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COUPLING FLUID FLOW WITH POROUS MEDIA FLOW*

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Abstract. The transport of substances back and forth between surface water and groundwater is a very serious problem. We study herein the mathematical model of this setting consisting of the Stokes equations in the fluid region coupled with the Darcy equations in the porous medium, coupled across the interface by the Beavers–Joseph–Saffman conditions. We prove existence of weak solutions and give a complete analysis of a finite element scheme which allows a simulation of the coupled problem to be uncoupled into steps involving porous media and fluid flow subproblems. This is important because there are many "legacy" codes available which have been optimized for uncoupled porous media and fluid flow.

Key words. coupled porous media and fluid flow, Stokes and Darcy equations, Beavers–Joseph–Saffman condition, weak solutions, finite element scheme, error estimates

AMS subject classifications. 35Q35, 65N30, 65N15, 76D07, 76S05

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1. Introduction and the model.

There are many serious problems currently facing the world in which the coupling between groundwater and surface water is important. These include questions such as predicting how pollution discharged into streams, lakes, and rivers makes its way into the water supply. This coupling is also important in technological applications involving filtration.

The aim of our research is to begin the study of the following problem: an incompressible fluid in a region Ω_1 can flow both ways across an interface Γ_I into a domain Ω_2 which is a porous medium saturated with the same fluid. The mathematical theory and numerical analysis of each subproblem is well developed, and reliable codes are available. Nevertheless, the mathematical theory of the coupled problem seems to be not completely understood. The model of this situation which is most accessible to large scale computations consists of the Navier–Stokes equations (or Stokes equations) in the fluid region coupled across an interface with the Darcy equations for the filtration velocity in the porous medium. This leads to mathematical difficulties arising from the coupled system of equations of different orders in different regions. See Jäger and Mikelić [16], Payne and Straughan [22] for the beginning of analytical studies of this problem. (For the Brinkman model of porous media flow this difficulty does not occur; see Jäger and Mikelić [17], Angot [1].) The second issue concerns the correct transmission conditions on the interface. The Beavers–Joseph–Saffman interface conditions [3, 25] are now well established. The third difficulty is technical: where the interface meets the other boundaries, there are incompatibilities between the imposed boundary conditions.

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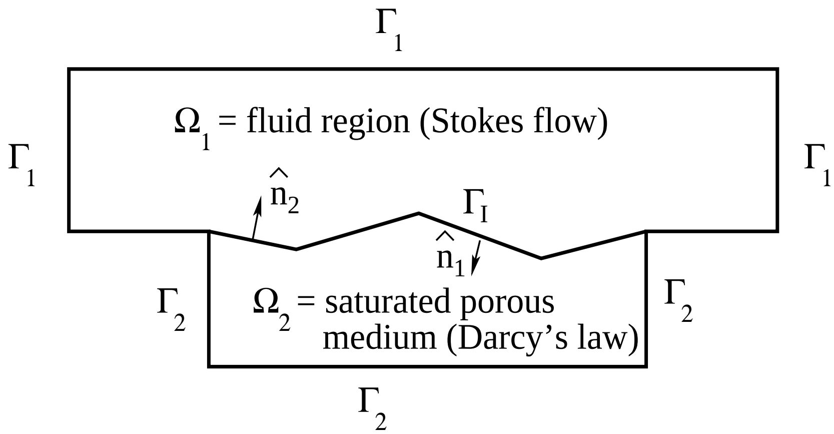

FIG. 1. *The model problem.*

One goal of this report is to find a variational formulation (section 2) for which weak solutions can be guaranteed to exist (section 3) and which can be used as a basis for a domain decomposition strategy for its approximate solution. The main goal is then to develop a finite element procedure with mathematical support (section 4). The method we study imposes the interface conditions using Lagrange multipliers. Thus, it can be used in a heterogeneous domain decomposition procedure in which each subproblem is alternately or simultaneously solved with codes (possibly "legacy" codes) developed and optimized for the physics of fluid motion and of porous media flow. In section 4 we give a complete analysis of this convergent finite element procedure. Because of the importance of the coupled problem, there are many computations of coupled surface water-groundwater flows in the applied literature, using various ad hoc interface decoupling strategies. See, for example, Salinger, Aris, and Derby [26], Gartling, Hickox, and Givler [14], and Prasad [23] for recent and interesting computational studies of the coupled problem.

The coupling strategy via Lagrange multipliers we consider herein has been proven in other applications and we are working towards practical tests of our ideas.

1.1. The model. The model we consider consists of Stokes flow in the fluid region Ω_1 and Darcy's law in the porous medium domain Ω_2. These are separated by an interface Γ_I. Here $\Omega_j \subset \mathbb{R}^d$ ($d = 2$ or 3) are bounded domains with outward unit normal vectors $\hat{n}_j$, $j = 1, 2$. Let $\Gamma_j := \partial\Omega_j \setminus \Gamma_I$. Each interface and boundary is assumed to be polygonal ($d = 2$) or polyhedral ($d = 3$). Figure 1 gives a schematic representation of the geometry.

The fluid velocities and pressures in Ω_1 and Ω_2 are denoted by

$$u_j : \Omega_j \to \mathbb{R}^d, \text{ fluid velocity in } \Omega_j,$$
$$p_j : \Omega_j \to \mathbb{R}\ , \text{ fluid pressure in } \Omega_j.$$

It is important to keep in mind that the velocities and pressures play different mathematical (and physical) roles in the fluid region and in the porous medium.

Recall that the deformation rate tensor $\mathbf{D}$ and stress tensor $\mathbf{T}$ associated with (u_1, p_1) are defined by

$$\mathbf{D}(u_1) := \frac{1}{2}\left(\frac{\partial u_{1i}}{\partial x_j} + \frac{\partial u_{1j}}{\partial x_i}\right), \qquad \mathbf{T}(u_1, p_1) := -p_1\mathbf{I} + 2\mu\mathbf{D}(u_1),$$

where μ is the viscosity. Assuming Stokes flow, (u_1, p_1) satisfies on Ω_1

$$\begin{cases} -\nabla \cdot \mathbf{T}(u_1, p_1) = f_1 \quad \text{in } \Omega_1 \quad \text{(conservation of momentum)}, \\ \nabla \cdot u_1 = 0 \quad \text{in } \Omega_1 \quad \text{(conservation of mass)}, \\ u_1 = 0 \quad \text{on } \Gamma_1 \quad \text{(no slip)}. \end{cases} \tag{1.1}$$

Assuming Darcy's law and no flow through Γ_2, (u_2, p_2) satisfies on Ω_2

$$\begin{cases} u_2 = -k\nabla p_2 \quad \text{in } \Omega_2 \quad \text{(Darcy's law)}, \\ \nabla \cdot u_2 = f_2 \quad \text{in } \Omega_2 \quad \text{(conservation of mass)}, \\ u_2 \cdot \hat{n}_2 = 0 \quad \text{on } \Gamma_2 \quad \text{(no flow)}, \end{cases} \tag{1.2}$$

where k is a symmetric and uniformly positive definite tensor representing the rock permeability divided by the fluid viscosity. The source f_2 is assumed to satisfy the solvability condition

$$\int_{\Omega_2} f_2 \, dx = 0, \tag{1.3}$$

which makes physical sense due to the no-flow boundary condition on $\partial\Omega$ and to (1.4) below. The mixed formulation (1.2) is the most natural one for computations in the porous medium region since it leads to direct approximation of the velocity.

1.2. Interface conditions. The problems (1.1)–(1.2) must be coupled across Γ_I by the correct interface conditions. *Mass conservation* across Γ_I is expressed by

$$u_1 \cdot \hat{n}_1 + u_2 \cdot \hat{n}_2 = 0 \quad \text{on} \quad \Gamma_I. \tag{1.4}$$

The second interface condition is *balance of normal forces* across Γ_I. Recall from, e.g., Serrin [28], that the Cauchy stress vector or traction vector $\vec{t}$ is the force on $\partial\Omega_1$ acting on the fluid volume inside Ω_1 and that

$$\vec{t}(u_1, p_1) = \hat{n}_1 \cdot \mathbf{T}(u_1, p_1)$$

(see Figure 2). Thus, the force on Γ_I exerted by the fluid volume is $-\vec{t}$. The only force in Ω_2 acting on Γ_I is the Darcy pressure p_2. Continuity of forces gives

$$-\vec{t}(u_1, p_1) \cdot \hat{n}_1 = p_2 \quad \text{on} \quad \Gamma_I.$$

This gives the interface condition

$$p_1 - 2\mu\hat{n}_1 \cdot \mathbf{D}(u_1) \cdot \hat{n}_1 = p_2 \quad \text{on} \quad \Gamma_I. \tag{1.5}$$

Finally, since the fluid model is viscous, a condition on the tangential fluid velocity on Γ_I must be given. Let $\hat{\tau}_j$, $j = 1, d-1$, denote an orthonormal system of tangent vectors on Γ_I. The simplest assumption is no-slippage along Γ_I, i.e., $u_1 \cdot \hat{\tau}_j = 0$, $j = 1, d-1$. This is not in good accord with experiment. The boundary condition in best agreement with experimental evidence evolved from the work of Beavers and Joseph [3] and states that

(slip velocity along Γ_I) is proportional to (shear stress along Γ_I).

Mathematically, this can be represented by

$$(u_1 - u_2) \cdot \hat{\tau}_j = \left(\frac{\sqrt{\tilde{k}_j}}{\mu\alpha_1} \right) (-\vec{t}(u_1, p_1)) \cdot \hat{\tau}_j, \qquad j = 1, d-1, \quad \text{on} \quad \Gamma_I,$$

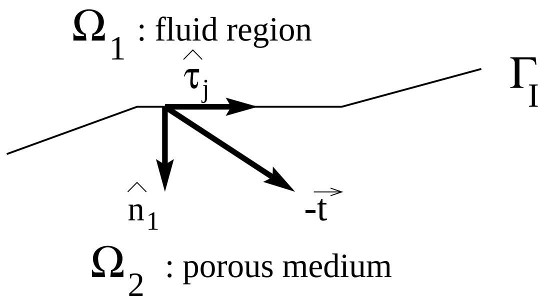

FIG. 2. *The traction vector on* Γ_I.

where $\tilde{k}_j = \hat{\tau}_j \cdot \mu k \cdot \hat{\tau}_j$. However, it is still unclear if this leads to a well-posed problem and it has been observed that the term on the left-hand side "$u_2 \cdot \hat{\tau}_j$" is much smaller than the other terms. Thus, its inclusion in this linear approximation is unclear. The most accepted interface condition was derived by Saffman [25] using a statistical approach and the Brinkman approximation and also by Jones [18] (also see Jäger and Mikelić [17]). This condition, which drops this term, is now known as the Beavers–Joseph–Saffman law and is thus given by

$$u_1 \cdot \hat{\tau}_j = -\frac{\sqrt{\tilde{k}_j}}{\alpha_1}\, 2\hat{n}_1 \cdot \mathbf{D}(u_1) \cdot \hat{\tau}_j, \qquad j = 1, d-1, \text{ on } \Gamma_I. \tag{1.6}$$

Here the form $\sqrt{\tilde{k}_j}/\alpha_1$ for the friction constant arises from dimensional analysis and experimental evidence. The parameter α_1 must be experimentally determined; it seems to depend on many particular features of Γ_I, including its geometry. See, e.g., Beavers and Joseph [3], Payne and Straughan [22], Saffman [25], and Jäger and Mikelić [16, 17] (among roughly 500 papers studying or using this interface condition) for more information.

