base. In the part formed by the porous medium 10 layers have been considered.

The viscosity of water is equal to $10^{-3}$ m$^2$/s and the fluid depth is 3 m. On the lateral boundary of the porous medium we have considered a Dirichlet condition imposing the piezometric head equal to 3 m. Finally $\boldsymbol{K}$ has been considered isotropic and constant, in particular $K_i = 10^{-3}$ m/s, $i = 1, 2, 3$, $S_0$ is equal to $10^{-4}$ m$^{-1}$ and the time step is set to $\Delta t = 30$ sec.

The tolerance for the stopping criterium and the relaxation parameter are still $tol = 10^{-5}$ and $\theta = 0.5$, respectively.

The results obtained in correspondence of four different mesh sizes are shown in Table 2. We can see that the number of iteration is essentially independent of the mesh parameter $h$.

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