2. Weak formulation of the coupled problem.

This section is devoted to developing suitable weak formulations of the problem (1.1)–(1.6). The weak formulations have two important purposes. One formulation is used to show well-posedness of (1.1)–(1.6). This is already nontrivial because of the incompatibility of the boundary and interface conditions where Γ_I, Γ_1, and Γ_2 meet. Thus, the conditions at these points must be interpreted correctly. A second closely related weak form is developed which is suitable for efficiently splitting the coupled problem into two subproblems. In this formulation the coupling conditions (1.4)–(1.5) are viewed as constraints and imposed via Lagrange multipliers.

Notation. For a subdomain $G \subset \mathbb{R}^d$, the $L^2(G)$ inner product (or duality pairing) and norm are denoted $(\cdot,\cdot)_G$ and $\|\cdot\|_G$, respectively, for scalar, vector, and tensor valued functions. For example, for tensor valued functions $A, B : G \to \mathbb{R}^{d\times d}$,

$$(A, B)_G := \sum_{i,j=1}^{d} \int_G A_{ij}(x) B_{ij}(x) dx = \int_G A : B \; dx.$$

For a connected open subset of the boundary $\Gamma \subset \partial\Omega_1 \cup \partial\Omega_2$, we write $\langle\cdot,\cdot\rangle_\Gamma$ and $\|\cdot\|_\Gamma$ for the $L^2(\Gamma)$ inner product (or duality pairing) and norm, respectively, for scalar valued functions λ, μ and vector valued functions u, v:

$$\langle \lambda, \mu \rangle_\Gamma := \int_\Gamma \lambda\mu \; ds, \qquad \langle u, v \rangle_\Gamma := \int_\Gamma \sum_{i=1}^{d} \; u_i v_i \; ds.$$

The Sobolev spaces $H^k(\Omega) = W^{k,2}(\Omega)$ are defined in the usual ways for $\Omega = \Omega_1$ or Ω_2 with the usual norm and seminorm $\|\cdot\|_{k,\Omega}$ and $|\cdot|_{k,\Omega}$, respectively. Let

$$X_1 := \{v_1 \in (H^1(\Omega_1))^d : v_1 = 0 \text{ on } \Gamma_1\}, \quad M_1 := L^2(\Omega_1)$$

denote the usual velocity-pressure spaces on Ω_1. The norm on X_1 is given by

$$\|v_1\|_{X_1} := |v_1|_{1,\Omega_1} := \|\nabla v_1\|_{\Omega_1}.$$

The velocity space X_2 on Ω_2 [24, 15, 7] is the subspace of

$$H(\text{div}; \Omega_2) = \{v_2 \in (L^2(\Omega_2))^d : \nabla \cdot v_2 \in L^2(\Omega_2)\}$$

consisting of functions with zero normal trace on Γ_2 and equipped with the norm

$$\|v_2\|_{H(\text{div};\Omega_2)} := (\|v_2\|^2_{\Omega_2} + \|\nabla \cdot v_2\|^2_{\Omega_2})^{1/2}.$$

It is well known [24, 15, 7] that for all $v_2 \in H(\text{div}; \Omega_2)$, $v_2 \cdot \hat{n}_2 \in H^{-1/2}(\partial\Omega_2)$ and there exists a positive constant C such that

$$\|v_2 \cdot \hat{n}_2\|_{-1/2,\partial\Omega_2} \leq C\|v_2\|_{H(\text{div};\Omega_2)}. \tag{2.1}$$

The restriction of $v_2 \cdot \hat{n}_2$ to Γ_2, however, may not lie in $H^{-1/2}(\Gamma_2)$. We define the velocity-pressure spaces on Ω_2 as follows [30], [7, sect. III.1]:

$$X_2 := \{v_2 \in H(\text{div}; \Omega_2) : \langle v_2 \cdot \hat{n}_2, w \rangle_{\partial\Omega_2} = 0 \text{ for all } w \in H^1_{0,\Gamma_I}(\Omega_2)\}, \quad M_2 := L^2(\Omega_2),$$

where

$$H^1_{0,\Gamma_I}(\Omega_2) = \{w \in H^1(\Omega_2) : w = 0 \text{ on } \Gamma_I\}.$$

Defining $X := X_1 \times X_2$, a typical $v \in X$ takes the form (v_1, v_2) with $v_i \in X_i$. The norm on X is, as usual,

$$\|v\|_X := (\|v_1\|^2_{X_1} + \|v_2\|^2_{X_2})^{1/2} \qquad \text{for all } v \in X.$$

If $V \subset X$ is any closed subspace, then $\|\cdot\|_X$ is also the induced norm on V. Similarly, let

$$M := \left\{ q = (q_1, q_2) : q_i \in M_i \text{ and } \sum_{i=1}^{2} (q_i, 1)_{\Omega_i} = 0 \right\},$$

with norm

$$\|q\|_M := (\|q_1\|^2_{M_1} + \|q_2\|^2_{M_2})^{1/2}.$$

The coupling across Γ_I between the subproblems in Ω_1 and Ω_2 occurs in the interface conditions (1.4)–(1.5). The procedure for uncoupling the two subproblems is to pick one (we pick the second) and introduce the Lagrange multiplier λ:

$$p_1 - 2\mu\hat{n}_1 \cdot \mathbf{D}(u_1) \cdot \hat{n}_1 = \lambda = p_2 \qquad \text{on } \Gamma_I. \tag{2.2}$$

Considering λ to be known data for each subproblem, the weak formulation is then derived in the usual manner as follows. Beginning with a classical solution of (1.1), multiplying by a sufficiently smooth $v_1 \in X_1$, and integrating by parts gives

$$\begin{aligned}(f_1, v_1)_{\Omega_1} &= (-2\mu\nabla \cdot \mathbf{D}(u_1) + \nabla p_1, v_1)_{\Omega_1} \\ &= 2\mu(\mathbf{D}(u_1), \mathbf{D}(v_1))_{\Omega_1} - (p_1, \nabla \cdot v_1)_{\Omega_1} \\ &+ \langle\{p_1 - 2\mu\hat{n}_1\mathbf{D}(u_1)\hat{n}_1\}, v_1 \cdot \hat{n}_1\rangle_{\Gamma_I} \\ &+ \sum_{j=1}^{d}\langle\{-2\mu\hat{n}_1\mathbf{D}(u_1)\hat{\tau}_j\}, v_1 \cdot \hat{\tau}_j\rangle_{\Gamma_I}.\end{aligned}$$

The first term in the braces $\{\cdot\}$ is replaced by λ using (2.2) and the second by $(\mu\alpha_1/\sqrt{\tilde{k}_j})\ u_1 \cdot \hat{\tau}_j$ using (1.6). Therefore, introducing the bilinear forms

$$a_1(u_1, v_1) := 2\mu(\mathbf{D}(u_1), \mathbf{D}(v_1))_{\Omega_1} + \sum_{j=1}^{d-1} \frac{\mu\alpha_1}{\sqrt{\tilde{k}_j}}\ \langle u_1 \cdot \hat{\tau}_j, v_1 \cdot \hat{\tau}_j\rangle_{\Gamma_I} \text{ for all } u_1, v_1 \in X_1,$$

and

$$b_1(v_1, q_1) := -(q_1, \nabla \cdot v_1)_{\Omega_1} \qquad \text{for all } v_1 \in X_1, q_1 \in M_1,$$

we obtain for all $v_1 \in X_1$ and $q_1 \in M_1$

$$\begin{aligned}a_1(u_1, v_1) + b_1(v_1, p_1) + \langle\lambda, v_1 \cdot \hat{n}_1\rangle_{\Gamma_I} &= (f_1, v_1)_{\Omega_1}, \\ b_1(u_1, q_1) &= 0.\end{aligned}$$

In the porous medium region, multiplication of the first equation in (1.2) by $v_2 \in X_2$, integration over Ω_2, and integration by parts gives

$$0 = (k^{-1}u_2 + \nabla p_2, v_2)_{\Omega_2} = (k^{-1}u_2, v_2)_{\Omega_2} - (p_2, \nabla \cdot v_2)_{\Omega_2} + \langle\lambda, v_2 \cdot \hat{n}_2\rangle_{\Gamma_I},$$

where, by (2.2), p_2 is replaced by λ in the last term. Introducing

$$a_2(u_2, v_2) := (k^{-1}u_2, v_2)_{\Omega_2}, \quad b_2(v_2, p_2) := -(p_2, \nabla \cdot v_2)_{\Omega_2},$$

we have

$$\begin{aligned}a_2(u_2, v_2) + b_2(v_2, p_2) + \langle\lambda, v_2 \cdot \hat{n}_2\rangle_{\Gamma_I} &= 0 \qquad \text{for all } v_2 \in X_2, \\ b_2(u_2, q_2) &= -(f_2, q_2) \qquad \text{for all } q_2 \in M_2.\end{aligned}$$

The linking across Γ_I occurs through the condition $u_1 \cdot \hat{n}_1 + u_2 \cdot \hat{n}_2 = 0$ on Γ_I and the definition (2.2) of λ. This linkage is the key to the well-posedness of the coupled problem and it hinges on the choice of the space Λ for the Lagrange multipliers. Define

$$b_I(v, \lambda) := \langle v_1 \cdot \hat{n}_1 + v_2 \cdot \hat{n}_2, \lambda\rangle_{\Gamma_I} : X \times \Lambda \to \mathbb{R},$$

where Λ is not yet specified. The flux continuity condition (1.4) on Γ_I is then

$$b_I(v, \lambda) = 0 \qquad \text{for all } \lambda \in \Lambda.$$

Since $v_2 \in H\,(\text{div}, \Omega_2)$, it holds that $v_2 \cdot \hat{n}_2 \in H^{-1/2}(\partial\Omega_2)$. We wish to pick $\Lambda \subset L^2(\Gamma_I)$ to be the largest space for which the pairing $\langle v_2 \cdot \hat{n}_2, \lambda\rangle_{\Gamma_I}$ is well defined. We show in Lemma 2.1 below (see also [20]) that

$$v_2 \cdot \hat{n}_2|_{\Gamma_I} \in (H^{1/2}_{00}(\Gamma_I))^*,$$

where $H^{1/2}_{00}(\Gamma_I)$ is the completion of the smooth functions with compact support in Γ_I with respect to the norm

$$\|\mu\|_{1/2,\partial\Omega_2} := \left(\|\mu\|^2_{\partial\Omega_2} + \int_{\partial\Omega_2}\int_{\partial\Omega_2} \frac{|\mu(t_1)-\mu(t_2)|^2}{|t_1-t_2|^d} ds_{t_1} ds_{t_2}\right)^{1/2}.$$

It is well known that $H^{1/2}_{00}(\Gamma_I)$ is the interpolation space

$$H^{1/2}_{00}(\Gamma_I) = [L^2(\Gamma_I), H^1_0(\Gamma_I)]_{1/2}.$$

Any function $\mu \in H^{1/2}_{00}(\Gamma_I)$ has the property that its extension by zero to $\partial\Omega_j$ gives a function $\tilde{\mu}_j \in H^{1/2}(\partial\Omega_j)$ with

$$\|\tilde{\mu}_j\|_{1/2,\partial\Omega_j} \le C\|\mu\|_{H^{1/2}_{00}(\Gamma_I)}, \quad j = 1, 2. \tag{2.3}$$

See Lions and Magenes [19] for background information on $H^{1/2}_{00}(\Gamma_I)$.

Accordingly, choose

$$\Lambda := H^{1/2}_{00}(\Gamma_I)\ (\subset L^2(\Gamma_I)).$$

LEMMA 2.1. *The bilinear form $b_I(\cdot,\cdot)$ is continuous on $X \times \Lambda$.*

Proof. First note that $v_j \cdot \hat{n}_j \in H^{-1/2}(\partial\Omega_j), j = 1, 2$. Let $\mu \in H^{1/2}_{00}(\Gamma_I)$ and let $\tilde{\mu}_j$ be its extension by zero to $\partial\Omega_j$. We have, for $j = 1, 2$,

$$\begin{aligned}\int_{\Gamma_I} v_j \cdot \hat{n}_j \mu\, ds = \int_{\partial\Omega_j} v_j \cdot \hat{n}_j \tilde{\mu}_j\, ds &\le \|v_j \cdot \hat{n}_j\|_{-1/2,\partial\Omega_j}\|\tilde{\mu}_j\|_{1/2,\partial\Omega_j}\\ &\le C\|v\|_X\|\mu\|_\Lambda,\end{aligned}$$

using (2.1) and (2.3) in the last inequality. ∎

Further, define

$$\begin{aligned}a(u,v) &:= \sum_{i=1}^{2} a_i(u_i, v_i) : X \times X \to \mathbb{R},\\ b(v,p) &:= \sum_{i=1}^{2} b_i(v_i, p_i) : X \times M \to \mathbb{R},\\ \ell(v) &:= (f_1, v_1)_{\Omega_1}, \quad g(q) := -(f_2, q_2)_{\Omega_2}.\end{aligned}$$

Then, (1.1)–(1.6) has the following weak formulation: find $(u, p, \lambda) \in X \times M \times \Lambda$ satisfying

$$\begin{cases} a(u,v) + b(v,p) + b_I(v,\lambda) = \ell(v) & \text{for all } v \in X,\\ b(u,q) = g(q) & \text{for all } q \in M,\\ b_I(u,\mu) = 0 & \text{for all } \mu \in \Lambda.\end{cases} \tag{2.4}$$

We next derive another weak formulation using the space V of functions in X with trace-continuous normal velocities:

$$V := \{v \in X : b_I(v,\mu) = 0 \text{ for all } \mu \in \Lambda\}.$$

The connection between the two formulations (2.4) and (2.5) is considered in Remark 3.1 in section 3. Note that, due to Lemma 2.1, V is a closed subspace of X, e.g., Brezzi and Fortin [7]. The next lemma indicates that a trace-continuous normal velocity has a well-defined divergence on the whole domain. Let

$$\Omega := \text{ interior}(\overline{\Omega}_1 \cup \overline{\Omega}_2).$$

For a given $v = (v_1, v_2) \in X$, define $\tilde{v} \in (L^2(\Omega))^d$ by $\tilde{v}|_{\Omega_j} := v_j$, $j = 1, 2$. To simplify notation we will omit the tilde in this construction since the meaning whether it is v or $\tilde{v}$ is clear from the context.

Lemma 2.2. *If $v \in V$, then $v \in H(\mathrm{div};\Omega)$.*

Proof. Define

$$g(x) = \nabla \cdot v_j(x) \quad \text{for } x \in \Omega_j,\ j = 1, 2.$$

We will show that $g = \nabla \cdot v$. Since $v_j \in H(\mathrm{div};\Omega_j)$, $j = 1, 2$, we can apply the divergence theorem in each Ω_j. This gives, for all $\phi \in C_0^\infty(\Omega)$,

$$\begin{aligned}\int_\Omega v\nabla\phi\ dx &= \int_{\Omega_1} v_1\nabla\phi\ dx + \int_{\Omega_2} v_2\nabla\phi\ dx \\ &= -\int_{\Omega_1} (\nabla\cdot v_1)\phi\ dx - \int_{\Omega_2} (\nabla\cdot v_2)\phi\ dx \\ &\quad + \int_{\Gamma_I} (v_1\cdot\hat{n}_1 + v_2\cdot\hat{n}_2)\phi\ dx.\end{aligned}$$

The last term vanishes since $\phi \in C_0^\infty(\Omega)$ implies $\phi_{|\Gamma_I} \in H_{00}^{1/2}(\Gamma_I)$. Thus,

$$\int_\Omega v\nabla\phi\ dx = -\int_\Omega g\phi\ dx.$$

Since $\nabla\cdot v_j \in L^2(\Omega_j)$, $g \in L^2(\Omega)$, and hence g is the weak L^2 divergence of $v \in V$. ∎

We next define the subspace Z,

$$Z := \{v \in V : b(v,q) = 0 \qquad \text{for all } q \in M\}.$$

Lemma 2.3. *The space Z is a closed subspace of V and X. Moreover, if $v \in Z$, then $\nabla\cdot v = 0$, a.e. $x \in \Omega$.*

Proof. Let $v \in Z$. Since $Z \subset V$, we know by Lemma 2.2 that $v \in H(\mathrm{div};\Omega)$. Thus, for any $q \in M$

$$b(v,q) = -\int_\Omega \nabla\cdot v\ q\ dx.$$

We claim that $\nabla\cdot v \in M$. Indeed, $\nabla\cdot v \in L^2(\Omega)$ and $\nabla\cdot v$ has zero mean value over Ω:

$$\int_\Omega \nabla\cdot v\ dx = \int_{\partial\Omega} v\cdot\hat{n}\ ds = 0$$

using the divergence theorem. Thus, $\nabla\cdot v \in M$. The second part of the lemma follows by setting $q = \nabla \cdot v$.

The space Z is a closed subspace of V since

$$\begin{aligned} b(v,q) &= -\int_\Omega \nabla\cdot v\ q\ dx \le \|\nabla\cdot v\|_\Omega \|q\|_\Omega \\ &\le \|v\|_X\|q\|_M, \end{aligned}$$

i.e., $b(\cdot,\cdot)$ is continuous on $V\times M$. ◻

Since V is a closed subspace of X, we can write the following variational formulation: find $(u,p)\in V\times M$ satisfying

$$\left\{ \begin{array}{l} a(u,v)+b(v,p)=\ell(v) \qquad \text{for all } v\in V, \\ b(u,q)=g(q) \qquad \text{for all } q\in M. \end{array} \right. \tag{2.5}$$

We end this section noting that, under the solvability condition (1.3), any solution of (2.5) satisfies the mass conservation equations in (1.1) and (1.2). Indeed, define $f\in L^2(\Omega)$ such that $f=0$ on Ω_1 and $f=f_2$ on Ω_2. If (u,p) is a solution to (2.5), then $\nabla\cdot u\in L^2(\Omega)$ due to Lemma 2.2. The second equation in (2.5) implies that $\nabla\cdot u - f = c$, where c is a constant. The divergence theorem gives

$$c|\Omega| = \int_\Omega (\nabla\cdot u - f)\,dx = \int_{\partial\Omega} u\cdot\hat{n}\,ds - \int_\Omega f\,dx = -\int_{\Omega_2} f_2\,dx = 0$$

using (1.3). Therefore $\nabla\cdot u = 0$ on Ω_1 and $\nabla\cdot u = f_2$ on Ω_2.

3. Analysis of the weak formulation. This section is devoted to a proof of existence of weak solutions to (1.1)–(1.6) based on the weak formulations (2.4) and (2.5). Existence depends on our choice of the Lagrange multiplier space $\Lambda = H_{00}^{1/2}(\Gamma_I)$ so that the problem is neither over nor underconstrained.

We begin with a few simple but useful estimates. Let

$$W_2 := \{v_2\in X_2 : \nabla\cdot v_2 = 0,\ \text{a.e. } x\in\Omega_2\}\subset X_2$$

denote the (closed) subspace of div-free functions in X_2.

LEMMA 3.1. *For $v_i\in H^1(\Omega_i)^d\cap X_i$ $(i=1,2)$ we have*

$$C_1\|v_i\|_{\Omega_i} \le \|v_i\|_{X_i} \le C_2\|v_i\|_{1,\Omega_i}. \tag{3.1}$$

Furthermore, for $i=1,2$, there holds

$$|a_i(u_i,v_i)| \le C_3\|u_i\|_{X_i}\|v_i\|_{X_i} \textit{ for all } u_i,v_i\in X_i, \tag{3.2}$$

$$a_1(v_1,v_1) \ge C_4\|v_1\|_{X_1}^2 \textit{ for all } v_1\in X_1, \tag{3.3}$$

$$a_2(v_2,v_2) \ge C_5\|v_2\|_{X_2}^2 \textit{ for all } v_2\in W_2, \tag{3.4}$$

$$|b_i(v_i,p_i)| \le C_6\|v_i\|_{X_i},\|p_i\|_{M_i} \textit{ for all } v_i\in X_i, p_i\in M_i, \tag{3.5}$$

$$|a(u,v)| \le C_3\|u\|_X\|v\|_X \textit{ for all } u,v\in X, \tag{3.6}$$

$$|b(v,p)| \le C_6\|v\|_X\|p\|_M \textit{ for all } v\in X,\ p\in M, \tag{3.7}$$

$$a(v,v) \ge \min\{C_4,C_5\}\|v\|_X^2 \textit{ for all } v\in X_1\times W_2. \tag{3.8}$$

Proof. Inequalities (3.1) and (3.2) follow from the Poincaré–Friedrich inequality and the trace theorem. The Korn inequality implies (3.3) while (3.4) and (3.5) are immediate. Inequalities (3.6), (3.7), and (3.8) follow by combining earlier ones. ◻

The next lemma establishes the Ladyzhenskaya–Babuška–Brezzi condition required for the formulation (2.5) in $V \times M$.

LEMMA 3.2. *There is a constant $\beta > 0$ such that*

$$\inf_{q\in M\setminus\{0\}} \sup_{v\in V\setminus\{0\}} \frac{b(v,q)}{\|v\|_X \ \|q\|_M} \geq \beta. \tag{3.9}$$

Proof. Let $q \in M \setminus \{0\}$ be fixed but arbitrary. We construct a $v \in V$ satisfying

$$b(v,q) \geq \beta \|v\|_X \ \|q\|_M.$$

Given $q = (q_1, q_2) \in M$, the function $\tilde{q}(x)$ defined by $\tilde{q}|_{\Omega_i} = q_i$ has mean value zero over Ω; thus $\tilde{q} \in L^2_0(\Omega)$. Thus, (see, e.g., [15, 13]) there exists $\tilde{v} \in (H^1_0(\Omega))^d$ satisfying

$$\nabla \cdot \tilde{v} = \tilde{q}, \text{ in } \Omega, \ \tilde{v} = 0, \text{ on } \partial\Omega, \ \|\tilde{v}\|_{1,\Omega} \leq C_7 \|\tilde{q}\|_\Omega.$$

Given this $\tilde{v}$, define $v = (v_1, v_2) \in X$ by $v_i = \tilde{v}|_{\Omega_i}, (i = 1, 2)$. Since

$$\tilde{v} \in H^1_0(\Omega)^d, \text{ it follows that } v_1|_{\Gamma_1} = 0 \text{ and } v_2 \cdot \hat{n}_2|_{\Gamma_2} = 0.$$

Further, $v_1|_{\Gamma_I} = v_2|_{\Gamma_I} = \tilde{v}|_{\Gamma_I} \in (H^{1/2}_{00}(\Gamma_I))^d$ so that $v_i \cdot \hat{n}_i \in L^2(\Gamma_I)$ $(i = 1, 2)$ and

$$b_I(v, \mu) = \langle v_1 \cdot \hat{n}_1 + v_2 \cdot \hat{n}_2, \ \mu \rangle_{\Gamma_I} = 0$$

for all $\mu \in L^2(\Gamma_I)$. Thus, $v \in V$. Using (3.1) we find

$$\|v\|_X \leq C_2 \|\tilde{v}\|_{1,\Omega} \leq C_2 C_7 \|\tilde{q}\|_{0,\Omega} = C_2 C_7 \|q\|_M.$$

Finally, for this v

$$b(v,q) = \sum_{i=1}^{2} (-\nabla \cdot v_i, q_i) = -(\nabla \cdot \tilde{v}, \tilde{q})_\Omega \tag{3.10}$$

$$= \|\tilde{q}\|^2_{0,\Omega} \geq (C_2 C_7)^{-1} \|v\|_X \|q\|_M, \tag{3.11}$$

completing the proof with $\beta = (C_2 C_7)^{-1}$. ∎

To apply the abstract theory of mixed problems in, e.g., Girault and Raviart [15], Brezzi and Fortin [7], we must show $a(\cdot,\cdot)$ is coercive on the constraint set Z. This is accomplished in the next lemma.

LEMMA 3.3. *$a(\cdot,\cdot)$ is coercive on Z: there is an $\alpha > 0$ such that*

$$a(v,v) \geq \alpha \|v\|^2_X \qquad \textit{for all } v \in Z.$$

Proof. Note that by Lemma 2.3 if $v = (v_1, v_2) \in \ker(B)$, $\nabla \cdot v_2 = 0$, a.e. $x \in \Omega$, i.e., $v_2 \in W_2$. Coercivity now follows from (3.8) of Lemma 3.1. ∎

Lemmas 2.1, 3.2, and 3.3, together with the abstract theory of mixed problems [15, 7], immediately imply existence of a weak solution $(u, p) \in V \times M$ satisfying (2.5).

THEOREM 3.1. *There exists a unique solution $(u, p) \in V \times M$ to the problem* (2.5). ∎

To verify that the solution to (2.5) is also the solution to the formulation (2.4) in $X \times M \times \Lambda$ using the general saddle point problem theory [15, 7], we must verify the inf-sup condition

$$\inf_{0\neq\lambda\in\Lambda} \sup_{0\neq v\in X} \frac{b_I(v,\lambda)}{\|v\|_X \|\lambda\|_\Lambda} \geq \beta > 0. \tag{3.12}$$

Due to technical difficulties related to the restriction of $H^{-1/2}(\partial\Omega_2)$ functions to Γ_I, we are only able to show a modified inf-sup condition:

$$\inf_{0\neq\lambda\in\Lambda}\sup_{0\neq v\in X}\frac{b_I(v,\lambda)}{\|v\|_X\|\lambda\|_{1/2,\Gamma_I}}\geq\beta>0. \tag{3.13}$$

LEMMA 3.4. *The* inf-sup *condition* (3.13) *holds.*

Proof. Fix $\lambda\in H^{1/2}_{00}(\Gamma_I)$ and let $\tilde\lambda\in H^{1/2}(\partial\Omega_2)$ be its extension by zero to $\partial\Omega_2$. Since $H^{1/2}_{00}(\Gamma_I)\subset H^{1/2}(\Gamma_I)$, there exists $\hat\lambda_I\in H^{-1/2}(\Gamma_I)$ such that

$$\frac{\langle\hat\lambda_I,\lambda\rangle_{\Gamma_I}}{\|\hat\lambda_I\|_{-1/2,\Gamma_I}}\ \geq\ \frac{1}{2}\|\lambda\|_{1/2,\Gamma_I}. \tag{3.14}$$

We next define $\hat\lambda\in H^{-1/2}(\partial\Omega_2)$ by

$$\langle\hat\lambda,w\rangle_{\partial\Omega_2}:=\langle\hat\lambda_I,w\rangle_{\Gamma_I}\quad\text{for all } w\in H^{1/2}(\partial\Omega_2).$$

We then have

$$\|\hat\lambda\|_{-1/2,\partial\Omega_2}=\sup_{0\neq w\in H^{1/2}(\partial\Omega_2)}\frac{\langle\hat\lambda_I,w\rangle_{\Gamma_I}}{\|w\|_{1/2,\partial\Omega_2}}\leq\|\hat\lambda_I\|_{-1/2,\Gamma_I}. \tag{3.15}$$

Since the normal trace operator maps $H(\mathrm{div},\Omega_2)$ onto $H^{-1/2}(\partial\Omega_2)$ (see [15, Corollary 2.8]) and it is continuous (see (2.1)), by the open mapping theorem there exists $v_2\in H(\mathrm{div},\Omega_2)$ such that $v_2\cdot\hat n_2=\hat\lambda$ on $\partial\Omega_2$ and

$$\|v_2\|_{X_2}\leq C\|\hat\lambda\|_{-1/2,\partial\Omega_2}\leq C\|\hat\lambda_I\|_{-1/2,\Gamma_I}, \tag{3.16}$$

using (3.15) for the second inequality. We note that $v_2\in X_2$ since, for all $w\in H^1_{0,\Gamma_I}(\Omega_2)$,

$$\langle v_2\cdot\hat n_2,w\rangle_{\partial\Omega_2}=\langle\hat\lambda,w\rangle_{\partial\Omega_2}=\langle\hat\lambda_I,w\rangle_{\Gamma_I}=0.$$

Choosing $v=(0,v_2)\in X$ and using (3.14) and (3.16) we get

$$\begin{aligned}\frac{b_I(v,\lambda)}{\|v\|_X}&=\frac{\langle v_2\cdot\hat n_2,\tilde\lambda\rangle_{\partial\Omega_2}}{\|v_2\|_{X_2}}=\frac{\langle\hat\lambda,\tilde\lambda\rangle_{\partial\Omega_2}}{\|v_2\|_{X_2}}\\&=\frac{\langle\hat\lambda_I,\lambda\rangle_{\Gamma_I}}{\|v_2\|_{X_2}}\geq\frac{1}{C}\frac{\langle\hat\lambda_I,\lambda\rangle_{\Gamma_I}}{\|\hat\lambda_I\|_{-1/2,\Gamma_I}}\geq\beta\|\lambda\|_{1/2,\Gamma_I}.\end{aligned}$$ □

REMARK 3.1. *If the porous medium is entirely enclosed within the fluid region, then $\Gamma_I=\partial\Omega_2$. In this case there are no incompatible points and it is easy to extend slightly the proof of Lemma* 3.4 *to show that the stronger* inf-sup *condition* (3.12) *holds. In this case, the unique weak solution to* (2.5) *is also the unique weak solution to* (2.4) *and the two formulations are equivalent.*

4. Finite element discretization. This section considers the finite element discretization of the coupled problem. The interface conditions on Γ_I separate into tangential and normal conditions. This splitting on Γ_I introduces interesting features into the finite element procedure and its analysis.

Introduce upon Ω_j a mesh $\mathcal{T}_j^h$ $(j = 1, 2)$ with $\overline{\Omega}_j = \cup_{K\in\mathcal{T}_j^h} \overline{K}$. To simplify the notation we shall assume that the cells $K \in \mathcal{T}_j^h$ are affine equivalent, the grids $\mathcal{T}_1^h$ and $\mathcal{T}_2^h$ match at Γ_I, that Γ_I is polyhedral, and that no point of the interface boundary $\partial\Gamma_I$ belongs to the interior of an element face. We use the notation

$$\mathcal{E}_h(K) := \text{the set of all faces of the element } K,$$
$$\mathcal{E}_h(\Gamma_I) := \text{the set of all element faces } E \text{ with } E \subset \Gamma_I.$$

For the discretization of the fluid's variables we choose finite element spaces X_1^h and M_1^h which are assumed to be div-stable (also called LBB-stable),

$$\begin{cases} X_1^h \subset X_1, \ M_1^h \subset M_1, \text{ and} \\ \inf\limits_{0\neq q_1\in M_1^h} \ \sup\limits_{0\neq v_1\in X_1^h} \frac{b_1(v_1,q_1)}{\|v_1\|_{X_1}\|q_1\|_{M_1}} \ \geq \ \beta_1 > 0, \end{cases} \tag{4.1}$$

and to satisfy a discrete Korn inequality

$$(\mathbf{D}(v_1), \mathbf{D}(v_1))_{\Omega_1} \geq \alpha_1 |v_1|_{1,\Omega_1}^2 \quad \text{for all } v_1 \in X_1^h. \tag{4.2}$$

We assume that X_1^h and M_1^h include at least polynomials of degree r_1 and $r_1 - 1$, respectively, $(r_1 \geq 1)$. Specifically, we assume that there exist (quasi) interpolation operators

$$I_{X_1}^h : X_1 \cap (H^s(\Omega_1))^d \to X_1^h \quad \text{ and } \quad I_{M_1}^h : M_1 \cap H^s(\Omega_1) \ \to M_1^h$$

such that for all $K \in \mathcal{T}_1^h$

$$\begin{cases} |v_1 - I_{X_1}^h v_1|_{m,K} \leq C h_K^{s-m} |v_1|_{s,\delta(K)}, \quad m = 0, 1, \ 1 \leq s \leq r_1 + 1, \\ \|q_1 - I_{M_1}^h q_1\|_{0,K} \leq C h_K^s |q_1|_{s,\delta(K)}, \quad 0 \leq s \leq r_1. \end{cases} \tag{4.3}$$

Here $\delta(K)$ is equal to K in most cases of usual interpolation operators. However, in cases of quasi interpolation operators suited for H^1 functions like the Clement-operator [9] or the Scott–Zhang-operator [27], $\delta(K)$ denotes the vicinity of K consisting of all elements $\widetilde{K} \in \mathcal{T}_1^h$ that touch element K. We assume the grids $\mathcal{T}_1^h$ and $\mathcal{T}_2^h$ to be shape-regular in the usual sense such that cases with local grid refinement are allowed. For shape-regular grids, changes of the mesh size within the vicinity $\delta(K)$ of an element K are uniformly bounded by a constant C, i.e., in particular for $\mathcal{T}_1^h$,

$$C^{-1}\, h_K \leq h_{\widetilde{K}} \leq C\, h_K \qquad \text{for all } \widetilde{K} \subset \delta(K), \ \ \widetilde{K}, K \in \mathcal{T}_1^h. \tag{4.4}$$

This estimate is used to get rid of the $\delta(K)$-terms in final error estimates.

Examples of spaces satisfying (4.1)–(4.3) include the MINI elements [2], the Taylor–Hood elements [29], and the conforming Crouzeix–Raviart elements [10]. See, e.g., [15, 7], for a more complete list of such spaces.

REMARK 4.1. *The discrete Korn inequality* (4.2) *is inherited from the continuous inequality for all conforming elements. However, nonconforming spaces, in general, do not satisfy* (4.2)*; see* [12].

REMARK 4.2. *The* inf-sup *condition* (4.1) *differs from the usual one verified in the literature* [15, 7] *for various spaces because the pressure space* M_1^h *is not restricted to have zero mean over* Ω_1*, i.e.,* $M_1^h \subset L^2(\Omega_1)$*, not* $L_0^2(\Omega_1)$*. However, the usual discrete* inf-sup *condition is almost enough to prove* (4.1)*. The main extra ingredient* *needed is the existence of a (typically locally constructed, see* [7, section VI.4]*) operator* $P_1^h : X_1 \to X_1^h$ *(not necessarily the same as* $I_{X_1}^h$*) satisfying, for all* $K \in \mathcal{T}_1^h$ *and all* $v_1 \in X_1$,

$$\int_K \nabla \cdot (P_1^h v_1 - v_1)\, dx = 0 \quad \text{and} \quad \|P_1^h v_1\|_{1,\Omega_1} \le C_8 \|v_1\|_{1,\Omega_1}, \tag{4.5}$$

where C_8 *is a constant independent of* v_1 *and* h*. In, e.g.,* [7]*, such an operator is locally constructed for all the aforementioned spaces.*

The following lemma gives sufficient conditions for the discrete LBB-stability (4.1) of the spaces X_1^h and M_1^h.

LEMMA 4.1. *Suppose that an operator* $P_1^h : X_1 \to X_1^h$ *satisfying the condition* (4.5) *exists. Suppose also the spaces* $X_1^h \cap (H_0^1(\Omega_1))^d$ *and* $M_1^h \cap L_0^2(\Omega_1)$ *satisfy the usual discrete* inf-sup *condition. Then, the spaces* X_1^h *and* M_1^h *satisfy* (4.1).

Proof. Let $q_1^h \equiv q_0 \in \mathbb{R}$ be an arbitrary constant function of M_1^h. We first show that there exists a $v_1^h \in X_1^h$ such that

$$b_1(v_1^h, q_1^h) \ge \beta_0 \, \|v_1^h\|_{X_1} \|q_1^h\|_{M_1}$$

with a constant $\beta_0 > 0$ independent of v_1^h and h. To this end, let $\tilde{v}_1$ be a solution of the following problem: find $\tilde{v}_1 \in X_1$ satisfying

$$\nabla \cdot \tilde{v}_1 = q_1^h \;\text{ in } \Omega_1, \quad \tilde{v}_1 = g_1 \;\text{ on } \partial\Omega_1,$$

where g_1 is chosen suitably such that the compatibility condition $\langle g_1 \cdot \hat{n}_1, 1\rangle_{\partial\Omega_1} = (q_1^h, 1)_{\Omega_1} = q_0|\Omega_1|$ is fulfilled and $g_1 \in (H^{1/2}(\partial\Omega_1))^d$. By, e.g., [13, sect. III.3, Exercise 3.4], such a $\tilde{v}_1$ exists and satisfies the estimate

$$\|\tilde{v}_1\|_{1,\Omega_1} \le C_9 \{\|q_1^h\|_{\Omega_1} + \|g_1\|_{1/2,\partial\Omega_1}\}.$$

For the construction of g_1, let $\varphi_0 \in C(\partial\Omega_1)$ be such that $\varphi_0 \equiv 0$ on Γ_1, φ_0 is quadratic on Γ_I, and $\langle \varphi_0, 1\rangle_{\Gamma_I} = 1$. Then, we choose g_1 as $g_1 := |\Omega_1| q_0 \varphi_0 \hat{n}_1$. One can easily verify that g_1 belongs to $(H^{1/2}(\partial\Omega_1))^d$ and satisfies the compatibility condition as well as the estimate $\|g_1\|_{1/2,\partial\Omega_1} \le c(\Omega_1, \varphi_0)\|q_1^h\|_{\Omega_1}$. This implies

$$\|\tilde{v}_1\|_{1,\Omega_1} \le C_9\{1 + c(\Omega_1, \varphi_0)\}\, \|q_1^h\|_{\Omega_1} .$$

Defining $v_1^h := -P_1^h \tilde{v}_1$, we have

$$\frac{b_1(v_1^h, q_1^h)}{\|v_1^h\|_{X_1}\|q_1^h\|_{M_1}} = \frac{(\nabla \cdot \tilde{v}_1, q_1^h)_{\Omega_1}}{\|P_1^h \tilde{v}_1\|_{X_1}\|q_1^h\|_{M_1}} \ge \frac{\|q_1^h\|_{M_1}^2}{C_8\|\tilde{v}_1\|_{X_1}\|q_1^h\|_{M_1}} \ge \beta_0 \tag{4.6}$$

with $\beta_0 := (C_8 C_9\{1 + c(\Omega_1, \varphi_0)\})^{-1}$. Now, using this result and the assumed discrete inf-sup condition for the spaces $X_1^h \cap (H_0^1(\Omega_1))^d$ and $M_1^h \cap L_0^2(\Omega_1)$, we can show in the same way as in the proof of Theorem 1.12, section II.1.4 in [15] that the spaces X_1^h and M_1^h satisfy the inf-sup condition (4.1). ∎

For the discretization of the porous medium problem in Ω_2, we choose $X_2^h \times M_2^h \subset X_2 \times M_2$ to be any of the well-known mixed finite element spaces (see [7, section III.3]), the RT spaces [24, 21], the BDM spaces [6], the BDFM spaces [5], the BDDF spaces [4], or the CD spaces [8]. We assume that X_2^h and M_2^h contain at least polynomials of degree r_2 and l_2, respectively. It is known for these choices that

$$\nabla \cdot X_2^h = M_2^h$$

and that there exists an interpolation operator $I^h_{X_2} : (H^1(\Omega_2))^d \to X^h_2$ such that for all $v_2 \in (H^1(\Omega_2))^d$

$$(\nabla \cdot I^h_{X_2} v_2, w)_{\Omega_2} = (\nabla \cdot v_2, w)_{\Omega_2}, \quad w \in M^h_2. \tag{4.7}$$

Let $I^h_{M_2} : M_2 \to M^h_2$ be the L^2 orthogonal projection such that for all $q_2 \in M_2$

$$(I^h_{M_2} q_2, w)_{\Omega_2} = (q_2, w)_{\Omega_2}, \quad w \in M^h_2. \tag{4.8}$$

Our next lemma will collect some known useful results for these spaces. Their proof can be found in [7, section III.3].

Lemma 4.2. *There holds, for all $v_2 \in (H^1(\Omega_2))^d$,*

$$\langle I^h_{X_2} v_2 \cdot \hat{n}_2, \mu\rangle_E = \langle v_2 \cdot \hat{n}_2, \mu\rangle_E \quad \text{for all } \mu \in R_{r_2}(E) \text{ and for all } E \in \mathcal{E}_h(\Gamma_I), \tag{4.9}$$

where

$$R_{r_2}(E) := \begin{cases} \mathcal{P}_{r_2}(E) & \text{if } d = 2 \text{ or } E \text{ is a triangle}, \\ \mathcal{Q}_{r_2}(E) & \text{if } d = 3 \text{ and } E \text{ is a quadrilateral}, \end{cases} \tag{4.10}$$

where $P_{r_2}(E)$ and $Q_{r_2}(E)$ are the usual polynomial spaces (see, e.g., [7].) For the restrictions to the element faces,

$$v^h_2 \cdot \hat{n}_2|_E \in R_{r_2}(E) \quad \text{for all } v^h_2 \in X^h_2,\ E \in \mathcal{E}(K),\ K \in \mathcal{T}^h_2. \tag{4.11}$$

Further, the operators $I^h_{X_2}$ and $I^h_{M_2}$ satisfy, for all $K \in \mathcal{T}^h_2$,

$$\|q_2 - I^h_{M_2} q_2\|_{0,K} \le C h^s_K |q_2|_{s,K},\ 0 \le s \le l_2 + 1, \tag{4.12}$$

$$|v_2 - I^h_{X_2} v_2|_{m,K} \le C h^{s-m}_K |v_2|_{s,K},\ m \in \{0, 1\},\ 1 \le s \le r_2 + 1, \tag{4.13}$$

$$\|\nabla \cdot (v_2 - I^h_{X_2} v_2)\|_{0,K} \le C h^s_K |\nabla \cdot\ v_2|_{s,K},\ 0 \le s \le l_2 + 1. \tag{4.14}$$

□

4.1. The space V^h. Define the finite element spaces over Ω:

$$X^h := X^h_1 \times X^h_2,\ M^h := \left\{(q_1, q_2) \in M^h_1 \times M^h_2 : \int_{\Omega_1} q_1 dx + \int_{\Omega_2} q_2 dx = 0\right\}$$

and

$$\Lambda^h := \{\mu^h \in L^2(\Gamma_I) : \mu^h|_E \in \mathcal{R}_{r_2}(E) \text{ for all } E \in \mathcal{E}_h(\Gamma_I)\}.$$

Note that, since function $\mu^h \in \Lambda^h$ does not in general vanish on $\partial\Gamma_I$,

$$\Lambda^h \not\subset \Lambda.$$

With this Λ^h define

$$V^h := \{v = (v_1, v_2) \in X^h : b_I(v, \mu) = 0 \text{ for all } \mu \in \Lambda^h\}.$$

These choices result in an approximation which is nonconforming (since $\Lambda^h \not\subset \Lambda$) and exterior (since $V^h \not\subset V$).

Remark 4.3. *The space Λ^h is the normal trace of X^h_2 on Γ_I.*

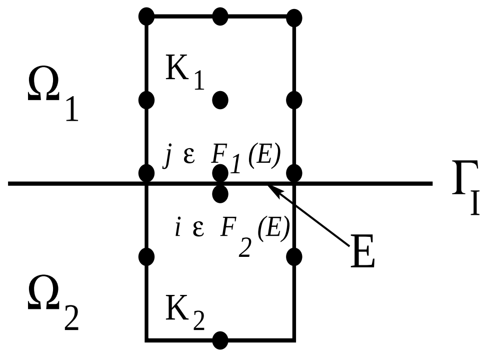

FIG. 3. *Degrees of freedom on* Γ_I.

We consider the following discrete problem: find $(u^h, p^h) \in V^h \times M^h$ satisfying

$$\begin{cases} a(u^h, v^h) + b(v^h, p^h) = \ell(v^h) \qquad \text{for all } v^h \in V^h, \\ b(u^h, q^h) = g(q^h) \qquad \text{for all } q^h \in M^h. \end{cases} \tag{4.15}$$

This is the natural discretization of (2.5). Since $V^h \not\subset V$, conservation of mass across Γ_I holds only in an approximate sense.

It is important to understand in exactly what sense mass conservation across Γ_I holds. To this end, a local characterization of the functions $v = (v_1, v_2) \in V^h$ is needed.

Characterization of $\boldsymbol{v = (v_1, v_2) \in V^h}$. If a function $v = (v_1, v_2) \in X^h$ belongs to V^h, then the nodal values of $v_2 \cdot \hat{n}_2 \in X_2^h$ on Γ_I are linked to those of $v_1 \cdot \hat{n}_1$ on Γ_I. To be specific, let $\mathcal{F}_i$ denote the set of nodes of $X_i^h, i = 1, 2$, and $\mathcal{F}_i(E)$ the set of nodes $j \in \mathcal{F}_i$ belonging to an element face E, and let $\phi_j^{(i)},\ j \in \mathcal{F}_i$ $(i = 1, 2)$, be the associated basis functions of X_i^h. Let $E \in \mathcal{E}_h(\Gamma_I)$ be an element face on Γ_I associated with elements $K_1 \subset \Omega_1$ and $K_2 \subset \Omega_2$,

$$E \in \mathcal{E}(K_1) \cap \mathcal{E}(K_2),\ K_i \in \Omega_i,$$

as depicted in Figure 3.

From the construction of the basis functions, we have for $v = (v_1, v_2) \in X^h$

$$v_i \cdot \hat{n}_i|_E = \sum_{j \in \mathcal{F}_i(E)} (v_j^{(i)} \phi_j^{(i)}) \cdot \hat{n}_i, \qquad i = 1, 2, \tag{4.16}$$

where $v_j^{(i)} \in \mathbb{R}$ are the nodal values of v_i. By (4.10)

$$\dim(R_{r_2}(E)) = \text{ cardinality}(\mathcal{F}_2(E))$$

so that there is a one-to-one *correspondence between nodes* $i \in \mathcal{F}_2(E)$ *and basis functions* $\lambda_{E,i} \in R_{r_2}(E)$ *such that*

$$R_{r_2}(E) = \text{ span } \{\lambda_{E,i} :\ i \in \mathcal{F}_2(E)\}. \tag{4.17}$$

Consider a degree of freedom associated with a node $i \in \mathcal{F}_2(E)$ that is precisely the nodal functional

$$N_i^{(2)}(v_2) := |E|^{-1} \langle v_2 \cdot \hat{n}_2,\ \lambda_{E,i} \rangle_E, \qquad |E| = \text{ measure } (E). \tag{4.18}$$

The basis functions are, by construction, dual with respect to these functionals:

$$N_i^{(2)}(\phi_j^{(2)}) = \delta_{ij} \qquad \text{for all } i, j \in \mathcal{F}_2. \tag{4.19}$$

From (4.18), (4.19), and the formula (4.16) for $v_i \cdot \hat{n}_i|_E$, we get

$$v_i^{(2)} = |E|^{-1} \langle v_2 \cdot \hat{n}_2,\ \lambda_{E,i} \rangle_E \quad \text{for all } i \in \mathcal{F}_2(E), E \in \mathcal{E}_h(\Gamma_I),\ v_2 \in X_2^h. \tag{4.20}$$

Consider the condition defining $V^h, b_I(v, \mu) = 0$ for all $\mu \in \Lambda^h$. Restricting μ to a generic basis function $\lambda_{E,i}$ for Λ^h gives

$$\langle v_2 \cdot \hat{n}_2, \lambda_{E,i} \rangle_E = -\langle v_1 \cdot \hat{n}_1, \lambda_{E,i} \rangle_E \qquad \text{for all } i \in \mathcal{F}_2(E),\ E \in \mathcal{E}_h(\Gamma_I). \tag{4.21}$$

Combining this with (4.20) gives

$$v_i^{(2)} = -|E|^{-1} \langle v_1 \cdot \hat{n}_1, \lambda_{E,i} \rangle_E \qquad \text{for all } i \in \mathcal{F}_2(E), E \in \mathcal{E}_h(\Gamma_I). \tag{4.22}$$

Inserting the expression of v_1 in terms of its nodal values (4.16) into (4.22) gives the following pointwise characterization of the space $v \in V^h$.

PROPOSITION 4.1. *Let $v = (v_1, v_2) \in X^h$ be given. Then $v \in V^h$ is equivalent to the following relation between the nodal values $v_i^{(1)}$ and $v_i^{(2)}$ of v_1 and v_2 on E being satisfied:*

$$v_i^{(2)} = -|E|^{-1} \sum_{j \in \mathcal{F}_1(E)} v_j^{(1)} \langle \phi_j^{(1)} \cdot \hat{n}_1,\ \lambda_{E,i} \rangle_E \quad \text{for all } i \in \mathcal{F}_2(E), \quad E \in \mathcal{E}_h(\Gamma_I). \qquad \square \tag{4.23}$$

REMARK 4.4. *The relation* (4.23) *can be interpreted to mean that the nodes*

$$i \in \bigcup_{E \in \mathcal{E}_h(\Gamma_I)} \mathcal{F}_2(E)$$

are "hanging nodes" in that values of the function $v \in V^h$ are determined by the corresponding values at the nodes $j \in \cup_{E \in \mathcal{E}_h(\Gamma_I)} \mathcal{F}_1(E)$.

4.2. Inf-sup conditions for the coupled problem. The discrete formulation (4.15) leads to the question of an inf-sup condition in $V^h \times M^h$. We show next that the usual fluid's velocity-pressure discrete inf-sup condition (4.1) in fact implies the needed $V^h \times M^h$ inf-sup condition.

LEMMA 4.3. *Suppose that (X_1^h, M_1^h) satisfies the discrete* inf-sup *condition* (4.1). *Then, (V^h, M^h) is LBB-stable as well. Specifically,*

$$\inf_{q^h \in M^h} \sup_{V^h \in V^h} \frac{b(v^h, q^h)}{\|v^h\|_X \ \|q^h\|_M} \ \geq \beta > 0. \tag{4.24}$$

Proof. Let $q^h = (q_1^h, q_2^h) \in M^h \subset M$ be given and let $\tilde{q} \in L_0^2(\Omega)$ denote the function with $\tilde{q}|_{\Omega_i} = q_i^h$. Then it is known, e.g., [13, 15, 7], that there exists $\tilde{v} \in H^1(\Omega)^d$ with

$$\nabla \cdot \tilde{v} = -\tilde{q} \text{ in } \Omega,\ \ \tilde{v} = 0 \text{ on } \partial\Omega,$$

satisfying

$$\|\tilde{v}\|_{1,\Omega} \leq C\|\tilde{q}\|_{0,\Omega}.$$

Define $v = (v_1, v_2) \in X$ by $v_i = \tilde{v}|_{\Omega_i},\ i = 1, 2$, so that

$$b(v, q^h) = -(\nabla \cdot \tilde{v}, \tilde{q})_\Omega = \|\tilde{q}\|^2_{0,\Omega} = \|q^h\|^2_M.$$

The above a priori bound on $\tilde{v}$ implies

$$b(v, q^h) \geq \frac{1}{C}\ \|\tilde{v}\|_{1,\Omega}\ \|q^h\|_M,$$

which implies an inf-sup condition, similar to (4.24), only over (V, M^h) rather than (V^h, M^h).

To prove the condition (4.24) over (V^h, M^h), we now construct (following Fortin's idea) an operator $\Pi^h : X_1 \times (X_2 \cap (H^1(\Omega_2))^d) \to V^h$ with

$$b(\Pi^h v - v, q^h) = 0 \text{ for all } q^h \in M^h \text{ and } \|\Pi^h v\|_X \leq C\|\tilde{v}\|_{1,\Omega}.$$

Indeed, if such an operator exists, then we have

$$\frac{1}{C}\|q^h\|_M \leq \frac{b(v, q^h)}{\|\tilde{v}\|_{1,\Omega}} = \frac{b(\Pi^h v, q^h)}{\|\tilde{v}\|_{1,\Omega}} \leq \frac{b(\Pi^h v, q^h)}{\frac{1}{C}\|\Pi^h v\|_X} \qquad \text{for all } q^h \in M^h,$$

which would prove (4.24).

Let $\Pi^h v = (\Pi^h_1 v, \Pi^h_2 v) \in X^h_1 \times X^h_2$. To define Π^h_1, note that since (X^h_1, M^h_1) is LBB-stable, by Lemma 1.1 in Chapter II section 1.1 of [15], there exists an operator $i^h_1 : X_1 \to X^h_1$ satisfying, for all $v_1 \in X_1$,

$$b_1(i^h_1 v_1 - v_1, q^h_1) = 0 \qquad \text{for all } q^h_1 \in M^h_1$$

and

$$\|i^h_1 v_1\|_{X_1} \leq C\|v_1\|_{X_1}.$$

Thus, define

$$\Pi^h_1 v := i^h_1 v_1 \in X^h_1.$$

Next, construct a $w_2 \in (H^1(\Omega_2))^d$ with

$$\left\{ \begin{array}{l} \nabla \cdot w_2 = \nabla \cdot v_2 \text{ in } \Omega_2, \\ w_2 = 0 \text{ on } \Gamma_2 \text{ and } w_2 = \Pi^h_1 v \text{ on } \Gamma_I. \end{array} \right. \tag{4.25}$$

Indeed, let $g \in L^2(\partial\Omega_2)$ be given by

$$g = \left\{ \begin{array}{l} 0 \text{ on } \Gamma_2, \\ \Pi^h_1 v \text{ on } \Gamma_I. \end{array} \right.$$

Since $\Pi^h_1 v = 0$ on $\partial\Gamma_I, \Pi^h_1 v \in H^{1/2}_{00}(\Gamma_I)^d$. Thus, $g \in H^{1/2}(\partial\Omega_2)^d$ and

$$\begin{aligned} \|g\|_{1/2,\partial\Omega_2} &\leq C\|\Pi^h_1 v\|_{1/2,\Gamma_I} \leq C\|\Pi^h_1 v\|_{1/2,\partial\Omega_1} \\ &\leq C\|\Pi^h_1 v\|_{1,\Omega_1} \leq C\|i^h_1 v_1\|_{X_1} \leq C\|v_1\|_{1,\Omega_1}. \end{aligned}$$

Thus, there exists an extension $z \in H^1(\Omega_2)^d$ with

$$z = g \text{ on } \partial\Omega, \|z\|_{1,\Omega_2} \leq C\|g\|_{1/2,\partial\Omega_2} \leq C\|v_1\|_{1,\Omega_1}.$$

Next, write $w_2 = z + w_0$, where w_0 satisfies

$$\nabla \cdot w_0 = \nabla \cdot (v_2 - z) \text{ in } \Omega_2, w_0 = 0 \text{ on } \partial\Omega_2.$$

The solution to this problem $w_0 \in H^1(\Omega)^d$ exists [15] and satisfies

$$\begin{aligned}\|w_0\|_{1,\Omega_2} &\leq C\|\nabla \cdot (v_2 - z)\|_{0,\Omega_2} \leq C(\|v_2\|_{1,\Omega_2} + \|z\|_{1,\Omega_2}) \\ &\leq C\{\|v_2\|_{1,\Omega_2} + \|v_1\|_{1,\Omega_1}\} \leq C\|\tilde{v}\|_{1,\Omega}.\end{aligned}$$

The function w_2, so constructed, satisfies (4.25) and

$$\|w_2\|_{1,\Omega_2} \leq C\|\tilde{v}\|_{1,\Omega}. \tag{4.26}$$

Finally, define $\Pi_2^h v$ as the finite element (quasi) interpolant of $w_2 \in X_2$,

$$\Pi_2^h v := I_{X_2}^h w_2 \in X_2^h.$$

From the assumed properties of $I_{X_2}^h$, (4.14) with $s = m = 1$, we get

$$\|I_{X_2}^h w_2\|_{1,K} \leq C\|w_2\|_{1,K},$$

so that (squaring and summing over $K \in \mathcal{T}_2^h$)

$$\begin{aligned}\|I_{X_2}^h w_2\|_{X_2}^2 &= \sum_{K \in \mathcal{T}_2^h} \{\|I_{X_2}^h w_2\|_{0,K}^2 + \|\nabla \cdot I_{X_2}^h w_2\|_{0,K}^2\} \\ &\leq C\|w_2\|_{1,\Omega_2}^2.\end{aligned}$$

This with (4.26) gives

$$\|\Pi_2^h v\|_{X_2} \leq C\|\tilde{v}\|_{1,\Omega},$$

which is one of the two required conditions on Π^h. Next, we show

$$b(\Pi^h v - v, q^h) = 0 \qquad \text{for all } q^h \in M^h.$$

Let $q^h = (q_1^h, q_2^h) \in M^h$. Then, for all $K \in \mathcal{T}_2^h$, $q_2^h|_K \in \mathcal{P}_{r_2}(K)$. We thus get from (4.7) and (4.25) that

$$(\nabla \cdot \Pi_2^h v, q_2^h) = (\nabla \cdot I_{X_2}^h w_2, q_2^h)_K = (\nabla \cdot w_2, q_2^h)_K = (\nabla \cdot v_2, q_2^h)_K.$$

Thus, by summing over K, we get

$$b_2(\Pi_2^h v, q_2^h) = b_2(v_2, q_2^h) \qquad \text{for all } q_2^h \in M_2^h. \tag{4.27}$$

Now, let $E \in \mathcal{E}_h(\Gamma_I)$ be an element face on the interface and let $\mu \in R_{r_2}(E)$. Then, (4.9) in Lemma 4.2 implies (noting that $\Pi_2^h v = I_{X_2}^h w_2$)

$$\langle \Pi_2^h v \cdot \hat{n}_2, \mu\rangle_E = \langle I_{X_2}^h w_2 \cdot \hat{n}_2, \mu\rangle_E = \langle w_2 \cdot \hat{n}_2, \mu\rangle_E = \langle \Pi_1^h v \cdot \hat{n}_2, \mu\rangle_E,$$

where the fact that $w_2 = \Pi_1^h v$ on Γ_I (see (4.25)) was used. Thus

$$\langle \Pi_1^h v \cdot \hat{n}_1 + \Pi_2^h v \cdot \hat{n}_2, \mu \rangle_E = 0 \qquad \text{for all } \mu \in R_{r_2}(E).$$

The definition of Λ^h and summing over $E \subset \Gamma_I$ now implies that

$$\langle \Pi_1^h v \cdot \hat{n}_1 + \Pi_2^h v \cdot \hat{n}_2, \mu^h \rangle_{\Gamma_I} = 0 \qquad \text{for all } \mu^h \in \Lambda^h. \tag{4.28}$$

In other words, $\Pi^h v = (\Pi_1^h v_1, \Pi_2^h v_2) \in V^h$. Since we have shown

$$b_j(\Pi_j^h v, q_j^h) = b_j(v_j, q_j^h),\ \ j = 1, 2,$$

it follows that

$$b(\Pi^h v, q^h) = b(v, q^h),$$

completing the proof. ∎

4.3. Approximation of the coupled problem in V^h. The finite element spaces X_1^h and X_2^h are well understood so the approximation properties of $X^h = X_1^h \times X_2^h$ are known and asymptotically optimal. On the other hand, the finite element space arising in the error analysis is V^h rather than X^h. If $X^h \times \Lambda^h$ satisfied a discrete inf-sup condition similar to (3.13), then the abstract theory of mixed methods [15, 7] would imply that the error in approximation in V^h would be comparable to that in $X^h \times \Lambda^h$. However, $\Lambda^h \not\subset \Lambda$ since functions in Λ^h do not vanish at $\partial\Gamma_I$ (a key condition in the continuous case). Therefore, we do not, in general, expect this discrete inf-sup condition to hold.

Thus, the approximation properties of

$$V^h = \{v^h \in X^h : \langle v_1^h \cdot \hat{n}_1 + v_2^h \cdot \hat{n}_2, \mu \rangle_{\Gamma_I} = 0 \text{ for all } \mu \in \Lambda^h\}$$

must be delineated by a direct construction. Herein, we shall construct an interpolation operator

$$I^h := W \to V^h,$$

where W is a subspace of V of sufficiently smooth functions. To that end, we choose s_i sufficiently large and define W as follows:

$$\begin{aligned} W := \{v = (v_1, v_2) \in X : v_i \in W_i := X_i \cap (H^{s_i}(\Omega_i))^d,\ i = 1, 2,& \\ \text{and } v_1 \cdot \hat{n}_2|_{\Gamma_I} = v_2 \cdot \hat{n}_2|_{\Gamma_I} \text{ in } L^2(\Gamma_I)\}.& \end{aligned} \tag{4.29}$$

The construction of I^h will be based on the finite element interpolation operators: $I_{X_i}^h : W_i \to X_i^h$ $(i = 1, 2)$. Define $I^h = (I_1^h v, I_2^h v) \in V^h$ via

$$I_1^h v = I_{X_1}^h v_1 \in X_1^h, \qquad I_2^h v = I_{X_2}^h v_2 - \delta_2^h \in X_2^h,$$

where the (small) correction $\delta_2^h \in X_2^h$ is chosen to enforce in a discrete sense continuity of the normal velocities across Γ_I in (4.29).

Construction of the correction δ_2^h enforcing $I^h v \in V^h$. By the choice of $I_{X_2}^h$ and Λ^h we get the following relation for all $\mu^h \in \Lambda^h$:

$$\begin{aligned} &\langle I_1^h v \cdot \hat{n}_1 + I_2^h v \cdot \hat{n}_2, \mu^h \rangle_{\Gamma_I} \\ &\quad = -\langle I_{X_1}^h v_1 \cdot \hat{n}_2, \mu^h \rangle_{\Gamma_I} + \langle v_2 \cdot \hat{n}_2, \mu_h \rangle_{\Gamma_I} - \langle \delta_2^h \cdot \hat{n}_2, \mu^h \rangle_{\Gamma_I} \\ &\quad = \langle (v_1 - I_{X_1}^h v_1) \cdot \hat{n}_2, \mu^h \rangle_{\Gamma_I} - \langle \delta_2^h \cdot \hat{n}_2, \mu^h \rangle_{\Gamma_I}. \end{aligned} \tag{4.30}$$

To construct δ_2^h we shall first construct $\delta_2 \in X_2 \cap (H^1(\Omega_2))^d$ such that

$$\delta_2 = v_1 - I^h_{X_1} v_1 \text{ on } \Gamma_I, \text{ and } \|\delta_2\|_{1,\Omega_2} \le C|v_1 - I^h_{X_1} v_1|_{1,\Omega_1}. \tag{4.31}$$

To this end, let g_2 extend $v_1 - I^h_{X_1} v_1$ by zero to $\partial\Omega_2$:

$$g_2 := \begin{cases} v_1 - I^h_{X_1} v_1 \text{ on } \Gamma_I, \\ 0 \text{ on } \Gamma_2 = \partial\Omega_2 \setminus \Gamma_I. \end{cases}$$

Since $(v_1 - I^h_{X_1} v_1) = 0$ on $\partial\Gamma_I, (v_1 - I^h_{X_1} v_1) \in H^{1/2}_{00}(\Gamma_I)$ so $g_2 \in H^{1/2}(\partial\Omega_2)^d$. Further, we have the bound

$$\begin{aligned} \|g_2\|_{1/2,\partial\Omega_2} &\le C\|v_1 - I^h_{X_1} v_1\|_{1/2,\Gamma_I} \le C\|v_1 - I^h_{X_1} v_1\|_{1/2,\partial\Omega_1} \\ &\le C\|v_1 - I^h_{X_1} v_1\|_{1,\Omega_1} \le C|v_1 - I^h_{X_1} v_1|_{1,\Omega_1}. \end{aligned}$$

Since $H^{1/2}(\partial\Omega_2)^d$ is the range of the trace operator on $H^1(\Omega_2)^d$, we can find a $\delta_2 \in H^1(\Omega_2)^d$ extending g_2 onto Ω_2 and satisfying

$$\|\delta_2\|_{1,\Omega_2} \le C\|g_2\|_{1/2,\partial\Omega_2} \le C|v_1 - I^h_{X_1} v_1|_{1,\Omega_1}.$$

Define δ_2^h as the interpolant of this extension:

$$\delta_2^h := I^h_{X_2} \delta_2. \tag{4.32}$$

The property (4.9) of $I^h_{X_2}(\cdot)$ implies that for $\mu^h \in \Lambda^h$

$$\langle \delta_2^h \cdot \hat{n}_2, \mu^h \rangle_{\Gamma_I} = \langle \delta_2 \cdot \hat{n}_2, \mu^h \rangle_{\Gamma_I} = \langle (v_1 - I^h_{X_1} v_1) \cdot \hat{n}_2, \mu^h \rangle_{\Gamma_I}.$$

Combining this with (4.30) gives

$$\langle I_1^h v \cdot \hat{n}_1 + I_2^h v \cdot \hat{n}_2, \mu^h \rangle_{\Gamma_I} = 0 \qquad \text{for all } \mu^h \in \Lambda^h, \tag{4.33}$$

implying that $(I_1^h v, I_2^h v) \in V^h$. Thus, this completes the construction of $I^h : W \to V^h$. We shall need an estimate of the correction term $\|\delta_2^h\|_{X_2}$ developed as follows.

From the interpolation error estimates we get, for every $K \in \mathcal{T}_2^h$,

$$\|\delta_2^h\|_{1,K} \le \|\delta_2\|_{1,K} + \|\delta_2 - I^h_{X_2}\delta_2\|_{1,K} \le C\|\delta_2\|_{1,K}.$$

Thus, (summing over $K \subset \Omega_2$)

$$\|\delta_2^h\|_{X_2} = \left\{ \|\delta_2^h\|^2_{0,\Omega_2} + \|\nabla \cdot \delta_2^h\|^2_{0,\Omega_2} \right\}^{1/2} \le \left\{ \sum_{K \in \mathcal{T}_2^h} \|\delta_2^h\|^2_{1,K} \right\}^{1/2},$$

which implies

$$\|\delta_2^h\|_{X_2} \le C\|\delta_2\|_{1,\Omega_2} \le C|v_1 - I^h_{X_1} v_1|_{1,\Omega_1}. \tag{4.34}$$

Bound (4.34) now gives interpolation error estimates for $I_1^h v = I^h_{X_1} v_1$ and $I_2^h v = I^h_{X_2} v_2 - \delta_2^h$:

$$\begin{aligned} \|v - I^h v\|_X &\le |v_1 - I_1^h v|_{1,\Omega_1} + \|v_2 - I_2^h v\|_{X_2} \\ &\le C|v_1 - I^h_{X_1} v_1|_{1,\Omega_1} + \|v_2 - I^h_{X_2} v_2\|_{X_2}. \end{aligned} \tag{4.35}$$

Combining these with the estimates for $I^h_{X_j}$ (see (4.3)),

$$|v_1 - I^h_{X_1} v_1|_{1,\Omega_1} \le C \left\{ \sum_{K \in \mathcal{T}_1^h} \left(h_K^{r_1} |v_1|_{r_1+1,\delta(K)}\right)^2 \right\}^{1/2},$$

$$\|v_2 - I^h_{X_2} v_2\|_{X_2} \le C \left\{ \sum_{K \in \mathcal{T}_2^h} \left(h_K^{r_2+1} (|v_2|_{r_2+1,K} + |\nabla \cdot v_2|_{r_2+1,K})\right)^2 \right\}^{1/2},$$

and using (4.4) and the fact that an element $\widetilde{K}$ can belong at most to a finite number $n(\widetilde{K}) \le C$ of local patches $\delta(K)$ leads to the following result.

PROPOSITION 4.2. *For all $v \in W \subset V$ (given by* (4.29)*), the interpolation operator $I^h : W \to V^h$ satisfies*

$$\|v - I^h v\|_X \le C \left\{ \sum_{K \in \mathcal{T}_1^h} (h_K^{r_1} |v_1|_{r_1+1,K})^2 + \sum_{K \in \mathcal{T}_2^h} \left(h_K^{r_2+1} \; (|v_2|_{r_2+1,K} + |\nabla \cdot v_2|_{r_2+1,K})\right)^2 \right\}^{1/2}. \qquad \square$$

4.4. Discretization error estimates. Since, as noted above, $\Lambda^h \not\subset \Lambda$ and $V^h \not\subset V$, the associated discretizations of *either* saddle point formulations contain an extra *consistency error* which must be estimated using the earlier constructions. Indeed, the abstract error estimates from Brezzi and Fortin [7, Chap. II, sect. 2.6, Proposition 2.16] give the following.

LEMMA 4.4. *Let $(u, p) \in V \times M$ be the solution of the weak formulation* (2.5) *of the coupled problem. Let $(u^h, p^h) \in V^h \times M^h$ be the solution of the discrete problem* (4.15)*. Let the finite element spaces be chosen as in subsection* 4.1*, satisfying LBB-stability (subsection* 4.2*) and approximability (subsection* 4.3*). Then,*

$$\|u - u^h\|_X + \|p - p^h\|_M \le C \left\{ \inf_{v^h \in V^h} \|u - v^h\|_X + \inf_{q^h \in M^h} \|p - q^h\|_M \right\} + \mathcal{H}^h,$$

where

$$\mathcal{H}^h := \sup_{v^h \in V^h} \frac{|a(u, v^h) + b(v^h, p) - \ell(v^h)|}{\|v^h\|_X}$$

is the consistency error. $\square$

The error analysis thus depends on obtaining a bound on the consistency error term $\mathcal{H}^h$. To this end, suppose the weak solution (u, p) to the coupled problem is smooth enough (to be made precise soon) and that $\lambda \in H^s(\Gamma_I)$ (for some s depending on the smoothness of (u, p)), where λ is defined in (2.2).

The variational formulation (2.4) of (u, p, λ) in (X, M, Λ) implies that

$$a(u, v^h) + b(v^h, p) + \langle \lambda, v_1^h \cdot \hat{n}_1 + v_2^h \cdot \hat{n}_2 \rangle_{\Gamma_I} = \ell(v^h) \qquad \text{for all } v^h \in X^h.$$

Thus, if we define the consistency error functional

$$\theta(v^h) := a(u, v^h) + b(v^h, p) - \ell(v^h), \qquad v^h \in X^h,$$

it follows that

$$\theta(v^h) = -\langle v_1^h \cdot \hat{n}_1 + v_2^h \cdot \hat{n}_2, \lambda\rangle_{\Gamma_I} \qquad \text{for all } v^h \in V^h \subset X^h.$$

LEMMA 4.5 (consistency error estimate). *For all $v^h \in V^h$, there holds*

$$|\theta(v^h)| \le C \left\{ \sum_{E\in\mathcal{E}_h(\Gamma_i)} (h_E^s |\lambda|_{s,E})^2 \right\}^{1/2} \|v^h\|_X, \tag{4.36}$$

for $0 \le s \le r_2 + 1$.

Proof. Let $\mu^h \in \Lambda^h$ denote the $L^2(\Gamma_I)$ projection of λ into Λ^h. Since Λ^h consists of *discontinuous* piecewise polynomials, the orthogonality relation for μ^h holds edge by edge:

$$\langle \lambda - \mu^h, w\rangle_E = 0 \qquad \text{for all } w \in R_{r_2}(E),\ \text{for all } E \in \mathcal{E}_h(\Gamma_I). \tag{4.37}$$

From the definition of V^h it follows that, for all $v^h \in V^h$,

$$\begin{aligned}\theta(v^h) &= \langle v_1^h \cdot \hat{n}_1 + v_2^h \cdot \hat{n}_2,\ \mu^h - \lambda\rangle_{\Gamma_I} \\ &= \langle v_1^h \cdot \hat{n}_1, \mu^h - \lambda\rangle_{\Gamma_I} + \sum_{E\in\mathcal{E}_h(\Gamma_I)} \langle \mu^h - \lambda, v_2^h \cdot \hat{n}_2\rangle_E.\end{aligned}$$

By Lemma 4.2 we have that

$$w = v_2^h \cdot \hat{n}_2|_E \in R_{r_2}(E) \qquad \text{for all } E \in \mathcal{E}(K),\ K \in \mathcal{T}_2^h,$$

which implies

$$\langle \mu^h - \lambda, v_2^h \cdot \hat{n}_2\rangle_E = 0 \qquad \text{for all } E \in \mathcal{E}_h(\Gamma_I).$$

Thus, $\theta(v^h) = \langle v_1^h \cdot \hat{n}_1, \mu^h - \lambda\rangle_{\Gamma_I}$, for all $v^h \in V^h$, and it follows that

$$\begin{aligned}|\theta(v^h)| &\le \sum_{E\in\mathcal{E}_h(\Gamma_I)} \|v_1^h\|_{0,E} \|\lambda - \mu^h\|_{0,E} \\ &\le \left(\sum_{E\in\mathcal{E}_h(\Gamma_I)} \|\lambda - \mu^h\|_{0,E}^2 \right)^{1/2} \|v_1^h\|_{0,\Gamma_I}. \end{aligned} \tag{4.38}$$

By the trace theorem and the Poincaré–Friedrichs inequality,

$$\|v_1^h\|_{0,\Gamma_I} \le C\|v^h\|_X.$$

Since μ^h is the $L^2(E)$ projection of λ into $R_{r_2}(E)$ by (4.37), it follows that

$$\|\lambda - \mu^h\|_{0,E} \le C h_E^s |\lambda|_{s,E}, \qquad \text{for } 0 \le s \le r_2 + 1,\ E \in \mathcal{E}_h(\Gamma_I).$$

Using the last two bounds in (4.38) completes the proof. ∎

Lemma 4.4 immediately yields a bound on the consistency error term $\mathcal{H}^h$.

COROLLARY 4.1. *There holds*

$$\mathcal{H}^h \le C \left\{ \sum_{E\in\mathcal{E}_h(\Gamma_I)} (h_E^s |\lambda|_{s,E})^2 \right\}^{1/2}, \quad \textit{for}\ \ 0 \le s \le r_2 + 1. \qquad ∎$$

This bound can now be used in the abstract error estimate in Lemma 4.3 to yield a convergence theorem.

THEOREM 4.1. *Let the weak solution* (u,p) *to* (2.5) *be sufficiently smooth (that the norms in* (4.39) *are finite). Let* $(u^h,p^h)\in V^h\times M^h$ *be the finite element approximation to* (u,p)*. Then,*

$$\begin{aligned}\|u-u^h\|_X+\|p-p^h\|_M \le C\Bigg\{&\left\{\sum_{K\in\mathcal{T}_1^h}\left(h_K^{s_1}(|u_1|_{s_1+1,K}+|p_1|_{s_1,K})\right)^2\right\}^{1/2}\\ &+\left\{\sum_{K\in\mathcal{T}_2^h}\left(h_K^{\tilde{s}_2}|u_2|_{\tilde{s}_2,K}+h_K^{s_2}(|p_2|_{s_2,K}+|\nabla\cdot u_2|_{s_2,K})\right)^2\right\}^{1/2}\\ &+\left\{\sum_{E\in\mathcal{E}_h(\Gamma_I)}(h_E^{s_2}|\lambda|_{s_2,E})^2\right\}^{1/2}\Bigg\},\\ &1\le s_1\le r_1,\ 1\le\tilde{s}_2\le r_2+1,\ 0\le s_2\le l_2+1. \qquad\square\end{aligned}\tag{4.39}$$

REMARK 4.5. *Theorem* 4.1 *implies optimal error bounds in both the fluid region and in the porous medium region.*

REMARK 4.6. *We have just learned of the concurrent work of Discacciati, Miglio, and Quarteroni* [11] *on a closely related problem. They consider Stokes–Darcy coupling with a free slip condition on* Γ_I *(i.e.,* $\alpha_1=0$ *in* (1.6)*) and the formulation of the Darcy model as a Poisson problem rather than as a mixed method, and they obtain interesting results.*

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